Separation of Variable Techniques for PDEs
Problem Sheet 5

1. Find the most general separable solution of the partial differential equation

$$\frac{\partial^2 \phi}{\partial t \partial x} + x e^t \phi = 0.$$

2. The temperature, T, in a metal rod of length L satisfies the diffusion equation,

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2},$$

where D is a positive constant.

   (a) If the temperature at the ends of the bar is fixed, so that $T(0, t) = T_1$ and $T(L, t) = T_2$, find the steady solution, $T = T_s(x)$.

   (b) If the initial temperature is spatially-uniform and equal to T_1, determine the temperature for $t > 0$. What happens as $t \to \infty$?
   Hint: Define $\bar{T}(x, t) = T(x, t) - T_s(x)$ and solve for $\bar{T}$ using separation of variables.

3. Laplace's equation in two dimensions is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Find the solution in $x > 0$, $0 < y < L$ that satisfies $\phi = 0$ at $y = 0$ and $y = L$, $\phi \to 0$ as $x \to \infty$ and $\phi = y(L - y)$ at $x = 0$.

4. The small-amplitude motion of a stretched string of length L that is uniformly forced along its whole length at frequency ω satisfies the forced one-dimensional wave equation,

$$\frac{\partial^2 Y}{\partial t^2} = c^2 \frac{\partial^2 Y}{\partial x^2} + g \sin \omega t.$$

The string is fixed at $x = 0$ and $x = L$, so that $Y(0, t) = Y(L, t) = 0$.

   (a) Find the forced solution, $Y = Y_f(x, t)$ by writing $Y_f = \bar{Y}(x) \sin \omega t$ and finding $\bar{Y}(x)$. You should thereby determine the resonant frequencies, but can assume that the forcing frequency, ω, is *not* a resonant frequency.

   (b) If the string is undisturbed when $t = 0$, determine the solution for $t > 0$. In this case, you can just write down an integral expression for the Fourier coefficients. *Hint:* see the method for question 2.