Full worked solutions to selected examples from control section of the course

Introduction

These worked solutions are intended as a supplement to the tutorial examples covered in the seminars. Only the odd numbered examples are included – this is deliberate, we would like you to try the even numbered examples for yourselves and we will be happy to help if you are stuck on any of these.

Solutions to the even numbered examples will not be made generally available on moodle.

CONTROL: EXERCISE SHEET 0

1. Determine the Laplace transform $F(s)$ of $f(t)$, if:

   a) $f(t) = 0.5\frac{dx}{dt} + 4x$, and $x = 4$ when $t = 0$

   Solution: In these cases, it's best to be methodical. The question tells us that x=4 for t=0, so we cannot use the normal assumption that the system is at rest and x=0 at t=0.

   So the Laplace transform of $0.5\frac{dx}{dt}$ is $0.5\big(sX(s) - f(0)\big)s = 0.5sX(s) - 2$ from the table of Laplace Transforms.

   $4x$ has a Laplace transform of $4X(s)$.

   And therefore

   $$F(s) = (0.5s + 4)X(s) - 2$$

   b) $f(t) = \frac{d^2x}{dt^2} + 0.1\frac{dx}{dt} + 3x$, and $x = 10$ and $\frac{dx}{dt} = 2$ when $t = 0$

   Solution:

   The Laplace transform of $\frac{d^2x}{dt^2}$ is

   $$s^2X(s) - sx(0) - x^1(0)$$

   To explain the notation, $x^1(0)$ is the value of the first derivative with respect to time - $\frac{dx}{dt}$ at t=0

   From the question:

   $$\frac{d^2x}{dt^2} \to s^2X(s) - 10s - 2$$

   $$\frac{dx}{dt} \to sX(s) - 10$$

   $$Hence$$

   Hence:

   $$F(s) = s^2X(s) - 10s - 2 + 0.1(sX(s) - 10) + 3X(s)$$

   Simplifying:

   $$F(s) = (s^2 + 0.1s + 3)X(s) - 10s - 3$$

   c) $f(t) = \frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} + 0.1\frac{dx}{dt} + 3x$, and $x = 0, \frac{dx}{dt} = 0, \frac{d^2x}{dt^2} = 0$ when $t = 0$

   Using the same breakdown as before:

   The Laplace transform of $\frac{d^3x}{dt^3}$ is

$$s^3X(s) - sx(0) - x^1(0)$$

2. a) Use Laplace transforms to determine the solution to the following differential equation in the time domain (i.e. $x(t)$)

$$\frac{d^2x}{dt^2} + 0.1\frac{dx}{dt} + x = f(t)$$

Where $f(t)$ is a unit step and the initial conditions are taken to be zero

b) Determine the transfer function $G(s)$ of the system analysed in (a) taking $f(t)$ to be the input and $x(t)$ to be the output of the system.

3. Determine the transfer function for the following, where x_i is the input and x_o is the output

a) $\frac{d^2x_o}{dt^2} + 2\zeta\omega_n\frac{dx_o}{dt} + \omega_n^2x_o = x_i$

Solution:

Breaking up the expression into its component parts, and remembering that to derive a transfer function we take the initial conditions to be zero:

$$\frac{d^2x_o}{dt^2} \xrightarrow{\mathcal{L}} s^2X_o(s)$$

$$2\zeta\omega_n\frac{dx_o}{dt} \xrightarrow{\mathcal{L}} 2\zeta\omega_n sX_o(s)$$

$$\omega_n^2x_o \xrightarrow{\mathcal{L}} \omega_n^2X_o(s)$$

$$x_i \xrightarrow{\mathcal{L}} X_i(s)$$

So:

$$s^2X_o(s) + 2\zeta\omega_n sX_o(s) + \omega_n^2X_o(s) = (s^2 + 2\zeta\omega_n s + \omega_n^2)X_o(s) = X_i(s)$$

The transfer function (what goes out/what goes in):

$$\frac{X_o(s)}{X_i(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

b) $T\dot{x}_i + x_i = x_o$

$$sTX_i(s) + X_i(s) = X_o(s)$$

$$\frac{X_o(s)}{X_i(s)} = sT + 1$$

c) $\frac{d^4x_o}{dt^4} + 3\frac{d^3x_o}{dt^3} + 2\frac{d^2x_o}{dt^2} + 2\frac{dx_o}{dt} + x_o = 2\frac{dx_i}{dt} + 5x_i$

$$(s^4 + 3s^3 + 2s^2 + 2s + 1)X_o(s) = (2s + 5)X_i(s)$$

$$\frac{X_o(s)}{X_i(s)} = \frac{2s + 5}{s^4 + 3s^3 + 2s^2 + 2s + 1}$$

Answers:

1. a) $F(s) = (0.5s + 4)X(s) - 2$

b) $F(s) = (s^2 + 0.1s + 3)X(s) - 10s - 3$

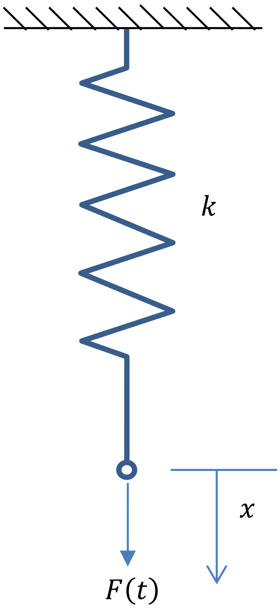
c) $F(s) = (s^3 + s^2 + 0.1s + 3)X(s)$

2. a) $x(t) = 1 - \frac{e^{-0.05t}}{\sqrt{1-(0.05)^2}} \sin\left(t\sqrt{1-(0.05)^2} + \cos^{-1} 0.05\right)$

   b) $G(s) = \frac{1}{(s^2+0.1s+1)}$

EXERCISE SHEET 1

1. Derive expressions for the transfer functions that relate input force $F(t)$, and output displacement $x(t)$, of the spring and mass systems shown in figures 1a and 1b.



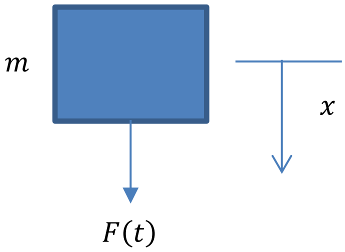


Figure 1(a). Spring System

Figure 1(b). Mass System

Solution:

For Figure 1a, the physics of the spring gives the relationship between the force, F, and x (the question states that x is the extension, so $F = 0$ for $x = 0$) :

$$Force\ (\downarrow) = k\ x$$

Note that at the moment, I am keeping track of direction, we will check through the signs later on.

Expressing this mathematically:

$$F(t) = k\ x(t)$$

(Check that signs are consistent with the definitions)

And with Laplace Transforms:

$$\boxed{F(s) = k\ X(s)}$$

For figure 1b, Newton's second law of motion states that the acceleration of the body will be the magnitude of the force divided my the mass:

$$\frac{d^2x}{dt^2} = \frac{F(t)}{m}$$

Again, check that signs are consistent – will a positive force give a positive acceleration?

From the tables, the Laplace transform of $\frac{d^n}{dt^n}\big(f(t)\big)$ is:

$$s^nF(s) - s^{n-1}f(0) - s^{n-2}f^1(0) \ldots - f^{n-1}(0)$$

So $\frac{d^2x}{dt^2}$ will become $s^2X(s) - sx(0) - x'(0)$ – and in this case we assume that initial conditions are zero, so the Laplace transform of the equation of motion is

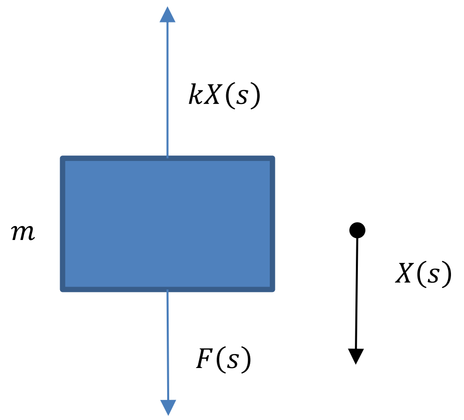
$$s^2\ X(s) = \frac{F(s)}{m}$$

So

$$\boxed{F(s) = ms^2\ X(s)}$$

c) Derive an expression for the transfer function $G(s)$ of a system that combines the spring and mass systems in parts (a) and (b).

The forces acting on the mass are the input force $F(s)$ and the tension in the spring, $k\,X(s)$. Here, it makes sense to sketch a free body diagram for the mass to get the directions of the forces (and hence the signs in the equation) right:



Combining the answers from (a) and (b):

$$F(s) = ms^2\,X(s) = F(s) - k\,X(s)$$
$$F(s) = ms^2\,X(s) + k\,X(s) = (ms^2 + k\,)X(s)$$

Since we are interested in the displacement (response) caused by a given force (input):

$$X(s) = \frac{F(s)}{ms^2 + k}$$

And hence the transfer function relating input (force) to output (displacement) is:

$$G(s) = \frac{Output}{Input} = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + k}$$

4. Numerical answers:

   a) $G(s) = \frac{X(s)}{F(s)} = \frac{1}{k_T}$ b) $G(s) = \frac{X(s)}{F(s)} = \frac{1}{Is^2 + k_T}$

5. Derive expressions for the transfer functions that relate input x_i, and output x_0, of the spring/mass systems shown in figures 1a and 1b.

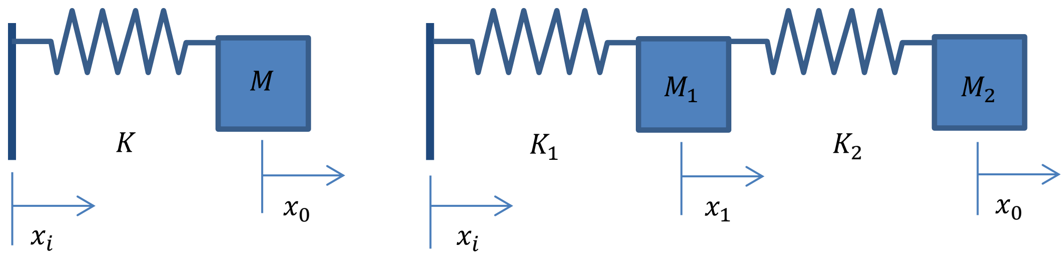


Figure 1a Figure 1b

Solution:

In Figure 1a, think about the force acting on the mass:

$$Force\ (\rightarrow) = K\,(x_i - x_o)$$

This force translates into an acceleration for the mass, M:

$$M\frac{d^2x_o}{dt^2} = K\,(x_i - x_o)$$

Putting the terms in x_o and $x_i$on separate sides of the equation:

$$M\frac{d^2x_o}{dt^2} + Kx_o = Kx_i$$

Laplace transforms give:

$$(Ms^2 + K)X_o(s) = KX_i(s)$$

To make this into a transfer function, remember that $G(s) = \frac{X_o(s)}{X_i(s)}$

$$\frac{X_o(s)}{X_i(s)} = \frac{K}{(Ms^2 + K)}$$

Figure 2b follows similar logic: The force on mass M_2 is given by:

$$Force\ (\rightarrow) = K_2(x_1 - x_o)$$

$$M_2\frac{d^2x_o}{dt^2} = K_2\,(x_1 - x_o) \quad (1)$$

There are two forces on mass M_1:

$$Force\ (\rightarrow) = K_1(x_i - x_1)$$
$$Force\ (\leftarrow) = K_2(x_1 - x_o)$$

$$M_1\frac{d^2x_1}{dt^2} = K_1(x_i - x_1) - K_2\,(x_1 - x_o) \quad (2)$$

Laplace transforms of equations (1) and (2) give:

$$M_2s^2X_o(s) = K_2\,\big(X_1(s) - X_o(s)\big) \quad (3)$$
$$M_1s^2X_1(s) = K_1\,\big(X_i(s) - X_1(s)\big) - K_2\,\big(X_1(s) - X_o(s)\big) \quad (4)$$

From (3):

$$X_1(s) = X_o(s)\left(\frac{M_2s^2 + K_2}{K_2}\right)$$

Substituting in (4) for $X_1(s)$:

$$M_1s^2X_o(s)\left(\frac{M_2s^2 + K_2}{K_2}\right) = K_1\,\left(X_i(s) - X_o(s)\left(\frac{M_2s^2 + K_2}{K_2}\right)\right) - K_2\,\left(X_o(s)\left(\frac{M_2s^2 + K_2}{K_2}\right) - X_o(s)\right)$$

$$M_1s^2X_o(s)\left(\frac{M_2s^2 + K_2}{K_2}\right) + K_1X_o(s)\left(\frac{M_2s^2 + K_2}{K_2}\right) + K_2X_o(s)\left(\frac{M_2s^2 + K_2}{K_2}\right) - K_2X_o(s) = K_1X_i(s)$$

$$\big(M_1s^2X_o(s) + K_1X_o(s) + K_2X_o(s)\big)\left(\frac{M_2s^2 + K_2}{K_2}\right) - K_2X_o(s) = K_1X_i(s)$$

Simplifying gives:

$$\frac{M_1M_2s^4X_o(s) + (K_1M_2 + K_2M_2 + M_1K_2)s^2X_o(s)}{K_2} + K_1X_o(s) + K_2X_o(s) - K_2X_o(s) = K_1X_i(s)$$

Which becomes:

$$M_1M_2s^4X_o(s) + (K_1M_2 + K_2M_2 + M_1K_2)s^2X_o(s) + K_1K_2X_o(s) = K_1K_2X_i(s)$$

And the transfer function is:

$$\frac{X_0}{X_i} = \frac{K_1K_2}{M_1M_2s^4 + (M_2K_1 + M_1K_2)s^2 + K_1K_2}$$

5. Derive expressions for the appropriate transfer functions for the tank systems shown in figures 5 a), b) and c). taking the input and output to be as indicated in the following table:

System	Input	Output
3a	q_i	h_1
3b	q_i	h_2
3c	q_i	h_3

Where A_1, A_2, and A_3 are tank cross sectional areas; h_1, h_2, and h_3 are the liquid heights as indicated; q_i, q, and q_o are volume flow rates; and R_1, R_2, and R_3 are linearised flow resistances.
For systems 3a) and 3b) it should be noted that the volume flow rate (q) through the restrictor tap (denoted by a cross) is given by:

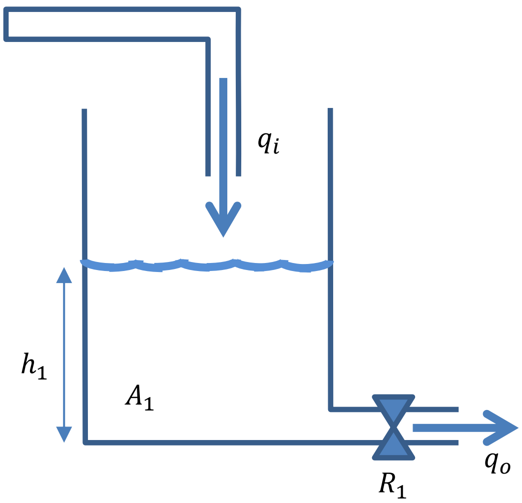
$$q = \frac{h}{R}$$

where h is the height of liquid in the tank and R is the linearised flow resistance.
In system 3c) the volume flow rate q through the restrictor tap is related to the difference in liquid "head" across it by an equation of the form:

$$q = \frac{h_1 - h_2}{R_1}$$

where h_1, and h_2 are the liquid heights in two adjacent, connected tanks, and R_1 is the linearised flow resistance between the connected tanks.

Figure 5 a)



Continuity – volume of water in the tank is given by

$$V = A_1h_1$$

Rate of change of V= what goes in – what goes out:

$$\frac{dV}{dt} = A_1\frac{dh_1}{dt} = q_i - q_o$$

From the question,

$$q_o = \frac{h_1}{R_1}$$

Thus:

$$A_1\frac{dh_1}{dt} = q_i - \frac{h_1}{R_1}$$

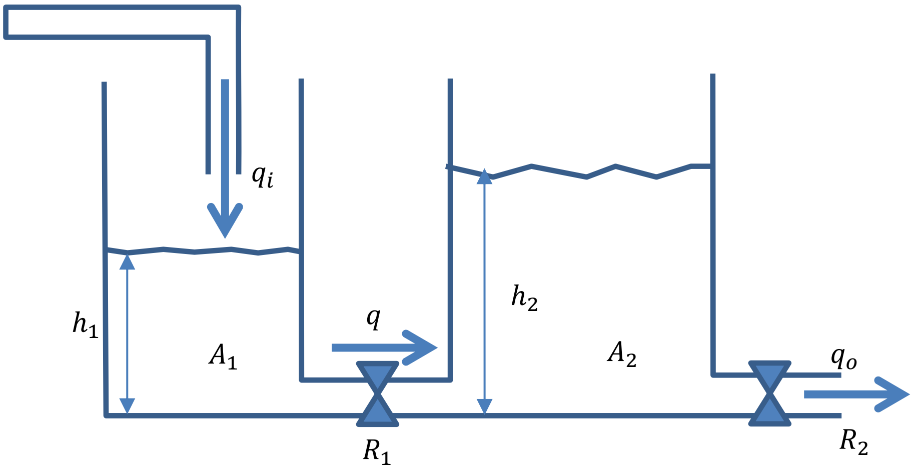
Laplace transforms give:

$$A_1sH_1(s) = Q_i(s) - \frac{H_1(s)}{R}$$

And the transfer function for the system is:

$$G(s) = \frac{H_1(s)}{Q_i(s)} = \frac{R_1}{1 + A_1 R_1 s}$$

Figure 5 c)



In this question, we want a transfer function for $\frac{h_2}{q_i}$:

$$\frac{dV_1}{dt} = A_1\frac{dh_1}{dt} = q_i - q = q_i - \frac{h_1 - h_2}{R_1}$$

$$\frac{dV_2}{dt} = A_2\frac{dh_2}{dt} = q - q_o = \frac{h_1 - h_2}{R_1} - \frac{h_2}{R_2}$$

Laplace transforms give:

$$A_1 s H_1(s) = Q_i(s) - \frac{H_1(s) - H_2(s)}{R_1}$$

$$(R_1 A_1 s + 1)H_1(s) = R_1 Q_i(s) + H_2(s) \qquad (1)$$

$$A_2 s H_2(s) = \frac{H_1(s) - H_2(s)}{R_1} - \frac{H_2(s)}{R_2} \qquad (2)$$

Substituting for $H_1(s)$ in (2) gives:

$$A_2 s H_2(s) = \frac{R_1 Q_i(s) + H_2(s)}{R_1(R_1 A_1 s + 1)} - \frac{H_2(s)}{R_1} - \frac{H_2(s)}{R_2}$$

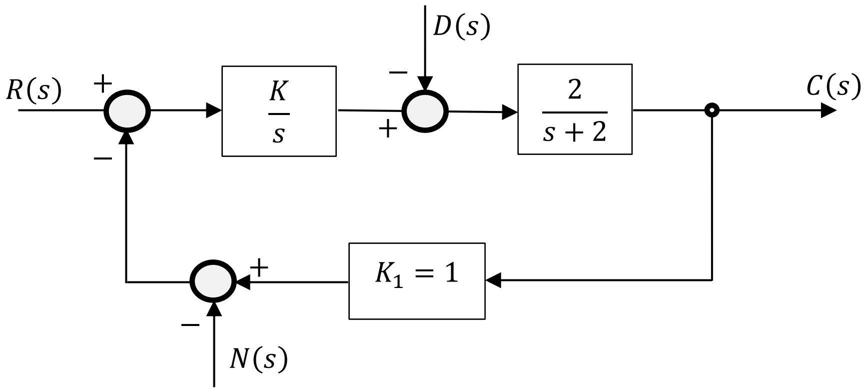
Rearranging gives the transfer function:

$$\frac{H_2(s)}{Q_i(s)} = \frac{R_2}{A_1 A_2 R_1 R_2 s^2 + (A_1(R_1 + R_2) + A_2 R_2)s + 1}$$

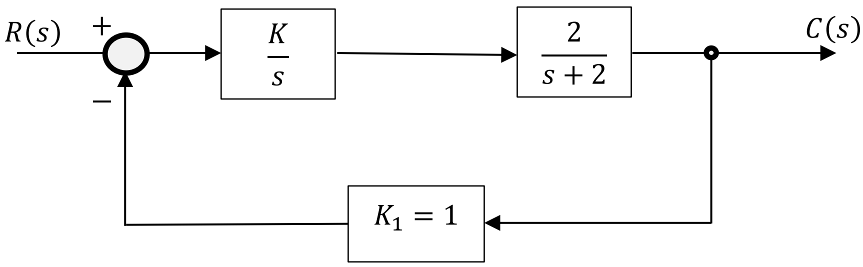
CONTROL: Exercise Sheet 2

SHEET 2: Block Diagram Manipulation

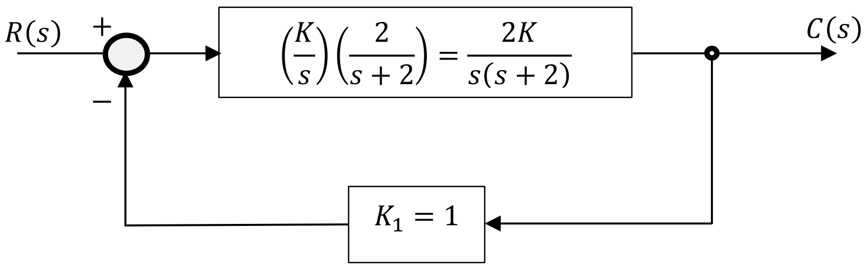
1.



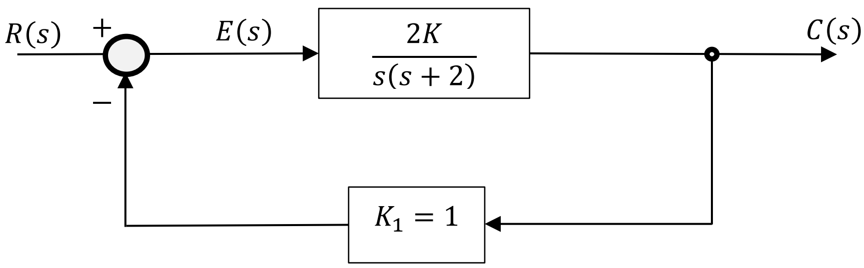
Step 1: In this case we are only interested in $\frac{C(s)}{R(s)}$ so the first thing to do is to redraw the diagram omitting the distubance and reference signals:



Step 2: Combine the forward transfer function (blocks in series multiply):



Step 3: Work out the inputs and outputs



Resolving the summing junction gives: $E(s) = R(s) - C(s)$

Then the forward transfer function gives:

$$C(s) = E(s)\left(\frac{2K}{s(s+2)}\right) = \left(R(s) - C(s)\right)\left(\frac{2K}{s(s+2)}\right)$$

$$C(s)\left(s(s+2)\right) = 2K\left(R(s) - C(s)\right)$$

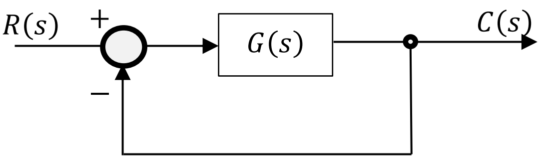
Collect the terms in C(s) on the left and R(s) on the right:

$$C(s)(s^2 + 2s + 2K) = 2K\left(R(s)\right)$$

This gives the transfer function:

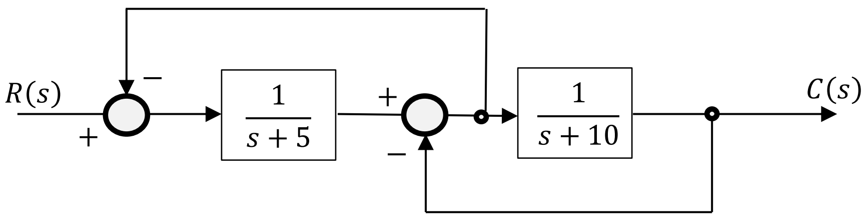
$$\frac{C(s)}{R(s)} = \frac{2K}{s^2 + 2s + 2K}$$

2.



Answer: $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

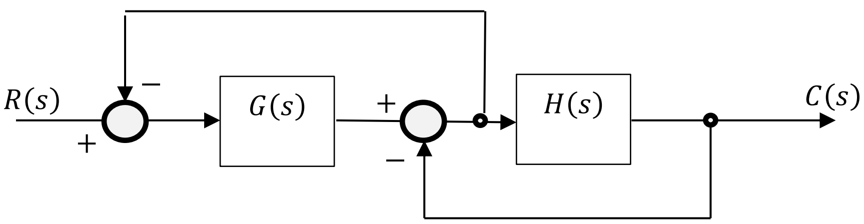
3.



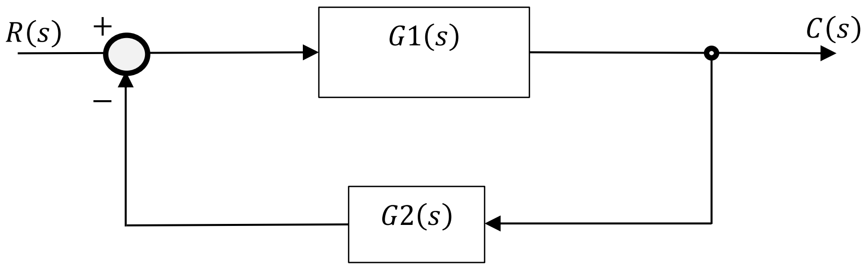
Step 1: assign transfer functions as follows to make drawing and keeping track easier:

$$G(s) = \frac{1}{s+5} \qquad H(s) = \frac{1}{s+10}$$

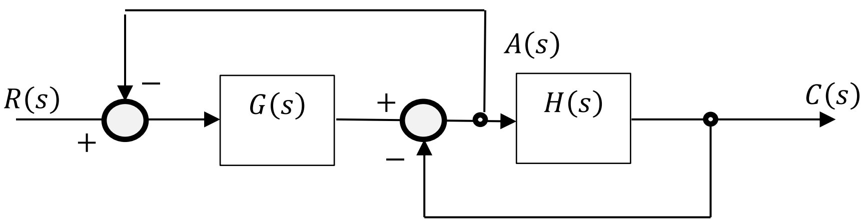
The block diagram becomes:



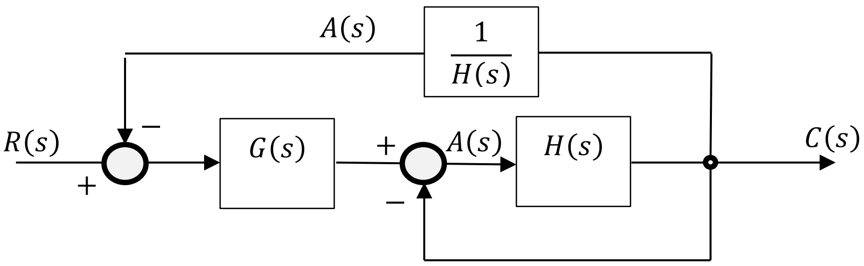
Step 2: The end goal is to have a forward transfer function and a feedback transfer function looking like:



So to achieve this, the feedback loops have to start at $C(s)$ and finish at the summing junction at the left.

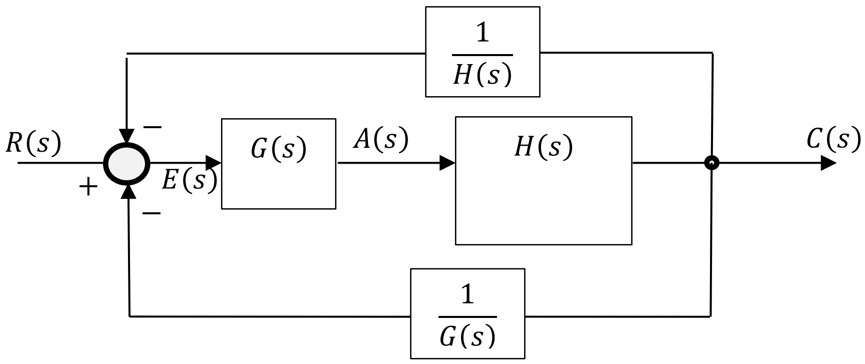


So if we move the origin of the upper feedback loop to the right and divide by $H(s)$, the diagram becomes:



To verify that this will give the same answer, consider the intermediate signal $A(s)$ produced by the summing junction. In the original block diagram, $A(s)$ is fed back and subtracted from $R(s)$. In the new block diagram, A(s) is multiplied by H(s) and subsequently divided by H(s). As these are linear functions, with no discontinuity, this means that A(s) is still the second input to the summing junction.

Step 3: Using the same logic, move the lower feedback loop to the left of G(s) and feed into the first summing junction. Be careful with the signs:



Again, to verify: If the input to the first block, G(s) is E(s) then A(s) is given by:

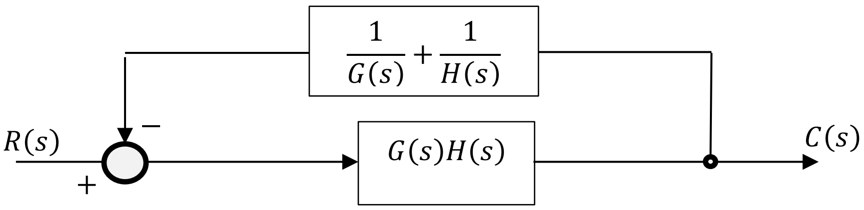
Top diagram: $A(s) = G(s)\left(R(s) - \frac{C(s)}{H(s)}\right) - C(s)$

Bottom diagram: $A(s) = G(s)E(s) = G(s)\left(R(s) - \frac{C(s)}{G(s)} - \frac{C(s)}{H(s)}\right)$

$$A(s) = G(s)E(s) = G(s)\left(R(s) - \frac{C(s)}{H(s)}\right) - C(s)$$

So we have shown that these are equivalent:

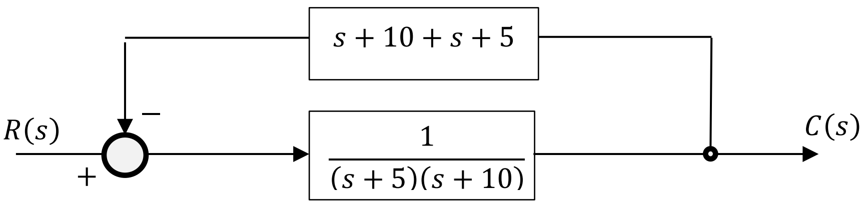
Step 4: Combine feedback transfer functions remember that elements in parallel follow the signs in the summing junction:



Step 5: Put s values back in:

$$G(s) = \frac{1}{s+5} \qquad \frac{1}{G(s)} = s+5 \qquad H(s) = \frac{1}{s+10} \qquad \frac{1}{H(s)} = s+10$$

The block diagram becomes:



Step 5: Algebra

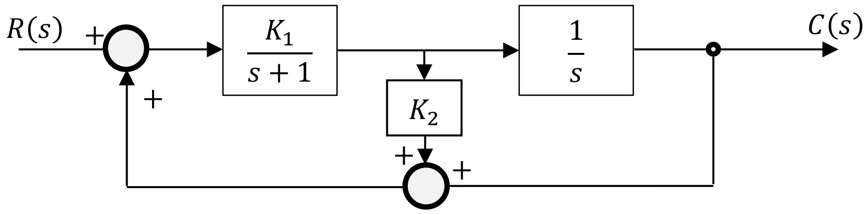
$$C(s) = \left(R(s) - (2s+15)C(s)\right)\left(\frac{1}{(s+5)(s+10)}\right)$$

$$C(s)\left(s^2 + 15s + 50 + (2s+15)\right) = R(s)$$

The transfer function is therefore:

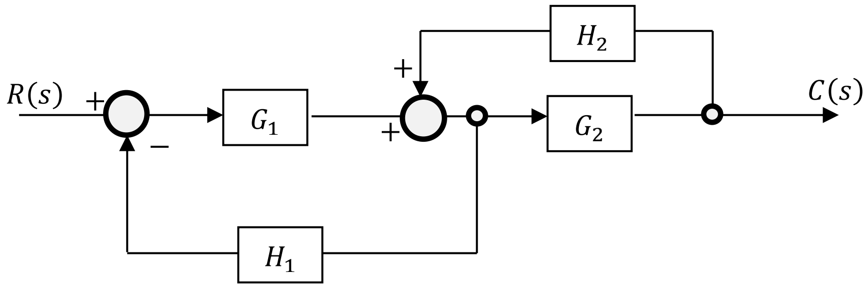
$$\frac{C(s)}{R(s)} = \frac{1}{(s^2 + 17s + 65)}$$

4.

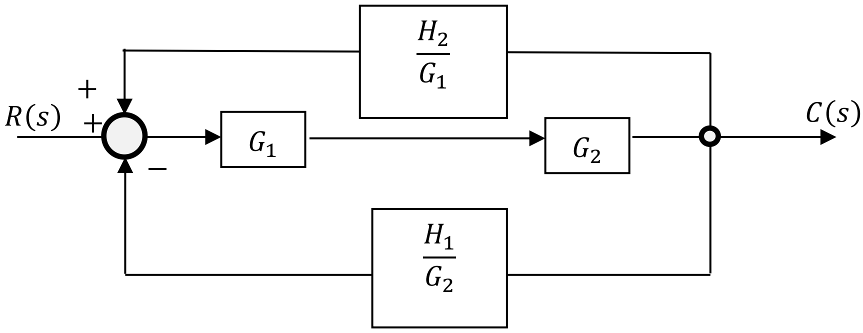


Answer: $\frac{C(s)}{R(s)} = \frac{K_1}{s(s+1)-K_1(K_2s+1)}$

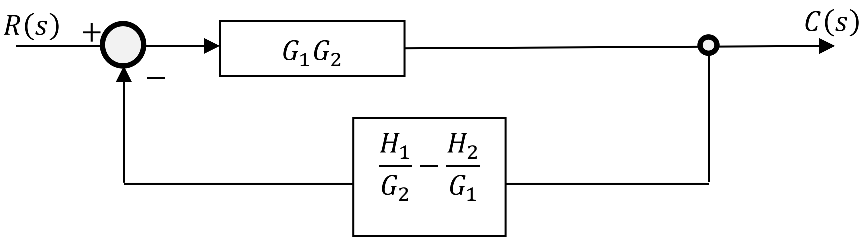
5.



This is quite similar to No. 3 and the approach to take is similar. The upper loop needs to go to the left and the lower loop to the right. Using the same logic as before:



The feedback loops can then be added together to give:



And then

$$C(s) = \left(R(s) - \left(\frac{H_1}{G_2} - \frac{H_2}{G_1}\right) C(s)\right) G_1G_2$$

$$C(s)(1 + H_1G_1 - H_2G_2) = G_1G_2\,R(s)$$

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2}{(1 + H_1 G_1 - H_2 G_2)}$$

6. A system of two tanks similar to the second laboratory experiment (yet different) is shown in figure 7.

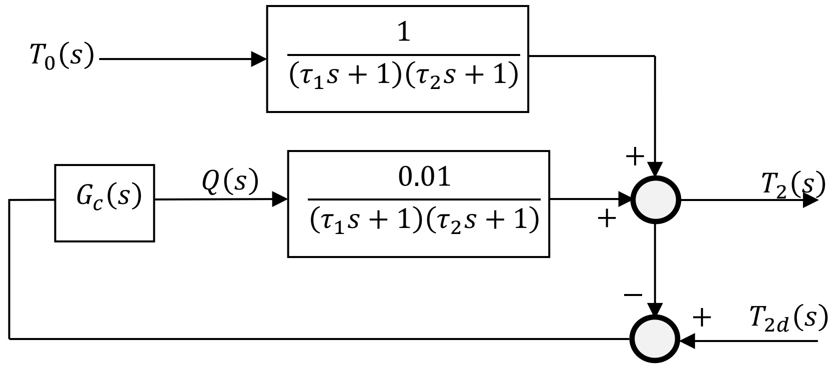


Figure 7

T represents temperature and Q is the heat input. Determine the overall transfer function for the system $G(s) = \frac{T_2(s)}{T_0(s)}$

Answer:

$$G(s) = \frac{T_2(s)}{T_0(s)} = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1) + 0.01 G_c(s)}$$

7. A control system to maintain the speed of a motor is shown in figure 7.

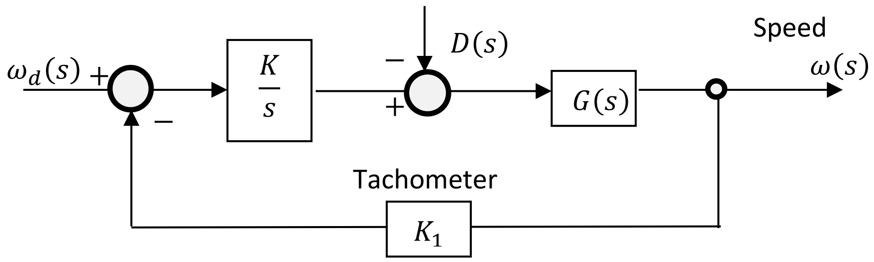
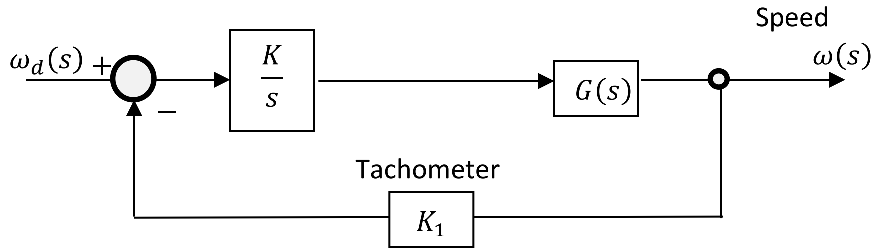


Figure 7.

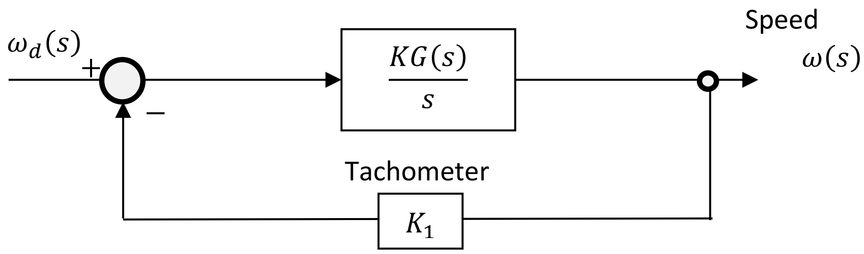
The motor has a transfer function of $G(s) = \frac{1}{s+3}$.

Determine the overall transfer function of the system with ω_d to ω.

Step 1: D(s) will have a different transfer function so we can remove it from the block diagram:



Step 2: Combine the blocks in the forward section:



Step 3: Resolve the transfer function:

$$\omega(s) = \frac{KG(s)}{s}\left(\omega_d(s) - K_1\omega(s)\right)$$

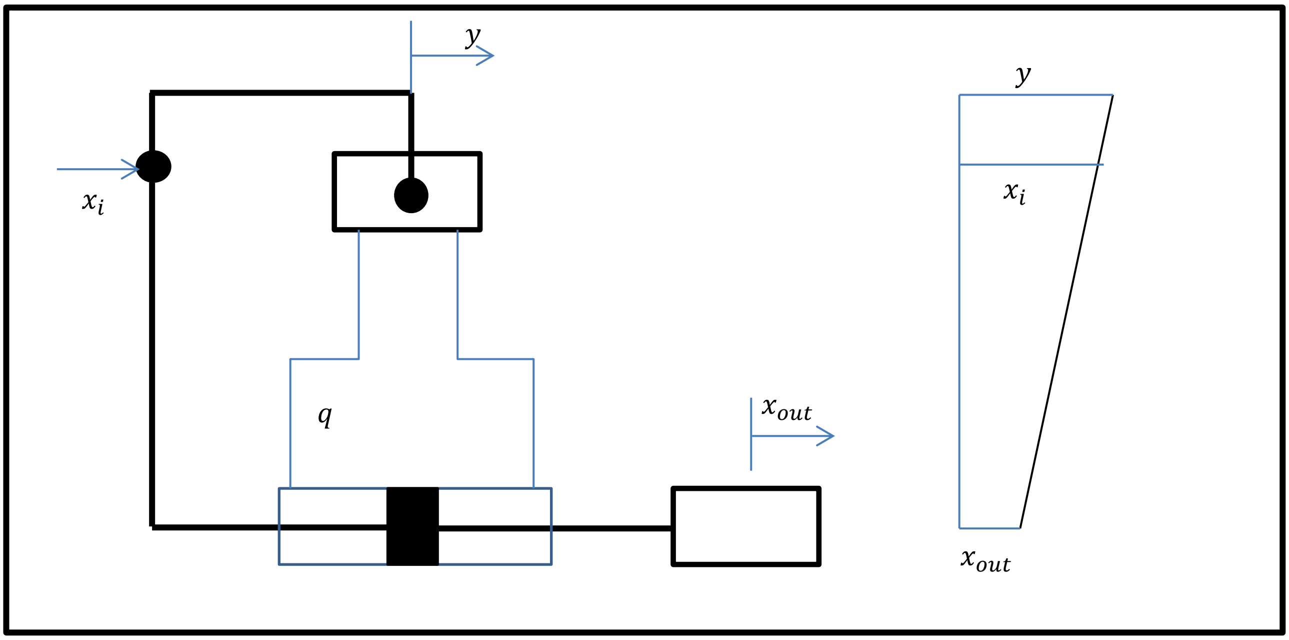
$$\omega(s)\left(1 + \frac{KK_1G(s)}{s}\right) = \frac{KG(s)}{s}\omega_d(s)$$

$$\frac{\omega(s)}{\omega_d(s)} = \frac{\frac{KG(s)}{s}}{1 + \frac{KK_1G(s)}{s}} = \frac{KG(s)}{s + KK_1G(s)}$$

Exercise Sheet 3

Q.1

a)



Step 1: Derive a relationship for x_i, y, x_{out} from the input lever and the feedback link:

From the information in the question,

$$\frac{y - x_i}{0.1} = \frac{x_i - x_{out}}{0.9}$$

(Similar triangles – see diagram on the left)

We are also told that the flow rate of hydraulic pump is given by:

$$q = 0.2y \; m^3 s^{-1}$$

And so the rate of change of load position, x_{out} is given by the flow rate (change in volume) divided by the piston area to give the change in length:

$$\frac{dx_{out}}{dt} = \frac{q}{0.01} ms^{-1}$$

Taking Laplace Transforms:

$$\frac{Y(s) - X_i(s)}{0.1} = \frac{X_i(s) - X_{out}(s)}{0.9}$$

Rearranging gives:

$$9\big(Y(s) - X_i(s)\big) = X_i(s) - X_{out}(s)$$

$$9Y(s) + X_{out}(s) = 10X_i(s)$$

$$Q(s) = 0.2Y(s)$$

$$sX_{out}(s) = \frac{Q(s)}{0.01} = \frac{0.2Y(s)}{0.01} = 20Y(s) \rightarrow Y(s) = \frac{sX_{out}(s)}{20}$$

Substituting for $Y(s)$:

$$\frac{9sX_{out}(s)}{20} + X_{out}(s) = 10X_i(s)$$

$$\frac{X_{out}(s)}{X_i(s)} = \frac{10}{1 + 0.45s}$$

Hence, using the form $\frac{\mu}{1+Ts}$, the gain is 10 and the time constant is 0.45.

Check: We can verify the gain using the final value theorem as follows:

If the input, $X_i(s)$, is a unit step at t=0

$$x_{out}(t \to \infty) = \lim_{s \to 0}\left(s\left(\frac{1}{s}\right)\left(\frac{10}{1+0.45s}\right)\right) = 10$$

b): Error at t=1.0s after a unit step input

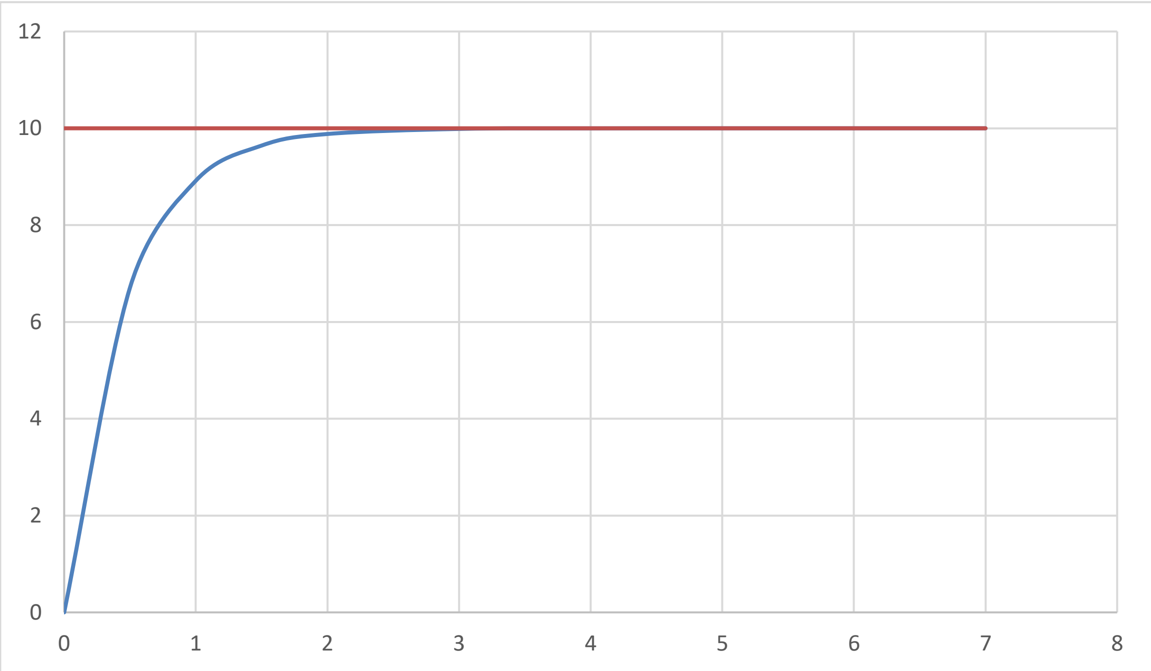
From the table of Laplace transforms, on page 10:

$$X_i(s) = \frac{1}{s}$$

$$X_{out}(s) = \frac{1}{s}\left(\frac{10}{1+0.45s}\right)$$

Taking inverse laplace transforms gives the output in the time domain:

$$x_{out}(t) = \mu\left(1 - e^{-t/T}\right)$$



Graph 1

This function (the blue line) is plotted in graph 1. The red line is the asymptote, y=10.

At t=1, the error is given by:

$$x_{out}(t=\infty) - x_{out}(t=1) = \mu\left(1 - \left(1 - e^{-1/T}\right)\right) = 10 \times e^{-1/0.45} = 1.08$$

c) For a ramp input given by:

$$X_i(s) = \frac{0.01}{s^2}$$

Using the transfer function calculated earlier:

$$\frac{X_{out}(s)}{X_i(s)} = \frac{10}{1+0.45s}$$

the output in the s-domain will be given by

$$X_{out}(s) = \frac{0.01}{s^2}\left(\frac{10}{1+0.45s}\right) = \frac{0.1}{s^2(1+0.45s)}$$

To find the velocity lag for $t \to \infty$:

Method 1: Time domain, taking inverse laplace transforms (number 9, on page 10 of the notes):

$$x_{out}(t) = 0.1t - \frac{0.1}{1/_{0.45}}\left(1 - e^{-t/_{0.45}}\right) = 0.1t - 0.045\left(1 - e^{-t/_{0.45}}\right)$$

And so steady state velocity lag = 0.045

Method 2: Final value theorem

$$Velocity\ lag\ (t \to \infty) = \lim_{s \to 0}\left(s\big(\mu X_i(s) - X_{out}(s)\big)\right) = \lim_{s \to 0} s\left(\frac{0.1}{s^2} - \frac{0.1}{s^2(1+0.45s)}\right)$$

$$= \lim_{s \to 0} s\left(\frac{0.1(1+0.45s) - 0.1}{s^2(1+0.45s)}\right) = = \lim_{s \to 0}\left(\frac{0.045s^2}{s^2(1+0.45s)}\right) = 0.045$$

Note: either method is correct and both take about the same amount of working: the final choice is up to you.

Question 2 numerical answers:

2 a) $\frac{KRH_i(s)+RQ_D(s)}{1+KR+ARs}$ b) K=0.3

Exercise Sheet 4

1. Figure 1 shows a mass-damper-spring system with an applied force *p(t)*.
   a. Derive the transfer function *G(s)* that relates the applied force *p(t)* to the velocity of the mass, *v(t)*. Let the Laplace Transform of *p(t)* and *v(t)* to be *P(s)* and *V(s)*, respectively.

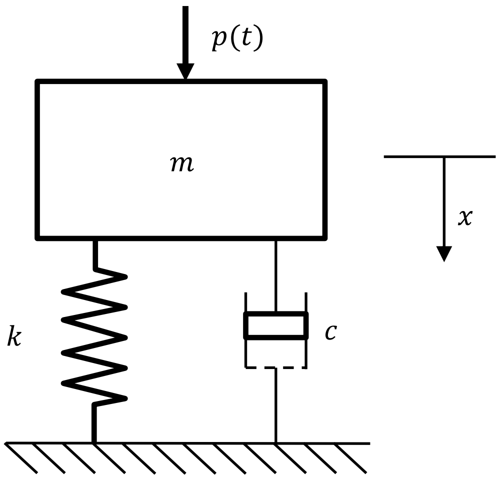


Figure 1.

The first step here is to determine the equation of motion in the time domain: if the velocity of the mass is $\dot{x}$ then the force due to the damper is $-c\dot{x}$ (note that it will always oppose the motion). Force due to the spring is $-kx$, so the net force acting on the mass will be:

$$\text{Net force} = p(t) - c\dot{x} - kx$$

Therefore if the acceleration of the mass is $\ddot{x}$:

$$m\ddot{x} = p(t) - c\dot{x} - kx$$

Rearranging gives the familiar form:

$$p(t) = m\ddot{x} + c\dot{x} + kx$$

And Laplace transforms give us:

$$P(s) = (ms^2 + cs + k)X(s)$$

This is fine – but the question asks for a transfer function in terms of the velocity, v. I find it easiest to work in terms of x to here, and then to substitute as follows:
If $v = \dot{x}$, then for a system that is initially at rest (number 1 in the table of Laplace transforms):

$$V(s) = sX(s)$$

So substituting ${}^{V(s)}/_{s}$ for $X(s)$:

$$P(s) = (ms^2 + cs + k)X(s) = \frac{(ms^2 + cs + k)V(s)}{s}$$

Rearranging gives the transfer function:

$$G(s) = \frac{V(s)}{P(s)} = \frac{s}{ms^2 + cs + k}$$

b. Determine the steady state velocity response of the mass when a step input force is applied to the system. The magnitude of the step input is a.

The input force has the Laplace transform:

$$P(s) = \frac{a}{s}$$

And so the velocity will be given by:

$$V(s) = G(s)P(s) = \frac{a}{s} \times \frac{s}{ms^2 + cs + k} = \frac{a}{ms^2 + cs + k}$$

Beware of the pitfall here – this expression can be solved using inverse Laplace transforms or using the final value theorem. Doing this in the time domain is more involved and requires some understanding to give a result:

Inverse Laplace:

$$V(s) = \frac{a}{ms^2 + cs + k}$$

From the table of Laplace transforms (number 16):

$$\frac{\omega}{\sqrt{1-\gamma^2}} e^{-\gamma\omega t} \sin\left(\omega t\sqrt{1-\gamma^2}\right) \vec{\mathcal{L}} \frac{\omega^2}{s^2 + 2\gamma\omega s + \omega^2}$$

$$V(s) = \frac{a}{k}\left(\frac{{}^{k}/_{m}}{s^2 + {}^{cs}/_{m} + {}^{k}/_{m}}\right)$$

So this system will have a natural frequency ${}^{k}/_{m}$ (in radians per second) and a damping ratio γ given by ${}^{c}/_{2k}$. In the time domain, the solution $\frac{\omega}{\sqrt{1-\gamma^2}} e^{-\gamma\omega t} \sin\left(\omega t\sqrt{1-\gamma^2}\right)$is only valid for $\gamma < 1$ but we are unable to determine this from the question.

Performing reverse Laplace transforms:

$v(t) = \frac{a}{k}\left(\frac{\omega}{\sqrt{1-\gamma^2}} e^{-\gamma\omega t} \sin\left(\omega t\sqrt{1-\gamma^2}\right)\right)$	$\gamma < 1$
$v(t) = \frac{at}{k} e^{-{}^{kt}/_{m}}$ (consider the limiting case below for $l = n + \delta$)*	$\gamma = 1$
$v(t) = \frac{a}{k}\left(\frac{1}{l-n}(e^{-nt} - e^{-lt})\right)$ Where l and n are $\frac{-{}^{c}/_{m} + \sqrt{({}^{c}/_{m})^2 - 4{}^{k}/_{m}}}{2}$ and $\frac{-{}^{c}/_{m} - \sqrt{({}^{c}/_{m})^2 - 4{}^{k}/_{m}}}{2}$	$\gamma > 1$

For all of these forms, the steady state velocity will be zero due to the dominant exponential term.

Using the Final value theorem:

$$\lim_{t\to\infty} v(t) = \lim_{\substack{t \\ s\to 0}} sV(s) = \frac{sa}{ms^2 + cs + k} = 0$$

c. Determine the steady state velocity response of the mass when a ramp input force $p(t) = \sigma t$, is applied to system.
   The first step here is to get the input from the table of Laplace transforms (no. 6, multiply by σ):

$$P(s) = \frac{\sigma}{s^2}$$

Multiplying the input by the transfer function gives the output:

$$V(s) = G(s)P(s) = \frac{\sigma}{s^2} \times \frac{s}{ms^2 + cs + k} = \frac{\sigma}{s(ms^2 + cs + k)}$$

Doing this in the time domain is feasible, but rather involved: the reverse Laplace transform of this for the underdamped case gives:

$$V(s) = \frac{\sigma}{k}\left(1 - \frac{e^{-\gamma\omega t}}{\sqrt{1-\gamma^2}} \sin\left(\omega t\sqrt{1-\gamma^2} + \varphi\right)\right)$$

Where $\omega = \sqrt{k/m}$, $\gamma = c/_{2\sqrt{km}}$ and $\varphi = \cos^{-1}\gamma$. As $t \to \infty$, the output becomes:

$$V(s) = \frac{\sigma}{k}$$

Using the final value theorem is more straightforward:

$$\lim_{t\to\infty} v(t) = \lim_{\substack{t \\ s\to 0}} sV(s) = \frac{s\sigma}{s(ms^2 + cs + k)} = \frac{\sigma}{ms^2 + cs + k} = \frac{\sigma}{k}$$

As can be seen, both methods give the same answer – my advice would be to be comfortable with the final value theorem as it takes less time and is generally less error prone.

Question 2: Numerical answers –

a)	$\frac{V(s)}{V_R(s)} = \frac{K_c}{ms^2 + cs + k + K_c}$
b)	$\frac{V(s)}{F_D(s)} = \frac{1}{ms^2 + cs + k + K_c}$

3. Figure 2a shows a system for controlling the azimuth angle of a large antenna aerial. The input signal is provided by the input potentiometer, which develops 0.05 Volts per degree change in input θ_i. The angular position of the aerial is measured by a similar potentiometer that also generates 0.05 Volts per degree change in the aerial

position θ_O. The resulting differential error voltage is fed into the power amplifier which delivers a current to the motor with a gain of 200 Amps/Volt. The servo motor develops a torque of 0.5Nm/Amp and the moment of inertia of the rotating parts of the motor is 0.2 kg m^2. The gear ratio of the reduction gear between the motor and the antenna turntable is 10:1 and the moment of inertia of the aerial assembly about the turntable axis is 10 kg m^2.

A viscous damping torque of 100Nm/(rad s^{-1}) opposes the rotation of the aerial.

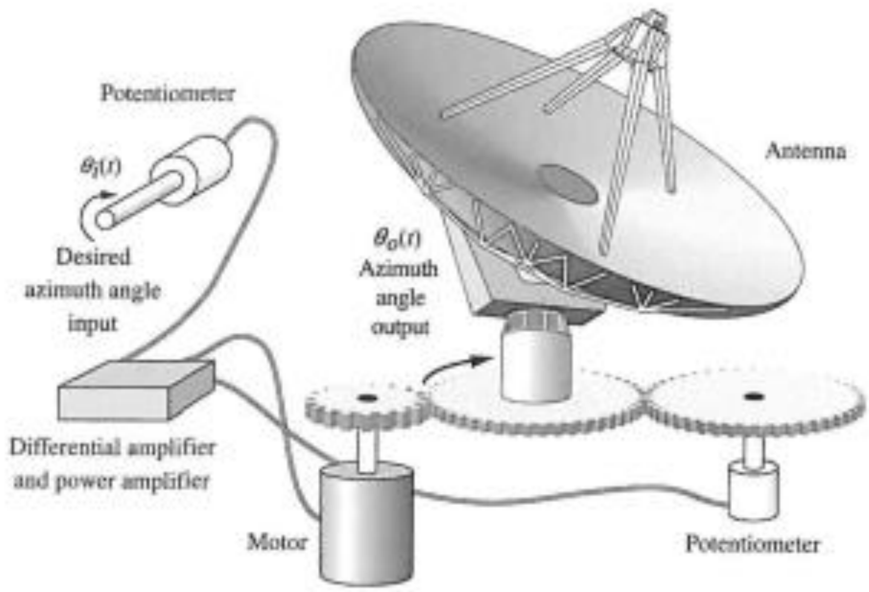
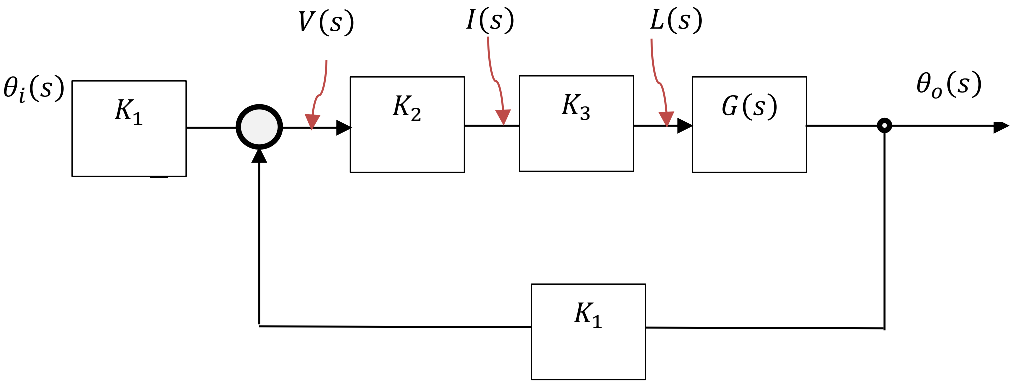


Figure 2a. Antenna Azimuth Control System (adapted from Nise, 2000)

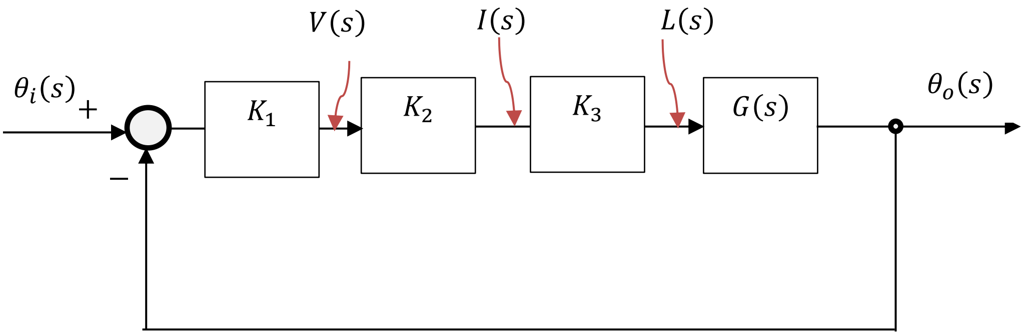
a) Draw the block diagram for the system and derive the overall transfer function relating θ_O and θ_i.

Solution: First step is to draw the system and the feedback loop:

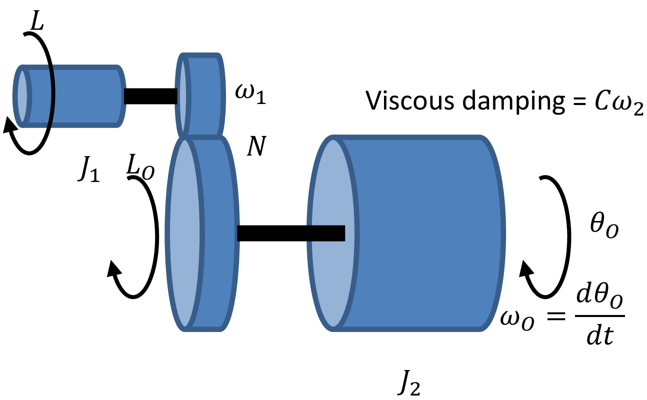
System Element	Transfer function	Value
Input Potentiometer (scalar constant)	K_1	0.05 Volts per degree (2.86 Volts per Radian)
Feedback Potentiometer (scalar constant)	K_1	0.05 Volts per degree (identical to input pot.)
Power Amplifier (scalar constant)	K_2	200 Amps/Volt
Servo motor	K_3	0.5Nm/Amp
Gear train and antenna	$G(s)$	See below for working



The working becomes easier if you note that the input and feedback potentiometers are identical and so the block diagram can also be given as:



To find G(s), consider the physics of the motor and the antenna:



Torque applied:

$$L_O = NL$$

Using the formula for the referred moment of inertia through the gearbox:

$$L_O = (J_1N^2 + J_2)\frac{d\omega_O}{dt} + C\omega_O$$

Moving to the Laplace domain:

$$L_O(s) = (J_1N^2s^2 + J_2s^2)\theta_O(s) + Cs\theta_O(s)$$

$$\frac{\theta_O(s)}{L_O(s)} = \frac{\theta_O(s)}{NL(s)} = \frac{1}{(J_1N^2s^2 + J_2s^2) + Cs}$$

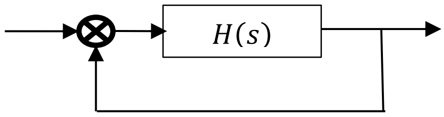
And the transfer function is therefore:

$$G(s) = \frac{\theta_O(s)}{L(s)} = \frac{N}{((J_1N^2 + J_2)s^2 + Cs)}$$

The transfer function of the forward loop is given by:

$$\frac{K_1K_2K_3N}{\left((J_1N^2+J_2)s^2+Cs\right)}$$

And using the result for unity feedback to calculate the system transfer function:



$$\equiv \frac{H(s)}{1+H(s)}$$

$$\frac{\theta_O(s)}{\theta_i(s)} = \frac{K_1K_2K_3N}{\left((J_1N^2+J_2)s^2+Cs+K_1K_2K_3N\right)}$$

Putting in the numbers from the question:

$$\frac{\theta_O(s)}{\theta_i(s)} = \frac{0.05\times 200\times 0.5\times 10\times (180/\pi)}{\left(30s^2+100s+0.05\times 200\times 0.5\times 10\times (180/\pi)\right)} = \frac{2865}{30s^2+100s+2865}$$

b) Calculate the system damping ratio γ.

Using the table of Laplace transforms,

$$A\left(\frac{\omega^2}{s^2+2\gamma\omega s+\omega^2}\right) = \frac{2865}{30s^2+100s+2865}$$

$$\omega^2 = \frac{2865}{30}$$

$$2\gamma\omega = \frac{100}{30}$$

$$\omega = 9.77$$

$$\gamma = \frac{5}{3\omega} = 0.171$$

c) Find the magnitude of the first overshoot which results from a step input $\theta_i = 10°$

Laplace transform of input function $\theta_i(s) = \frac{10}{s}$

Output is given by:

$$\theta_O(s) = \frac{10}{s}\left(\frac{2865}{30s^2+100s+2865}\right)$$

From the table of inverse Laplace Transforms and using the result of part (b):

$$\theta_O(t) = 10\left(1-\frac{e^{-\gamma\omega t}}{\sqrt{1-\gamma^2}}\sin\left(\omega t\sqrt{1-\gamma^2}+cos^{-1}\gamma\right)\right)$$

This function has a maximum where $\cos\left(\omega t\sqrt{1-\gamma^2}+cos^{-1}\gamma\right)$=0 and $\sin\left(\omega t\sqrt{1-\gamma^2}+cos^{-1}\gamma\right)$<0, i.e. $\left(\omega t\sqrt{1-\gamma^2}+cos^{-1}\gamma\right) = {}^{3\pi}/_{2}$

Using $\omega = 9.77$ and $\gamma = 0.171$:

$$t = \frac{\left({}^{3\pi}/_{2} - cos^{-1}\gamma\right)}{\omega\sqrt{1-\gamma^2}}$$

$$t = 0.34s$$

$$\theta_O(t) = 10\left(1-\frac{e^{-\gamma\omega t}}{\sqrt{1-\gamma^2}}\sin\left(\omega t\sqrt{1-\gamma^2}+cos^{-1}\gamma\right)\right) = 15.71°$$

And hence the overshoot is 5.7 degrees.

d) Find the steady state velocity error which results from the ramp input $\theta_i = 0.1t$ radians (for t>0).

Solution:

This can be done analytically:

$$\theta_i(s) = \frac{0.1}{s^2}$$
$$\theta_O(s) = \frac{0.1}{s^2}\left(\frac{2865}{30s^2 + 100s + 2865}\right)$$
$$\theta_O(t) = 0.1\left(t - \frac{2\gamma}{\omega} - \frac{e^{-\gamma\omega t}}{\sqrt{1-\gamma^2}}\sin\left(\omega t\sqrt{1-\gamma^2} + cos^{-1}\gamma\right)\right)$$

By inspection, $\lim_{t\to\infty}\left(\theta_i(t) - \theta_O(t)\right) = \frac{0.2\gamma}{\omega} = 0.0035$ using the values calculated in parts (b) and (c) as the final term tends to zero as t tends to infinity.

Alternatively, using the finite value theorem:

The error is given by:

$$E(s) = \theta_i(s) - \theta_O(s) = \frac{0.1}{s^2} - \frac{0.1}{s^2}\left(\frac{2865}{30s^2 + 100s + 2865}\right) = \frac{0.1}{s^2}\left(\frac{30s^2 + 100s}{30s^2 + 100s + 2865}\right)$$

$\lim_{s\to 0} sE(s) = \frac{1}{s}\left(\frac{3s^2+10s}{30s^2+100s+2865}\right) = \frac{10}{2865}$ =0.0035

SHEET 5: STABILITY OF FEEDBACK SYSTEMS

1. The characteristic equation of a feedback control system is

$$s^3 + (5+K)s^2 + 7s + 18 + 9K = 0$$

a. Determine the maximum positive value of K, below which the system is stable.

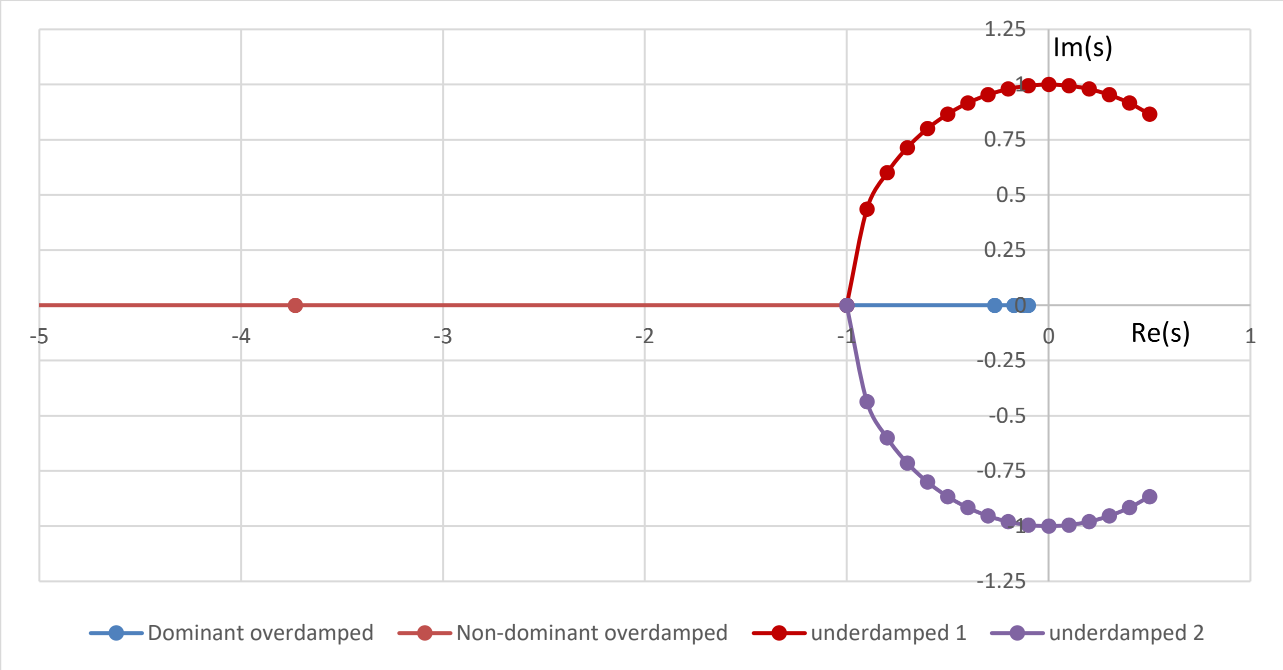
To get started here, we begin by writing out the Routh Table. Remember the rules for where to put the coefficients from the characteristic equation and how to calculate the entries in the 3rd and lower rows:

s^3	1	7	0	0
s^2	5+K	18+9K	0	0
s	$\frac{7(5+K) - (18+9K)}{5+K} = \frac{17-2K}{5+K}$	0	0	0
s^0	18+9K	0	0	0

From here, $\frac{17-2K}{5+K}$ will have a negative value for $K > 8.5$ and so the maximum positive value for K is 8.5 – anything above this will make the system unstable. The system will also be unstable for k<-2 but this was not asked for in the question.

b. Determine the frequency of oscillations at this value of K.

This part of the question is mathematically quite simple but does demand some understanding of the concept of the Root locus:



The figure above shows the root locus for a system with the characteristic equation

$$Q(s) = s^2 + 2\zeta s + 1$$

Note that in this case the natural frequency ω_n is 1. At the limit of stability (K=8.5 from the previous part of the question), the root locus will cross the imaginary (vertical) axis and the Real part of the root is zero. This will have a damping factor of zero, making the familiar form of the oscillatory system $(Q(s) = s^2 + 2\zeta\omega_n s + \omega_n^2)$ become:

$$Q(s) = s^2 + \omega_n^2$$

A third order system with a negative real root and two complex roots at the limit of stability will have a characteristic equation:

$$Q(s) = (s + a)(s^2 + \omega_n^2) = s^3 + as^2 + \omega_n^2 s + a\omega_n^2$$

From the question:

$$s^3 + (5 + K)s^2 + 7s + 18 + 9K = 0$$

From Part (a), K=8.5

$$s^3 + (13.5)s^2 + 7s + 94.5 = 0$$

$$\omega_n^2 = 7$$

And so the frequency of oscillations will be 2.65 rad s^{-1}

My tips to all students are to practise – it is highly unlikely that you will be given a supplementary question like this on a 4^{th} or higher order system, but this kind of question is a favourite in exams and textbooks, to test how well you understand the concepts underlying control theory.

2. A unity feedback control system is shown in Figure Q2.

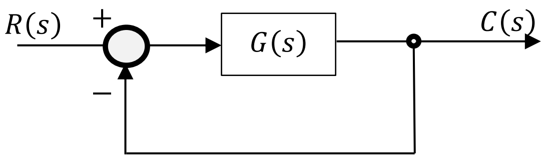


Figure Q2

Where r is a reference signal and c is the system response.

The forward loop transfer function is given by:

$$G(s) = \frac{3(s+4)(s+8)}{s(s+5)^2}$$

Determine the relative stability of the system.

3. A closed loop feedback control system is shown in figure Q3.

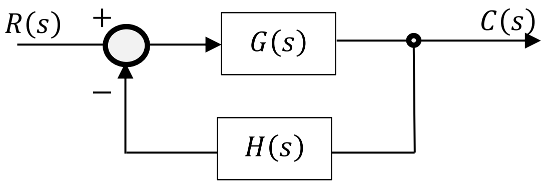


Figure Q3

Where r is a reference signal and c is a system response. The transfer functions for the forward and feedback loops are given by:

$$G(s) = \frac{K(s+40)}{s(s+10)} \qquad H(s) = \frac{1}{s+20}$$

Use the Routh-Hurwitz stability criterion to determine the values of K for which the closed loop system will be stable.

To get started on this question, first work out the overall transfer function:

$$C(s) = \big(R(s) - H(s)C(s)\big)G(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K(s+40)}{s(s+10)}\left(\frac{1}{1+\frac{K(s+40)}{s(s+10)(s+20)}}\right) = \frac{K(s+40)}{s(s+10)}\left(\frac{(s+20)}{(s+20)+\frac{K(s+40)}{s(s+10)}}\right)$$

$$\frac{C(s)}{R(s)} = \left(\frac{K(s+40)(s+20)}{s(s+10)(s+20)+K(s+40)}\right)$$

The characteristic equation is therefore:

$$Q(s) = s(s+10)(s+20) + K(s+40) = s^3 + 30s^2 + (200+K)s + 40K$$

Note: For all of this, don't worry about the numerator (the characteristic function) as this does not determine stability.

The Routh array will be:

s^3	1	200+K	0	0
s^2	30	40K	0	0
s	$\frac{6000+30K-40K}{30K} = \frac{600-K}{3K}$	0	0	0
s^0	40K	0	0	0

So from the table, the system will be stable for 0<K<600

4. The transfer function of a control system is as follows:

$$G(s) = \frac{1}{s^3 + 5s^2 + 20s + 6}$$

a) Is the system stable?
b) Use the final value theorem to calculate the unit step response of the system.

Answers:

a) Yes (if you think about it, there would be no point asking part (b) if the system was unstable …)
b) $\frac{1}{6}$

Show your working!

*- from page 16: if one root is $n + \delta$ and the other root is n, then

$$v(t) = \frac{a}{k}\left(\frac{1}{n+\delta-n}\left(e^{-nt} - e^{-(n+\delta)t}\right)\right) = \frac{a}{k\delta}\left(e^{-nt} - e^{-(n+\delta)t}\right) = \frac{a}{k\delta}\left(e^{-nt} - e^{-\delta t}e^{-nt}\right)$$

Using the result that for a very small vale of δt, $e^{-\delta t} = 1 - \delta t$, this becomes:

$$v(t) = \frac{a}{k\delta}e^{-nt}\left(1 - (1 - \delta t)\right) = \frac{at}{k}e^{-nt}$$

Where l and n are $\frac{-c/m + \sqrt{(c/m)^2 - 4k/m}}{2}$ and $\frac{-c/m - \sqrt{(c/m)^2 - 4k/m}}{2}$