Elastic-Plastic Deformations

Lecture 2 – Bending of Beams

Department of Mechanical, Materials & Manufacturing Engineering

MMME2053 – Mechanics of Solids

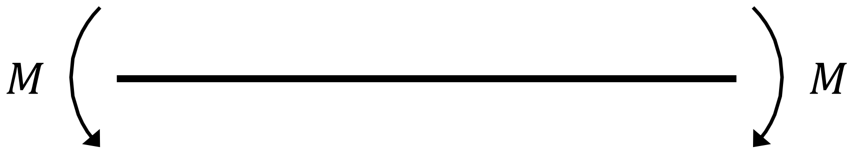
Elastic-Plastic Deformations

Learning Outcomes

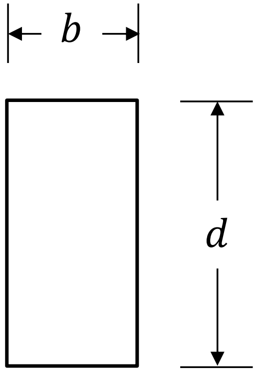
1. Know the shapes of uniaxial stress-strain curves and the elastic-perfectly-plastic approximation (knowledge);
2. Know the kinematic and isotropic material behaviour models used to represent cyclic loading behaviour (knowledge);
3. Understand elastic-plastic bending of beams (comprehension) and be able to use equilibrium, compatibility and σ-ε behaviour to solve these types of problems for deformation and stress state (application);
4. Understand elastic-plastic torsion of shafts (comprehension) and be able to use equilibrium, compatibility and τ-γ behaviour to solve these types of problems for deformation and stress state (application);
5. Be able to determine residual deformations and residual stresses in beams under bending and shafts under torsion (application).

Elastic-Plastic Bending of Beams

Part a of the figure below shows a beam which is subjected to a bending moment, M. The rectangular cross-sectional area of the beam is shown in part b of the figure.



(a)



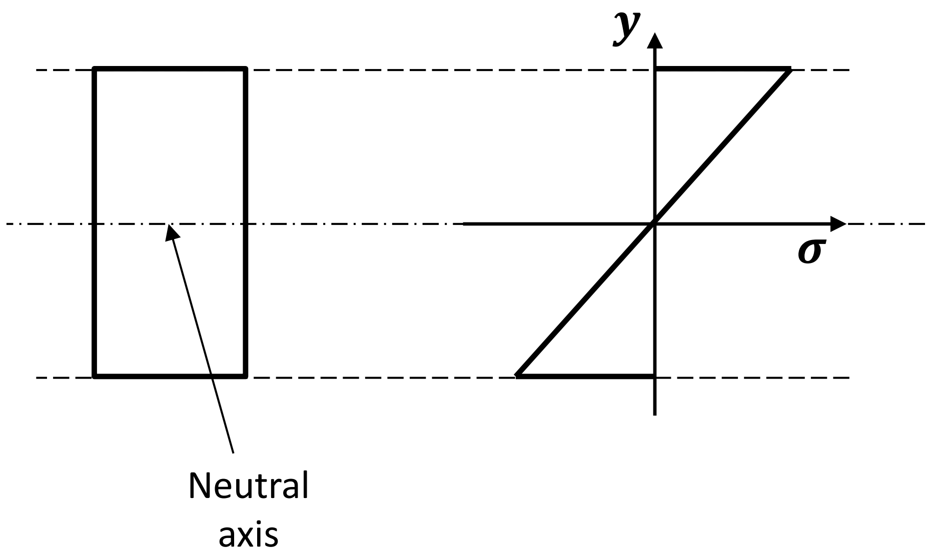
(b)

Stress Distribution

Assuming that the magnitude of the bending moment is not high enough to cause plasticity (yielding) within the beam, the elastic beam bending equation can be used to describe the stress distribution, as a function of y (distance from the neutral axis), as:

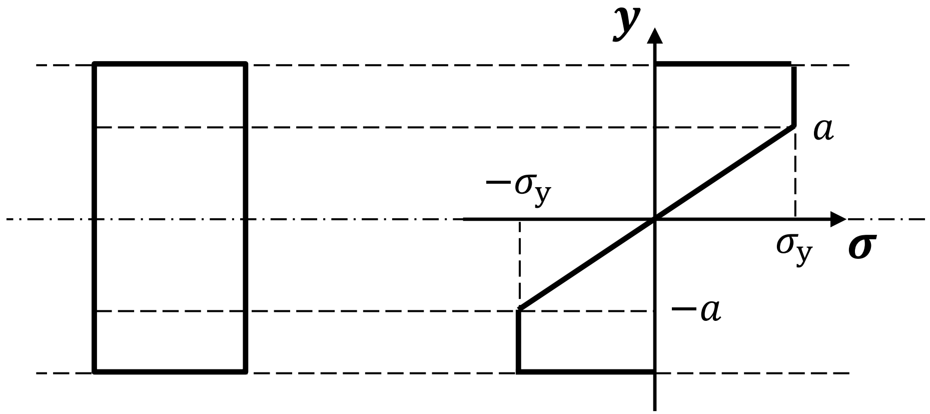
$$\sigma = \frac{My}{I}$$

In this elastic case, elastic the stress distribution throughout the cross-section, is linear, as shown in the figure below.



If the bending moment is increased to a magnitude which is just high enough to induce plasticity within the beam, this plasticity will occur at the positions furthest away from the neutral axis, i.e., at the positions of maximum y magnitude (top and bottom edges of the cross-section).

As the bending moment is further increased, the plasticity spreads from the outer edges, to further within the cross-section (towards the neutral axis) as shown in the figure below.



Moment Equilibrium

The applied bending moment, M, can be related to the to the position as which yielding occurs, a, as:

$$M = \int_A y\sigma \mathrm{d}A$$

$$= \int_{-d/2}^{d/2} y\sigma b \mathrm{d}y$$

Where:

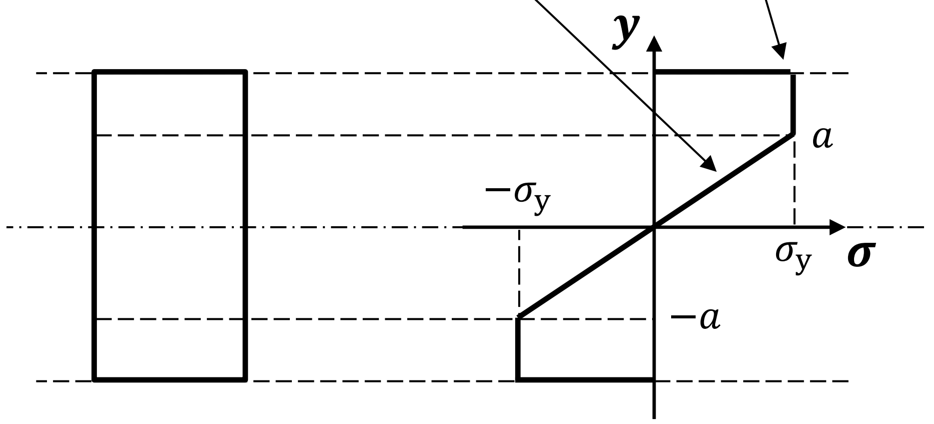
$$\mathrm{d}A = b\mathrm{d}y$$

Recognising the symmetry about the neutral axis in the stress distribution magnitudes, the above expression for M can be rewritten as:

$$M = 2\int_0^{d/2} y\sigma b \mathrm{d}y$$

Substituting the expressions for stress for each of the elastic ($0 > y > a$) and plastic ($a > y > d\,/\,2$) regions into this equation gives:

$$M = 2\left(\int_0^a y\left(\sigma_\mathrm{y}\frac{y}{a}\right) b\mathrm{d}y + \int_a^{d/2} y\sigma_\mathrm{y} b\mathrm{d}y\right)$$



$$\therefore M = 2b\sigma_\mathrm{y}\left(\frac{d^2}{8} - \frac{a^2}{6}\right)$$

Compatibility Requirement

In order for the radius of curvature of the beam, R, due to the applied bending moment, M, to be calculated, both compatibility and a stress-strain relationship are required.

As the region of the cross-section between $-a < y < a$ has only behaved elastically, the elastic beam bending equation can be applied. I.e.:

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

$$\therefore \frac{y}{R} = \frac{\sigma}{E} = \varepsilon$$

$$\therefore R = \frac{y}{\varepsilon}$$

As the beam behaves as one body, the entirety of the beam (both the elastic and plastic regions) must share this common radius of curvature, R.

Stress-Strain Relationship

Again, as the region of the cross-section between $-a < y < a$ has only behaved elastically, Hooke's law applies here, and so:

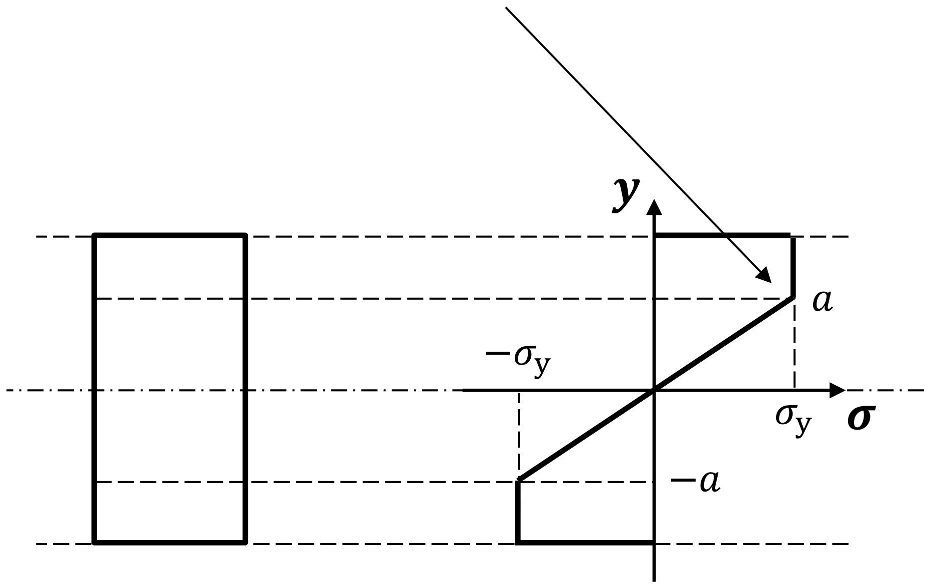
$$\sigma = E\varepsilon$$

Substituting the above equation into the expression for R on the previous slide:

$$R = \frac{Ey}{\sigma}$$

Substituting values for y and σ, from within the elastic region, into this equation, allows for R to be calculated.

A convenient value of y to use is a, for which the corresponding value of σ is σ_y.



Therefore:

$$R = \frac{Ea}{\sigma_y}$$

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Unloading

As plasticity has occurred within the beam during loading, on unloading the stress distribution and radius of curvature will not return to zero. Rather a residual stress distribution and residual radius of curvature will remain.

Assuming that the stress change which occurs on unloading is purely elastic, then the stress change can be calculated from the elastic beam bending equation as:

$$\Delta\sigma = \frac{\Delta M y}{I}$$

The maximum stress change will therefore occur at $y_{\max}$, and so:

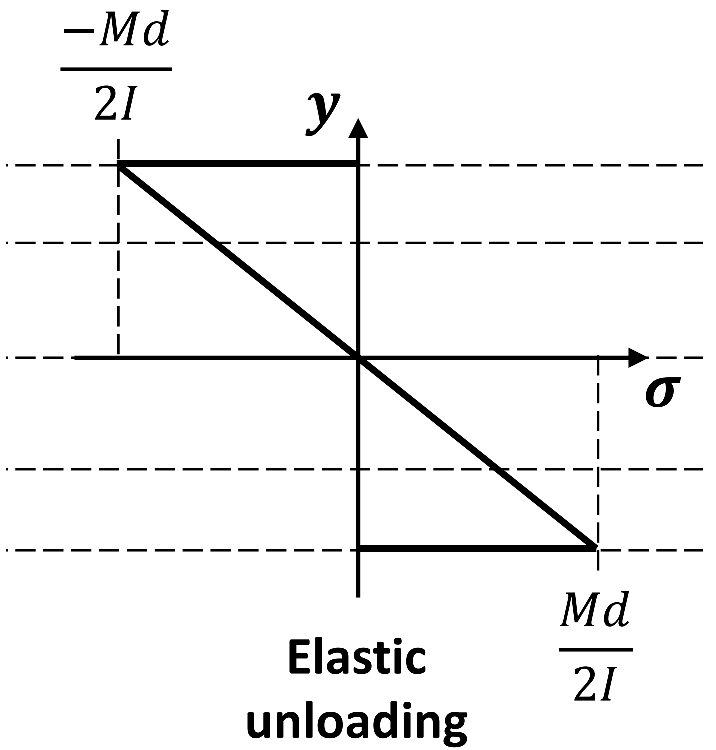
at $y = {}^{d}/_{2}$ (top edge):

$$\Delta\sigma_{(y={}^{d}/_{2})} = \frac{-Md}{2I}$$

at $y = -{}^{d}/_{2}$ (bottom edge):

$$\Delta\sigma_{(y={}^{d}/_{2})} = \frac{Md}{2I}$$

As the unloading behaviour has been assumed to be elastic, the stress variation between these two values, will be linear.



This residual stress distribution will be accompanied by a residual radius of curvature, which can be calculated by substituting unloaded beam values for y and σ into the expression derived for R, which again relate to a position that has only been subjected to elastic behaviour.