Mechanics of Solids – Deflection of Beams Notes

Learning Summary

1. Know how to derive the differential equation of the elastic line (i.e. deflection curve) of a beam (synthesis)
2. Be able to solve this equation by successive integration to yield the slope, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and the deflection, y, of a beam at any position, x, along its span (application)
3. Employ Macaulay's method, also called the method of singularities, to solve for beam slopes and deflections where there are discontinuities in the bending moment distribution arising from discontinuous loading (application)
4. Recognise and use different singularity functions in the bending moment expression for different loading conditions including point loads, uniformly distributed loads and point bending moments (comprehension)
5. Employ Macaulay's method for statically indeterminate beams (application)

1. Introduction

Whereas the design of engineering structures and components is very often dictated by the strength of the materials used and consequently the stresses within the structure, often the limiting factor is the allowable deflection. This is particularly important for engineering artefacts made from materials of lower stiffness, e.g. aluminium, plastics, composites, etc., but may also be critical for high stiffness structures comprising slender members. It is therefore important, as part of the design process, to be able to calculate maximum deflections in a structure in addition to the position at which they occur.

Here, following the derivation of the fundamental deflection equation for a beam, a flexible procedure is introduced, called Macaulay's Method, which allows for slopes and deflections to be calculated at any position along a beam span. In particular, the method allows us to deal with different types of loading, such as point loads, uniformly distributed loads and point bending moments, including discontinuities in these loads. Although not the only method for calculating deflections, as we will see in the Strain Energy Methods section of the module, it is a particularly powerful and flexible method.

2. Equation of the Elastic Line

Taking a generic curve, $y = f(x)$, two arbitrary points, A and B, can be chosen, as shown in Figure 1.

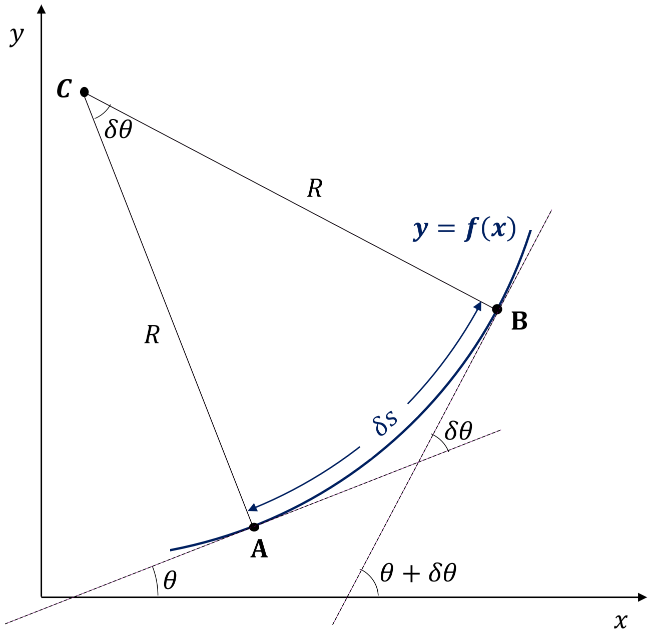

Figure 1

The gradients at these two arbitrary positions can be defined as:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\mathrm{A}} = \tan\theta$$

and

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\mathrm{B}} = \tan\left(\theta + \delta\theta\right)$$

Letting the normals to the curve at points A and B meet at point C, if points A and B are close, the lengths $\mathrm{A}C$ and $\mathrm{B}C$ are similar. I.e.:

$$\mathrm{A}C \ \approx \mathrm{B}C \ (= R)$$

Length AB can therefore be thought of as a small arc of a circle of radius, R.

Note that angle $\mathrm{A}C\mathrm{B} = \delta\theta$, since, as the tangent turns through angle $\delta\theta$, so does the normal. Therefore arc $\mathrm{AB} = \delta s = R\delta\theta$, which can be re-arranged to give:

$$\frac{1}{R} = \frac{\delta\theta}{\delta s}$$

As $\delta s \to 0$ (i.e. as points A and B become closer):

$$\frac{\delta\theta}{\delta s} \to \frac{\mathrm{d}\theta}{\mathrm{d}s}$$

$$\therefore \frac{1}{R} = \frac{\mathrm{d}\theta}{\mathrm{d}s} \tag{1}$$

It can be seen from Figure 2 that since δs is small, the arc AB $(= \delta s) \approx$ the chord AB.

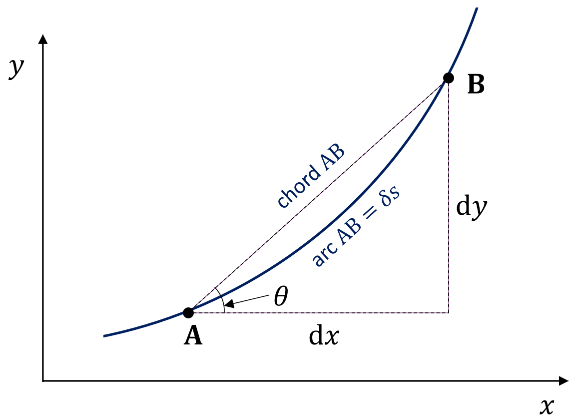

Figure 2

Therefore, when $\delta s \to 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta \tag{2}$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta \tag{3}$$

Differentiating equation (2) with respect to s gives:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}s}(\tan\theta)$$

Multiplying the left-hand side of this equation by $\frac{\mathrm{d}x}{\mathrm{d}x}$, and the right-hand side by $\frac{\mathrm{d}\theta}{\mathrm{d}\theta}$ gives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}\theta}(\tan\theta)\frac{\mathrm{d}\theta}{\mathrm{d}s}$$

Rearranging this and substituting in equation (3):

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(\cos\theta) = \sec^2\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}s}$$

$$\therefore \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = sec^3\theta \frac{\mathrm{d}\theta}{\mathrm{d}s} = \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2} \frac{\mathrm{d}\theta}{\mathrm{d}s} \tag{4}$$

where

$$\sec^3\theta = (\sec^2\theta)^{3/2} = (1 + \tan^2\theta)^{3/2} = \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2}$$

Rearranging equation (4):

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}}{\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2}}$$

Substituting this into equation (1) gives:

$$\frac{1}{R} = \frac{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}}{\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2}} \tag{5}$$

Application to a beam under bending

The section of a span of a beam, shown in Figure 3, is under pure bending, i.e. there is a constant bending moment along this section and no shear force.

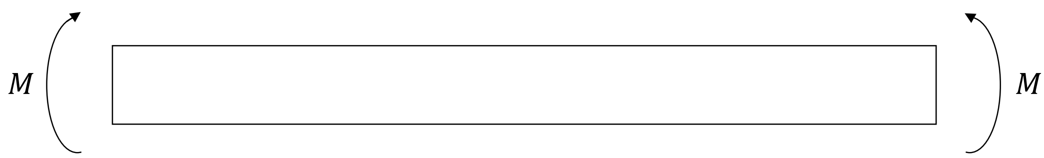

Figure 3

Under these pure bending conditions, the neutral axis (the axis on which there is zero stress) of the beam, also named the elastic line or deflection curve, is a circular arc with radius of curvature, R, as shown in Figure 4.

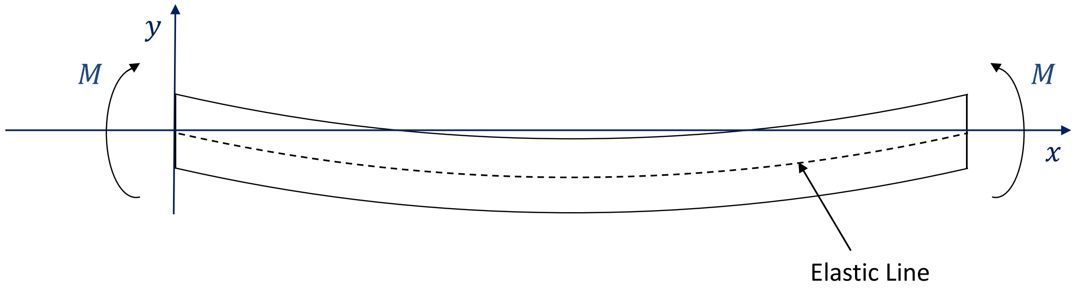

Figure 4

The transverse deflection of the elastic line is given by the co-ordinate, y, of any position along its length, x [n. b. do not confuse this 'y' definition for deflection with the 'y' denoting distance from the neutral axis in the beam bending equation, as shown in equation (6)]. The line denoting the neutral axis in Figure 4 is known as the 'elastic line' or the 'deflection curve' of the beam.

The elastic beam bending equation, which can be used to describe the bending moment, M, as a function of the radius of curvature, R, is:

$$\frac{M}{I} = \frac{\sigma}{\mathrm{y}} = \frac{E}{R} \tag{6}$$

$$\therefore \frac{1}{R} = \frac{M}{EI}$$

where E is Young's modulus and I is 2nd moment of area.

Substituting this into equation (5), which also represents the shape of an arc, gives:

$$\frac{M}{EI} = \frac{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}}{\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2}} \tag{7}$$

For small deflections, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is small. Therefore:

$$\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2} \approx 1$$

Equation (7) can therefore be simplified and rearranged to give:

$$\boldsymbol{EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = M} \tag{8}$$

This is the is the 2nd order differential equation of the elastic line, relating the deflection, y, to the applied bending moment, M, the Young's modulus, E, 2nd moment of area, I, and position along beam span, x. The product of E and I, i.e. EI, is termed the 'Flexural Rigidity' of the beam.

Successive integration of this equation, with respect to x, will yield the slope, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and the deflection, y, as function of position, x, along the beam.

This equation has been derived for the case of pure bending, i.e. constant bending moment along the section, and does not take into account deflections due to shear. For long slender beams, shear deflections can be neglected.

A complication arises where discontinuities in M exist, such as where there are point loads and/or point bending moments or where there is an abrupt change in distributed loading. Various methods have been developed to solve such problems with discontinuities. Here we introduce and develop the procedure called Macaulay's Method, a versatile solution procedure which can handle most discontinuities we are likely to encounter.

3. Macaulay's Method (also termed the Method of Singularity Functions)

Named after the mathematician W. H. Macaulay, Macaulay's Method uses a mathematical technique to deal with discontinuous loading. The bending moment expression $M(x)$, i.e. M as a function of x, is replaced with the step function $M\langle x-a\rangle$, in which a defines the position at which a discontinuity arises.

Figure 5 shows a simply supported beam carrying a point load, P, at the centre of its length. This load gives rise to a discontinuity in the bending moment expression.

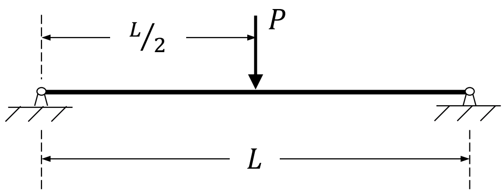

Figure 5

Figure 6 shows a free body diagram of this beam.

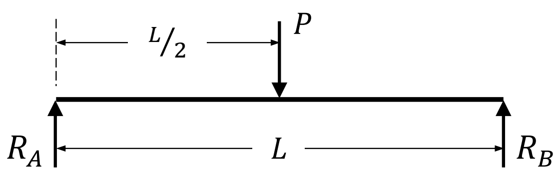

Figure 6

Considering each span between the loading discontinuity separately in order to determine expressions for bending moment, M.:

Span 1

Figure 7 shows the free body diagram of the beam sectioned within span 1 (i.e. before the loading discontinuity caused by P), taking the origin as the left-hand end. The unknown bending moment, M, and shear force, S, at this section, are shown in the diagram.

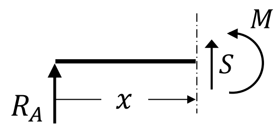

Figure 7

Taking moments about the section position in order to determine an expression for the bending moment, M, in span 1:

$$M = R_A x$$

Substituting this into equation (8):

$$EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = R_A x \tag{9}$$

Equation (9) applies to span 1 of the beam only.

Span 2

Figure 8 shows the free body diagram of the beam, now sectioned within span 2 (i.e. after the loading discontinuity caused by P). As before, the unknown bending moment, M, and shear force, S, at this section, are shown in the diagram.

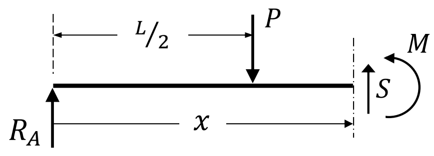

Figure 8

Taking moments about the section position in order to determine an expression for the bending moment, M, in span 2:

$$M + P\left(x - \frac{L}{2}\right) = R_A x$$

$$\therefore M = R_A x - P\left(x - \frac{L}{2}\right)$$

Substituting this into equation (8):

$$EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = R_A x - P\left(x - \frac{L}{2}\right) \tag{10}$$

Equation (10) applies to span 2 of the beam only.

It is interesting to note that the forms of equations (9) and (10) are similar, in that there is simply an extra term added to take account of the extra span of the beam (as we move past the loading discontinuity). In fact, due to this similarity, equation (10), i.e. the expression derived for the final span of the beam, can be applied to the full length of the beam by rewriting it in a slightly different form as follows:

$$EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = R_A x - P\left\langle x - \frac{L}{2}\right\rangle \tag{11}$$

Note the change of bracket shape. These $\langle\quad\rangle$ brackets are termed 'Macaulay Brackets' and due to these, equation (11) is now applicable to any position, x, in the beam shown in Figure 5, if we adopt 'Macaulay's convention'. Macaulay's convention states that whenever a Macaulay bracketed term becomes negative, the entire term it is part of is set to zero.

Adopting this convention, the general 2nd order differential expression for the beam, shown by equation (11) for the beam shown in Figure 5, can be integrated with respect to x to give the slope, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and integrated again to give the deflection, y, at any position x, along the length of the beam.

If, for example, the slope and deflection at the position of the point load shown in Figure 5 was required, the solution is as follows:

Application of equilibrium for the determination of the reaction forces at the support positions

Vertical Equilibrium:

$$P = R_A + R_B \tag{12}$$

Taking moments about the position of R_A:

$$R_B L = \frac{PL}{2}$$

$$\therefore R_B = \frac{P}{2}$$

Substituting this into equation (12):

$$R_A = \frac{P}{2} \tag{13}$$

Determination of expressions for slope, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and deflection, y, as functions of x

Once the 2nd order differential expression for the beam has been determined, in this case as given by equation (11), integration with respect t to x gives:

$$EI\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{R_A x^2}{2} - \frac{P\,\langle x - \frac{L}{2}\rangle^2}{2} + \mathrm{A} \tag{14}$$

where $\frac{\mathrm{d}y}{\mathrm{d}x}$ represents the slope at any position, x.

Integrating with respect to x again:

$$EIy = \frac{R_A x^3}{6} - \frac{P\,\langle x - \frac{L}{2}\rangle^3}{6} + \mathrm{A}x + \mathrm{B} \tag{15}$$

where y represents the deflection at any position, x.

Use of boundary conditions for the determination of the constants of integration

Boundary condition 1: at x = 0, y = 0 (i.e. at this support position there is no deflection)

Applying this to equation (15):

$$\mathrm{B} = 0$$

Note that the term related to P has been set to zero, as the contents of the Macaulay brackets is negative (i.e. $\langle 0 - \frac{L}{2}\rangle < 0$).

Boundary condition 2: at x = L, y = 0 (i.e. at this support position there is no deflection)

Applying this to equation (15):

$$0 = \frac{R_A L^3}{6} - \frac{P\left(\frac{L}{2}\right)^3}{6} + \mathrm{A}L$$

$$\therefore \mathrm{A} = \frac{L^2}{6}\left(\frac{P}{8} - R_A\right) \tag{16}$$

Note that for the term related to P, the Macaulay brackets have been changed to regular brackets, as Macaulay's convention has been applied and the contents of the Macaulay brackets is positive (i.e. $\langle L-\frac{L}{2}\rangle > 0$)).

Evaluation of slope and deflection at position of interest

As it is at $x=\frac{L}{2}$ that the slope and deflection is required, this value of x can be substituted into equations (14) and (15), and Macaulay's convention applied to give:

$$EI\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{R_A L^2}{8}+\mathrm{A}$$

and

$$EIy=\frac{R_A L^3}{48}+\frac{\mathrm{A}L}{2}$$

Substituting the expressions for R_A and A, from equations (13) and (16), respectively, into these, and rearranging for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and y, gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

and

$$y=-\frac{PL^3}{48EI} \tag{17}$$

As the beam is loaded and supported symmetrically, it makes sense that the slope at the centre position is zero. Additionally, as the single applied load is downwards, it would be expected that the deflection at the centre position of the beam would also be downwards and therefore negative according to the sign convention defined in section 2 of these notes, i.e. y is positive in the upwards direction.

4. Alternative Loading Types

Uniformly distributed load

Consider a uniformly distributed load (UDL), w Nm^{-1}, acting over part of a beam's span, as shown in Figure 9. The UDL runs from distance a from the origin (left-hand end) of the beam, all the way to the right-hand end of the beam. A discontinuity occurs at the position where the UDL commences.

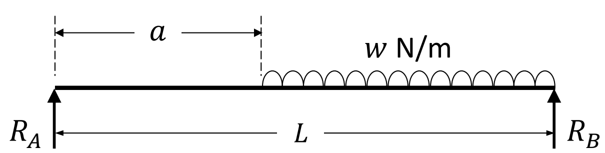

Figure 9

Figure 10 shows the resulting free-body diagram after sectioning the beam after the discontinuity.

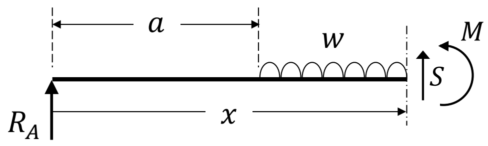

Figure 10

Taking moments about the section position:

$$M + \frac{w\langle x-a\rangle^2}{2} = R_A x \tag{18}$$

$$\therefore M = R_A x - \frac{w\langle x-a\rangle^2}{2}$$

And substituting this into equation (8) gives:

$$EI\frac{d^2y}{dx^2} = R_A x - \frac{w\langle x-a\rangle^2}{2}$$

As in section 3 of these notes, this 2nd order differential expression of the elastic line can now be integrated with respect to x in order to determine the slope, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and again in order to determine the deflection, y, as functions of x. Boundary conditions are then used to determine the constants of integration before evaluation of the slope and/or deflection at any position, x, along the beam.

As can be seen from equation (18), when taking moment equilibrium, the contribution of the UDL is calculated by first turning the UDL, w (unit Nm^{-1}), into a force (unit N) by multiplying it by the distance over which it acts, $x - a$ (unit m). This force is then turned into a moment by multiplying it by the distance to the centre position of the UDL, $\frac{\langle x-a\rangle}{2}$ (unit m). Note that Macaulay brackets are used in the length term in order to allow for Macaulay's method to be employed for the inclusion or elimination of the term depending on the position, x, being evaluated.

Discontinuous uniformly distributed load

A discontinuous UDL is shown in Figure 11.

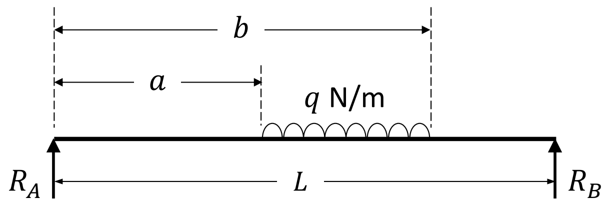

Figure 11

In this case, the UDL, q, runs from distance a from the origin (left-hand end) of the beam, up to distance b from the origin. Discontinuities therefore occur both at the position where the UDL commences, and at the position where it ends.

In order to progress towards a general bending moment expression for the beam, analogous to equation (18) for the continuous UDL, the applied discontinuous UDL, q, is extended to the end of the beam and an additional, negative, counterbalancing UDL superimposed over the newly extended part, as shown in Figure 12. The extended applied UDL, q, and the added counterbalancing UDL, $-q$, mathematically cancel each other out and therefore this gives a statically equivalent system to the original partially extended (discontinuous) UDL.

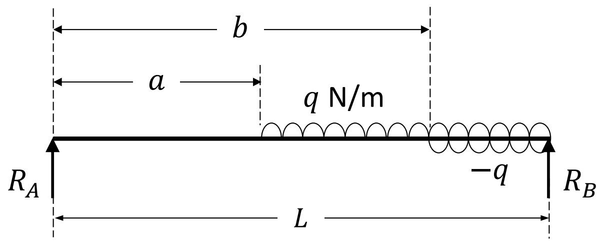

Figure 12

As before, the beam is then sectioned after the final discontinuity, and a free body diagram drawn, as shown in Figure 13.

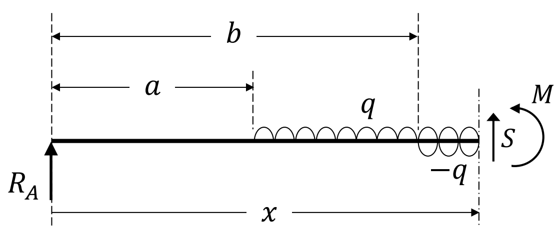

Figure 13

In the determination of a bending moment expression, each of the now continuous UDLs in Figure 13 are dealt with as in the determination of equation (18) from Figure 10. I.e.:

$$M+\frac{q\langle x-a\rangle^2}{2}=R_A x+\frac{q\langle x-b\rangle^2}{2}$$

$$\therefore M=R_A x+\frac{q\langle x-b\rangle^2}{2}-\frac{q\langle x-a\rangle^2}{2}$$

Substituting this into equation (8) gives:

$$EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=R_A x+\frac{q\langle x-b\rangle^2}{2}-\frac{q\langle x-a\rangle^2}{2}$$

As before, this 2nd order differential expression of the elastic line can be integrated with respect to x in order to determine the slope, $\frac{dy}{dx}$, and again in order to determine the deflection, y, as functions of x, and boundary conditions used to determine the constants of integration before evaluation of the slope and/or deflection at any position, x, along the beam.

Point bending moment

Consider a point bending moment, M_o Nm, acting at a distance a from the left-hand side of a beam which is simply supported at both ends, as shown in Figure 14. This point bending moment gives rise to a discontinuity in the bending moment expression.

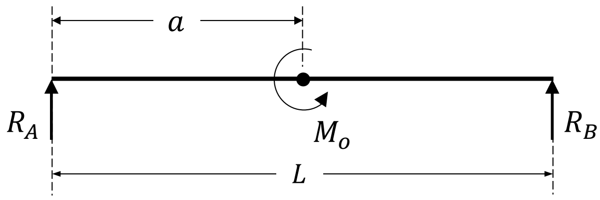

Figure 14

Figure 15 shows the resulting free-body diagram after sectioning the beam after the discontinuity.

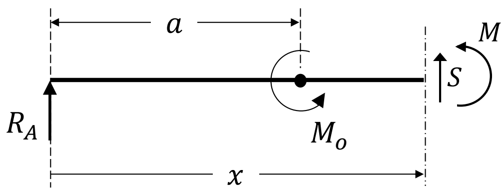

Figure 15

Taking moments about the section position:

$$M + M_o\langle x - a\rangle^0 = R_A x \tag{19}$$

$$\therefore M = R_A x - M_o\langle x - a\rangle^0$$

And substituting this into equation (8) gives:

$$EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = R_A x - M_o\langle x - a\rangle^0 \tag{20}$$

Again, equation (20) can be integrated with respect to x in order to determine the slope, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and again in order to determine the deflection, y, as functions of x, and boundary conditions used to determine the constants of integration before evaluation of the slope and/or deflection at any position, x, along the beam.

Note from equation (19), that the form of the discontinuity function for the point bending moment is $M_o\langle x - a\rangle^0$. This is the same as for a point load, except that the bracketed length is raised to the power zero. This is simply a mathematical convenience for facilitating Macaulay's method whilst maintaining the correct units of the moment, M_o. I.e. if a position in the beam where $x < a$ is considered for evaluation, then the contents of the Macaulay brackets is negative and the entire term set to zero. However, if a position in the beam where $x > a$ is considered for evaluation, then the contents of the Macaulay brackets is positive, and the term is included. As the length term, $\langle x - a\rangle^0$, is raised to the power of zero, it becomes 1, and so the term simplifies to M_o.

5. Summary of the Discontinuity Functions

We have seen that Macaulay's method can be used to find the slope and/or deflection at any position along a beam where point loads, uniformly distributed loads and/or point bending moments produce discontinuities in the bending moment expression. The method can also be used where there is a combination of these loads acting on a beam.

When developing the bending moment expression for a beam with load discontinuities, the singularity functions shown in Table 1 are used for each different type of load.

Table 1

Load Type	**Singularity Function**
Point Load, P	$P\langle x - a\rangle$
Continuous UDL, w – single discontinuity	$\frac{w\langle x - a\rangle^2}{2}$
Discontinuous UDL, q – double discontinuity	$\frac{q\langle x - b\rangle^2}{2} - \frac{q\langle x - a\rangle^2}{2}$
Point Bending Moment, M_o	$M_o\langle x - a\rangle^0$

Note that in these singularity functions, the exponent is 1 for a point load, 2 for a UDL and 0 for a point bending moment.

6. Worked Example – Beam with Point Load, Discontinuous Uniformly Distributed Load & Point Bending Moment

Problem

Figure 16 shows a steel simply supported beam of length, L = 1.5 m, carrying:

- a point bending moment, M_o = 3 kNm, at a distance of $\frac{L}{6}$ from the left-hand end
- a point load, P = 2 kN, at a distance of $\frac{L}{3}$ from the left-hand end
- a discontinuous uniformly distributed load, q = 4 kN/m, between distances of $\frac{L}{3}$ and $\frac{2L}{3}$ from the left-hand end

The Young's modulus, E, of the material is 200 GPa and the beam is of circular cross-section of diameter, D = 50 mm.

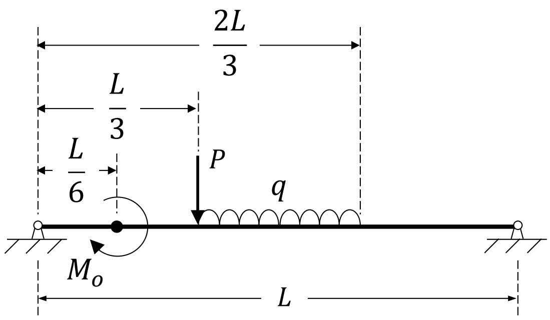

Figure 16

Use Macaulay's method to determine the slope and deflection of the beam at its centre position.

Solution

As the beam is of circular cross-section, the 2nd moment of area, I, is calculated as:

$$I = \frac{\pi D^4}{64} = \frac{\pi \times 0.05^4}{64} = 3.068 \times 10^{-7}\text{m}^4$$

Figure 17 shows a free-body-diagram of the beam which can be used to calculate the reaction forces.

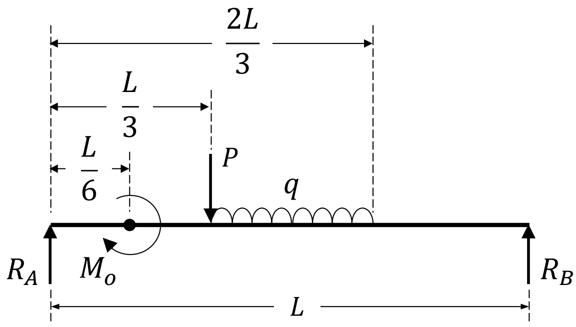

Figure 17

Taking vertical equilibrium:

$$R_A + R_B = P + \frac{qL}{3} \tag{21}$$

Taking moments about position A:

$$R_B L = M_o + \frac{PL}{3} + \frac{qL^2}{6}$$

$$\therefore R_B = \frac{M_o}{L} + \frac{P}{3} + \frac{qL}{6} = \frac{11}{3}\ \mathrm{kN}$$

Substituting this into equation (21) and rearranging for R_A:

$$R_A = P + \frac{qL}{3} - R_B = \frac{1}{3}\ \mathrm{kN}$$

Next, as there is a discontinuous uniformly distributed load, this needs to be extended to the end of the beam and an additional, negative, counterbalancing UDL superimposed over the newly extended part. Figure 18 shows the result of taking the left-hand end of the beam as the origin and sectioning the beam after the final discontinuity.

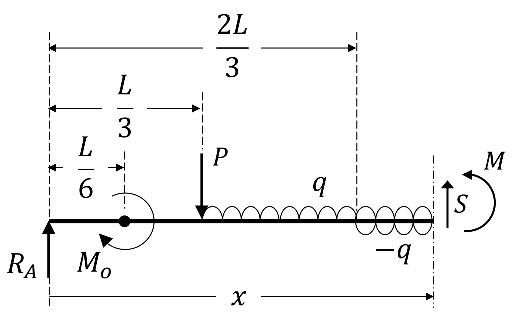

Figure 18

Taking moments about the section position (remembering to implement Macaulay's convention):

$$M + P\,\langle x - \frac{L}{3}\rangle + \frac{q\,\langle x - \frac{L}{3}\rangle^2}{2} = R_A x + M_o\,\langle x - \frac{L}{6}\rangle^0 + \frac{q\,\langle x - \frac{2L}{3}\rangle^2}{2}$$

$$\therefore M = M_o\,\langle x - \frac{L}{6}\rangle^0 + R_A x - P\,\langle x - \frac{L}{3}\rangle + \frac{q\,\langle x - \frac{2L}{3}\rangle^2}{2} - \frac{q\,\langle x - \frac{L}{3}\rangle^2}{2}$$

Substituting this into equation (8) gives:

$$EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = M_o\,\langle x - \frac{L}{6}\rangle^0 + R_A x - P\,\langle x - \frac{L}{3}\rangle + \frac{q\,\langle x - \frac{2L}{3}\rangle^2}{2} - \frac{q\,\langle x - \frac{L}{3}\rangle^2}{2}$$

Integrating with respect to x to determine an expression for the slope:

$$EI\frac{\mathrm{d}y}{\mathrm{d}x} = M_o\,\langle x - \frac{L}{6}\rangle + \frac{R_A x^2}{2} - \frac{P\,\langle x - \frac{L}{3}\rangle^2}{2} + \frac{q\,\langle x - \frac{2L}{3}\rangle^3}{6} - \frac{q\,\langle x - \frac{L}{3}\rangle^3}{6} + A \tag{22}$$

Integrating with respect to x again to determine an expression for the deflection:

$$EIy = \frac{M_o\,\langle x - \frac{L}{6}\rangle^2}{2} + \frac{R_A x^3}{6} - \frac{P\,\langle x - \frac{L}{3}\rangle^3}{6} + \frac{q\,\langle x - \frac{2L}{3}\rangle^4}{24} - \frac{q\,\langle x - \frac{L}{3}\rangle^4}{24} + \mathrm{A}x + \mathrm{B} \tag{23}$$

Using boundary conditions for the determination of the constants of integration:

Boundary condition 1: at x = 0, y = 0 (i.e. at this support position there is no deflection)

Applying this to equation (23):

$$\mathrm{B} = 0$$

Boundary condition 2: at x = L, y = 0 (i.e. at this support position there is no deflection)

Applying this to equation (23) and substituting in values for M_o, P, q, L, R_A and B:

$$\mathrm{A} = -\frac{583}{432}$$

The slope and deflection at $x = \frac{L}{2}$ can now be evaluated by using equations (22) and (23), respectively.

Substituting $x = \frac{L}{2}$ into (22) and (23) and applying Macaulay's convention gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{EI}\left(\frac{M_o L}{3} + \frac{R_A L^2}{8} - \frac{PL^2}{72} - \frac{qL^3}{1296} + A\right)$$

and

$$y = \frac{1}{EI}\left(\frac{M_o L^2}{18} + \frac{R_A L^3}{48} - \frac{PL^3}{1296} - \frac{qL^4}{31104} + \frac{\mathrm{A}L}{2} + \mathrm{B}\right)$$

Note that in each of the above expressions, the term related to the added counterbalancing UDL, i.e. $\frac{q\langle x-\frac{2L}{3}\rangle^3}{6}$ and $\frac{q\langle x-\frac{2L}{3}\rangle^4}{24}$, respectively, has been removed as $x < c$ and so the contents of the Macaulay brackets in these terms is negative and so the terms are set to zero. In all of the other terms with Macaulay brackets, these have been replaced with regular brackets as the contents are positive and so Macaulay's convention has been satisfied.

Substituting in values of E, I, L, M_o, P, q, R_A, A and B gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2.838 \times 10^{-3}\ \mathrm{rad}$$

and

$$y = -0.01026\ \mathrm{m} = \ -10.26\ \mathrm{mm}$$

7. Statically Indeterminate Problems

Macaulay's method can also be used to solve for the slopes and deflections of statically indeterminate beams. A beam is statically indeterminate when the reaction forces and/or bending moments cannot be determined by the equations of statics alone. An example of this is a clamped-clamped beam subjected to a point load, as shown in Figure 19.

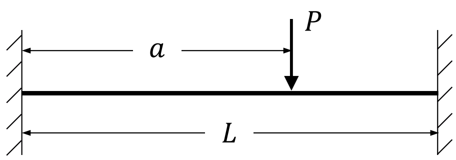

Figure 19

Figure 20 shows a free-body-diagram of the beam.

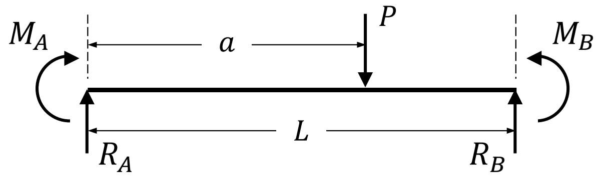

Figure 20

It can be seen from Figure 20 that the end reactions are R_A, M_A, R_B, and M_B; a reaction force and a reaction bending moment which restrain the displacement and rotation, respectively, at both ends of the beam. There are therefore four unknowns which cannot be solved for by equilibrium alone. We therefore continue as before, but this time without knowing the reactions.

Figure 21 shows the result of taking the left-hand end A as the origin and sectioning the beam after the discontinuity.

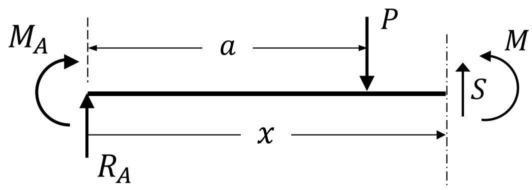

Figure 21

Taking moments about the section position:

$$M + P\langle x - a\rangle = R_A x + M_A$$

$$\therefore M = R_A x + M_A - P\langle x - a\rangle$$

Substituting this into equation (8):

$$EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = R_A x + M_A - P\langle x - a\rangle$$

Integrating with respect to x to determine an expression for the slope:

$$EI\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{R_A x^2}{2} + M_A x - \frac{P\langle x - a\rangle^2}{2} + \mathrm{A} \tag{24}$$

Integrating with respect to x again to determine an expression for the deflection:

$$EIy = \frac{R_A x^3}{6} + \frac{M_A x^2}{2} - \frac{P\langle x-a\rangle^3}{6} + \mathrm{A}x + \mathrm{B} \tag{25}$$

Equations (24) and (25) contain four unknowns, namely M_A and R_A and the integration constants A and B. In this case we can use four boundary conditions to solve for these four unknowns.

Boundary condition 1: at x = 0, y = 0 (i.e. at this clamp position there is no deflection)

Applying this to equation (25):

$$\mathrm{B} = 0$$

Boundary condition 2: at x = 0, $\frac{\mathrm{d}y}{\mathrm{d}x}$ = 0 (i.e. at this clamp position there is no rotation)

Applying this to equation (24):

$$\mathrm{A} = 0$$

Boundary condition 3: at x = L, y = 0 (i.e. at this clamp position there is no deflection)

Applying this to equation (25):

$$0 = \frac{R_A L^3}{6} + \frac{M_A L^2}{2} - \frac{P(L-a)^3}{6} \tag{26}$$

Boundary condition 4: at x = L, $\frac{\mathrm{d}y}{\mathrm{d}x}$ = 0 (i.e. at this clamp position there is no rotation)

Applying this to equation (24):

$$0 = \frac{R_A L^2}{2} + M_A L - \frac{P(L-a)^2}{2}$$

$$\therefore M_A = \frac{P(L-a)^2 - R_A L^2}{2L} \tag{27}$$

Substituting equation (27) into equation (26):

$$R_A = \frac{P(L^3 - 3La^2 + 2a^3)}{L^3}$$

Substituting this into equation (27):

$$M_A = -\frac{Pa(L^2 - 2La + a^2)}{L^2}$$

Substituting these expressions for R_A and M_A into equations (24) and (25):

$$EI\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{P(L^3 - 3La^2 + 2a^3)}{2L^3}x^2 - \frac{Pa(L^2 - 2La + a^2)}{L^2}x - \frac{P\langle x - a\rangle^2}{2} \tag{28}$$

and

$$EIy = \frac{P(L^3 - 3La^2 + 2a^3)}{6L^3}x^3 - \frac{Pa(L^2 - 2La + a^2)}{2L^2}x^2 - \frac{P\langle x - a\rangle^3}{6} \tag{29}$$

Special Case – Centrally Loaded Clamped-Clamped Beam

In the case of the beam shown in Figure 19 being loaded at the centre position, and evaluating for the slope and deflection at this position, i.e. $a = \frac{L}{2} = x$, equations (28) and (29) give:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and

$$y = -\frac{PL^3}{192EI}$$

As this beam is loaded and supported symmetrically, it makes sense that the slope at the centre position is zero. Additionally, as the single applied load is downwards, it would be expected that the deflection at the centre of the beam would also be downwards and therefore negative (according to the sign convention defined in section 2 of these notes, i.e. y is positive in the upwards direction).

It is interesting to note that clamping the ends of a beam which is carrying a single point load at its centre position, results in a deflection which is 25% of the deflection of a simply supported equivalent (see equation (17)).