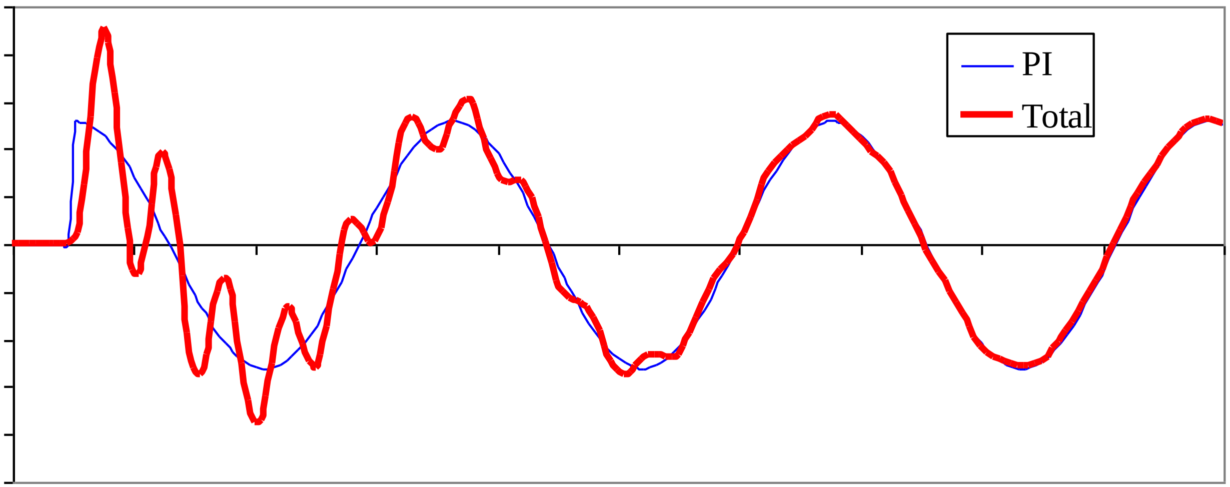
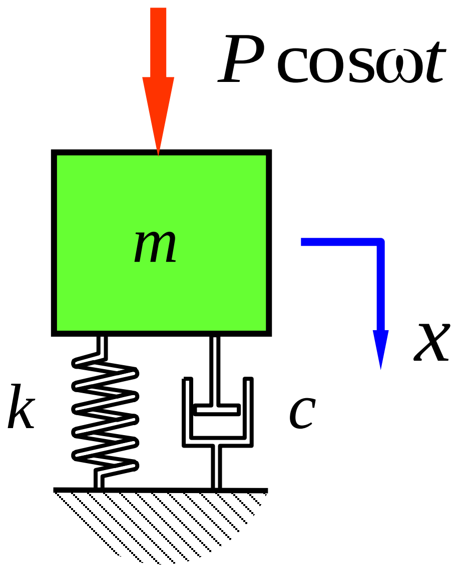
Section B

HARMONIC/FORCED EXCITATION OF SINGLE-DEGREE-OF-FREEDOM SYSTEMS

The response of a single-degree-of-freedom system due to excitation in the form of a cosine wave is shown below





There is an initial transient response that normally decays quickly. This is followed by a steady-state sinusoidal response at the same frequency as the excitation.

Forced Versus Harmonic Vibration

Forced Vibration is when an alternating force or motion is applied to a mechanical system, for example when a machine shakes momentarily due to an imbalance.

Harmonic Vibration is a type of Forced Vibration in which a force is repeatedly applied to a system. This is primarily what we will consider in this class.

(i) Solution of the Equation of Motion

$$M\ddot{z} + C\dot{z} + Kz = f(t)$$

The complete solution for $z(t)$ consists of

- The solution to the ***Complementary Function*** or ***Transient Response***
- The ***Particular Integral*** or ***Steady-State Response***

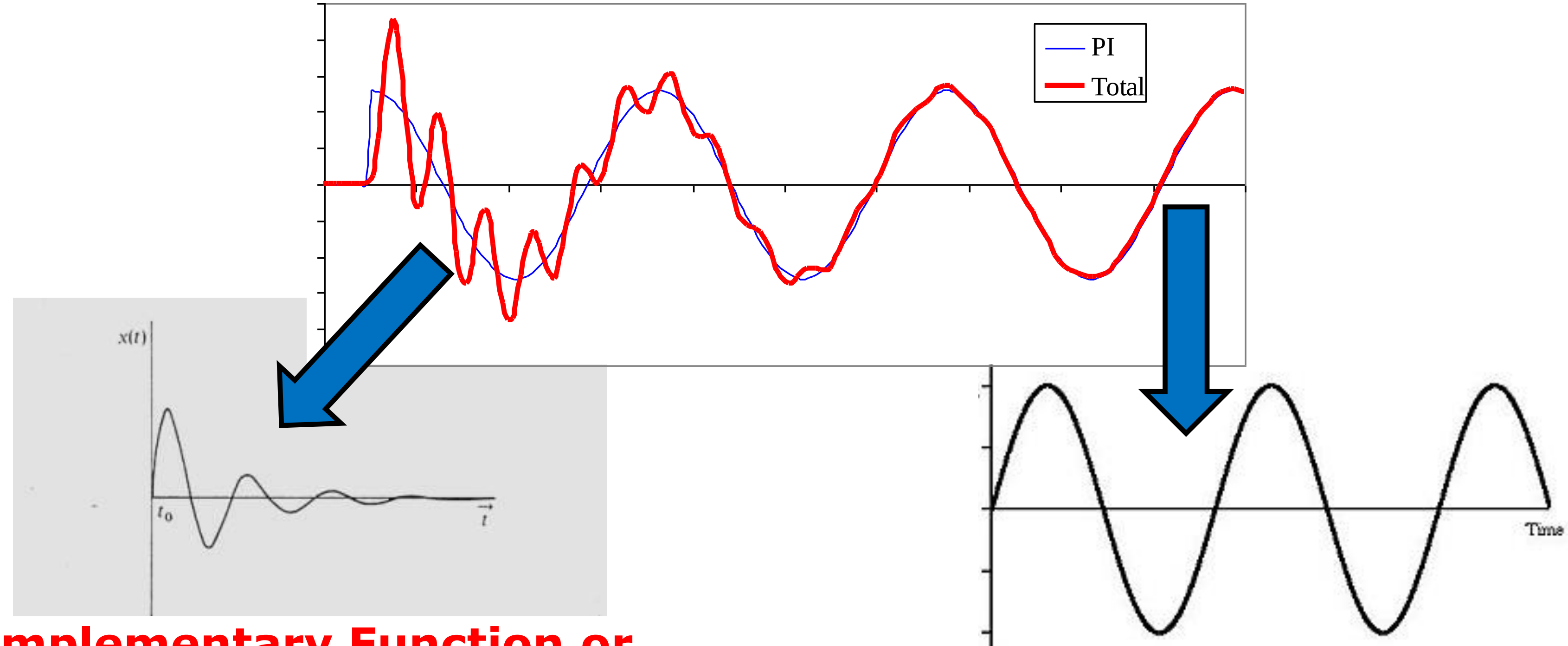
$$z(t) = z(t)_{TR} + z(t)_{SS}$$

The **Complementary Function** is simply the solution to the equivalent free vibration problem ($f(t) = 0$ from last week) considered last week and **provides the initial transient response** ($z(t)_{TR}$)

The **Particular Integral** provides the **steady-state part of the vibration** that continues for as long as the excitation remains

In most cases, the steady-state response ($z(t)_{SS}$)

is all we are interested in. This is what we will look at today.



Complementary Function or Transient Response

$$M\ddot{z} + C\dot{z} + Kz = 0$$

$$z(t)_{TR} = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

$$\lambda_{1,2} = \frac{-C \pm \sqrt{C^2 - 4KM}}{2M}$$

Particular Integral or Steady State Response

$$z(t)_{SS} = Z \cos(\omega t + \alpha)$$

$$M\ddot{z} + C\dot{z} + Kz = f(t)$$

$$M\ddot{z} + C\dot{z} + Kz = C_r\dot{r}(t) + K_r r(t)$$

$$M\ddot{z} + C\dot{z} + Kz = S_1 \cos\omega_1 t + S_2 \cos\omega_2 t$$

Total Displacement is then $z(t) = z(t)_{TR} + z(t)_{SS}$

Method 1 - Direct Substitution $M\ddot{z} + C\dot{z} + Kz = f(t)$

Consider harmonic excitation of the form $f(t) = F\cos\omega t$

For pure sinusoidal excitation, the response is also sinusoidal

- **The response has the same frequency as the excitation, but the two are likely to have a phase difference**

A suitable expression for the response is therefore

$$z(t)_{SS} = Z\cos(\omega t + \alpha) \tag{1}$$

- The **amplitude** of the vibration is Z
- α is the **phase angle** between response and excitation

The same mathematical approach can be extended

- to more complicated structures
- to more general forms of excitation
- to experimental testing and digital data analysis procedures

To find Z and α, substitute for $z(t)$ and its derivatives in the equation of motion, expand the various trigonometric terms and equate the coefficients of $\cos \omega t$ and $\sin \omega t$ and solve for Z and α

$$M \ddot{z} + C \dot{z} + K z = f(t)$$

$$z(t)_{SS} = Z \cos(\omega t + \alpha)$$

$$\dot{z}(t)_{SS} = -\omega Z \sin(\omega t + \alpha)$$

$$\ddot{z}(t)_{SS} = -\omega^2 Z \cos(\omega t + \alpha)$$

Hence

$$Z = \frac{F}{\sqrt{(K - M\omega^2)^2 + \omega^2 C^2}} \quad (2)$$

and

$$\alpha = \tan^{-1}\left(\frac{-\omega C}{K - M\omega^2}\right) \quad (3)$$

The following substitution is suggested in the Year 2 Maths module

$$M\ddot{z} + C\dot{z} + Kz = f(t)$$

$$z(t)_{SS} = A\sin(\omega t) + B\cos(\omega t)$$

$$\dot{z}(t)_{SS} = \omega A\cos(\omega t) - \omega B\sin(\omega t)$$

$$\ddot{z}(t)_{SS} = -\omega^2 A\sin(\omega t) - \omega^2 B\sin(\omega t)$$

This is equally valid and results in the same solution.

I recommend you use whichever one you find easier when expanding to other systems. But keep in mind that you will have to manipulate ***A*** and ***B*** to find Z and α, using these substitutions.

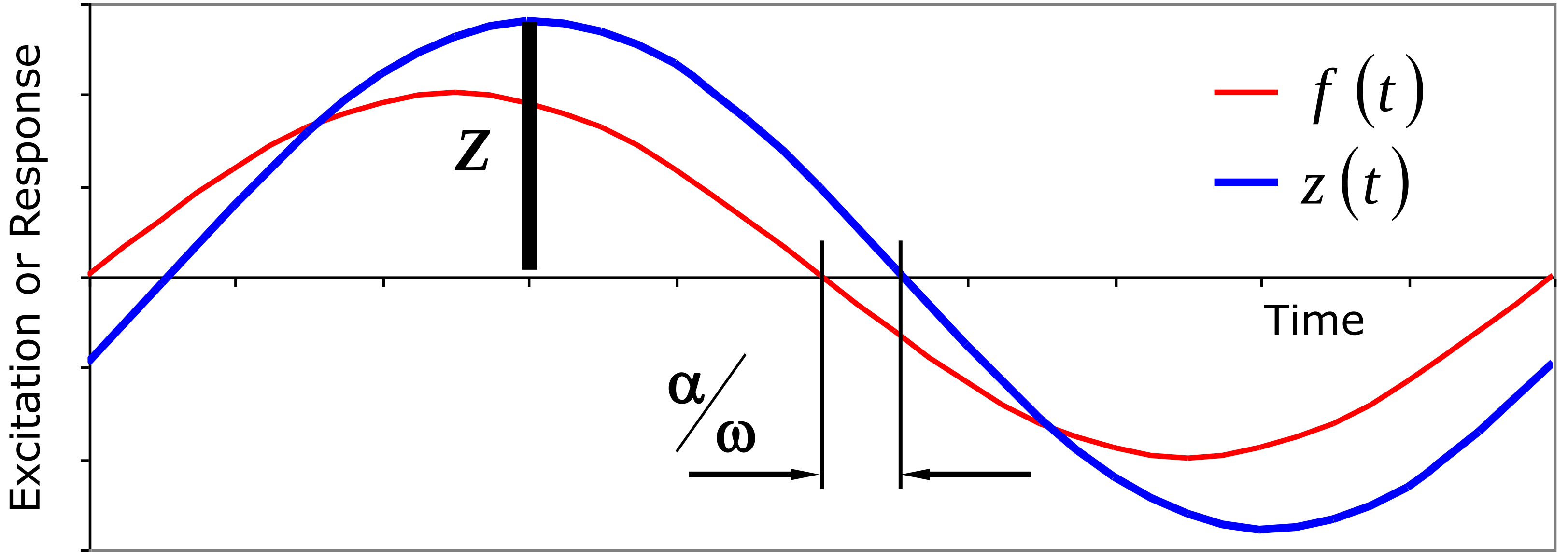
$$Z = \sqrt{A^2 + B^2} = \frac{F}{\sqrt{\left(K - M\omega^2\right)^2 + C^2\omega^2}}$$

$$\alpha = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{-C\omega}{K - M\omega^2}\right)$$

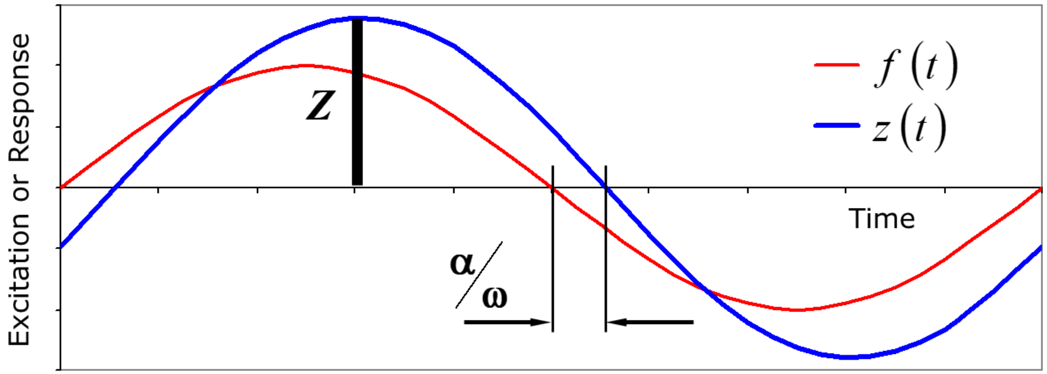
Q What if the excitation had been $f(t) = F \sin \omega t$?

A Choose the substitution $z(t)_{SS} = Z \sin(\omega t + \alpha)$

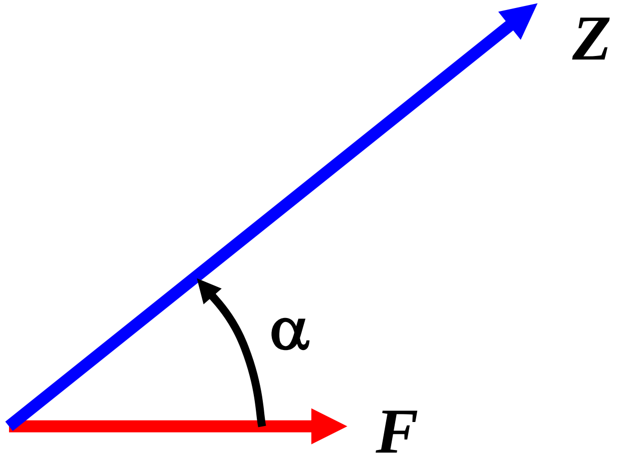
The expressions for Z and α are the same as before



Since the steady-state response to sinusoidal excitation is also sinusoidal and with the same frequency as the excitation, the key parameters to be identified are the **amplitude** and **phase angle**



Instead of time waveforms, sometimes the amplitude and phase relationship between excitation and response is with a **phasor diagram**



Method 2 - Complex Algebra $M\ddot{z} + C\dot{z} + Kz = f(t)$

This provides a mathematically convenient way of finding the amplitude and phase angle of the response and is the preferred method over the previously demonstrated "substitution" one.

The same mathematical approach can be extended

- to more complicated structures (shown later)
- to more general forms of excitation
- to experimental testing and digital data analysis procedures

The substitutions used are always the same

Put $$f(t) = F\,e^{\mathbf{i}\omega t} \qquad (4)$$

and $$z(t)_{SS} = Z\,e^{\mathbf{i}(\omega t + \alpha)}$$

$$= \left(Z\,e^{\mathbf{i}\alpha}\right)e^{\mathbf{i}\omega t}$$

$$= Z^*\,e^{\mathbf{i}\omega t} \qquad (5)$$

where Z^* is **COMPLEX** $\left[= Z\left(\cos\alpha + \mathbf{i}\sin\alpha\right)\right]$

$$z(t)_{SS} = Z^* e^{\mathbf{i}\omega t}$$

Substituting $\dot{z}(t)_{SS} = i\omega Z^* e^{\mathbf{i}\omega t}$ into the equation of motion, we get

$$\ddot{z}(t)_{SS} = -\omega^2 Z^* e^{\mathbf{i}\omega t}$$

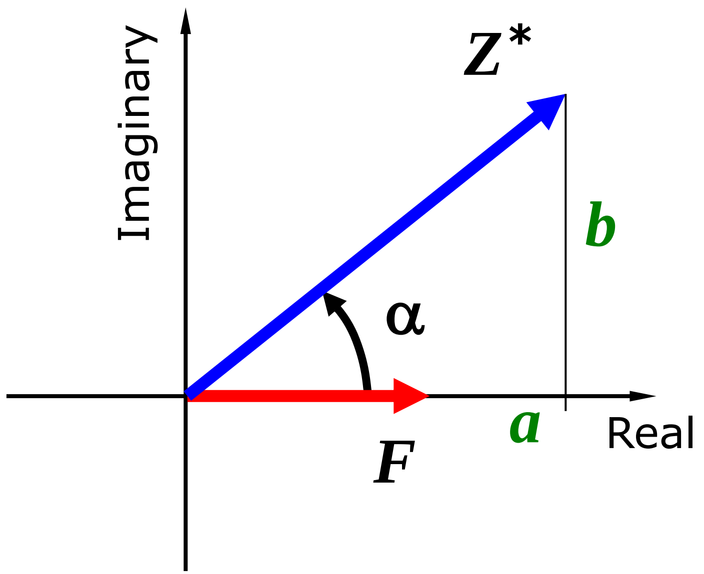
$$Z^* = \frac{F}{\left(K - M\omega^2\right) + \mathbf{i}\omega C}$$

The Moodle site includes a help page on the complex number algebra needed for this module. I will also cover this in general in the following next slides.

We need $\mathbf{Z^*}$ in the form $Z^* = a + \mathbf{i}\, b$

Amplitude $Z = \left|Z^*\right| = \sqrt{a^2 + b^2}$

Phase angle $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$



Unfortunately, the complex number expressions you will derive are not generally in the form

$$Z^* = a + \mathbf{i}\, b$$

The general case for you is the form

$$Z^* = \frac{c + \mathbf{i}\, d}{e + \mathbf{i} f}$$

Finding the amplitude of this ratio of complex numbers is easy. It's just

$$Z = \left| Z^* \right| = \sqrt{\frac{c^2 + d^2}{e^2 + f^2}}$$

Finding the phase lag is more difficult, since we first need to convert the expression into the form

$$Z^* = a + \mathbf{i}\, b$$

To do this, we multiply both numerator and denominator by the complex conjugate of the denominator. That is,

$$Z^* = \frac{c + \mathbf{i}\, d}{e + \mathbf{i}\, f} \times \frac{e - \mathbf{i}\, f}{e - \mathbf{i}\, f}$$

$$Z^* = a + \mathbf{i}\, b = \left(\frac{ce + df}{e^2 + f^2} \right) + \mathbf{i} \left(\frac{de\text{-}cf}{e^2 + f^2} \right)$$

$$a = \left(\frac{ce + df}{e^2 + f^2} \right) \qquad b = \left(\frac{de\text{-}cf}{e^2 + f^2} \right)$$

The phase angle is then given by $\alpha = \tan^{-1}\left(\frac{b}{a} \right)$ as before.

So for our problem…

$$Z^* = \frac{F}{\left(K - M\omega^2\right) + \mathbf{i}\,\omega C}$$

where

$$c = F$$

$$d = 0$$

$$e = K - M\omega^2$$

$$f = \omega C$$

$$Z = \left|Z^*\right| = \frac{c}{\sqrt{e^2 + f^2}} = \frac{F}{\sqrt{\left(K - M\omega^2\right)^2 + \omega^2 C^2}}$$

Same as before!!

For angle then

$$Z^{*} = \frac{F}{\left(K - M\omega^{2}\right) + \mathbf{i}\,\omega C} \times \frac{\left(K - M\omega^{2}\right) - \mathbf{i}\,\omega C}{\left(K - M\omega^{2}\right) - \mathbf{i}\,\omega C}$$

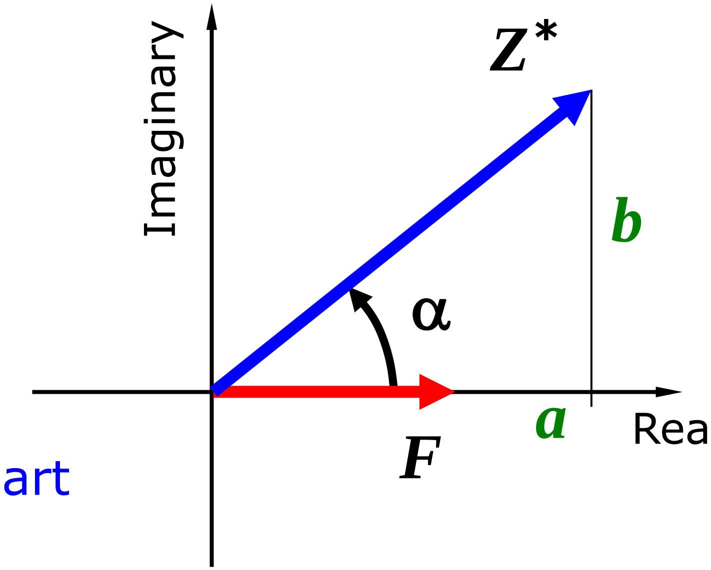
$$Z^{*} = \frac{F\left(K - M\omega^{2}\right) - \mathbf{i}\,F\omega C}{\left(K - M\omega^{2}\right)^{2} + \omega^{2} C^{2}}$$

$$\alpha = \tan^{-1}\left(\frac{-\omega C}{K - M\omega^{2}}\right)$$

Same as before!!

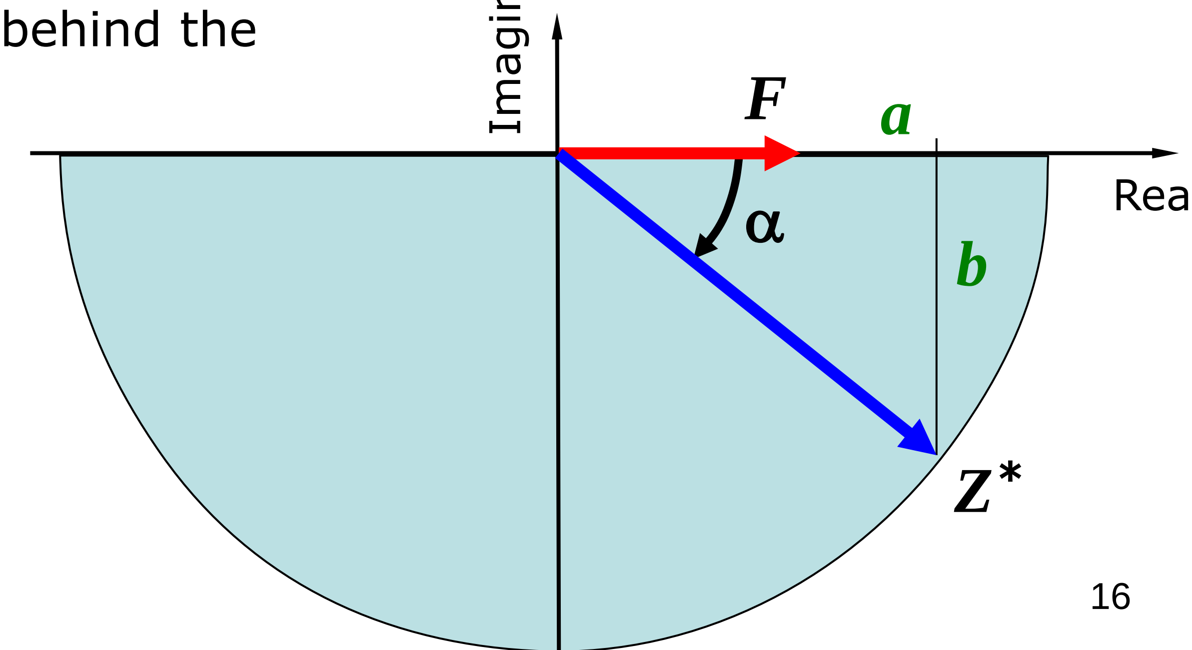
$$Z^* = a + \mathbf{i}\, b$$

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{-\omega C}{K - M\omega^2}\right)$$



In practice, notice that the imaginary part of Z^* is always negative

The phase angle is also negative, meaning that the response **LAGS** behind the excitation



(ii) Frequency characteristics of the response

To see how the excitation frequency affects the response, we will consider the ***Frequency Response Function*** (**FRF**)

Definition of FRF: **Response / Unit Applied Force**

Start with the general form of the equation of motion

$$M\,\ddot{z} + C\,\dot{z} + K\,z = f(t)$$

Dividing by $\boldsymbol{M}$ and noting that $\dfrac{C}{M} = 2\gamma\,\omega_n$ we get

$$\ddot{z} + 2\gamma\,\omega_n\,\dot{z} + \omega_n^2\,z = \frac{f(t)}{M}$$

$$\ddot{z} + 2\gamma\omega_n \dot{z} + \omega_n^2 z = \frac{f(t)}{M}$$

Put $f(t) = F\,\mathrm{e}^{\mathbf{i}\omega t}$ and

$$z(t)_{SS} = Z^* \mathrm{e}^{\mathbf{i}\omega t}$$

$$\dot{z}(t)_{SS} = i\omega Z^* \mathrm{e}^{\mathbf{i}\omega t}$$

$$\ddot{z}(t)_{SS} = -\omega^2 Z^* \mathrm{e}^{\mathbf{i}\omega t}$$

Hence the expression for the Frequency Response Function is

$$H(\omega) = \frac{Z^*}{F} = \frac{1}{M}\,\frac{1}{\left(\omega_n^2 - \omega^2\right) + \mathbf{i}\,2\gamma\omega_n\omega} \qquad (6)$$

Divide top and bottom by $\boldsymbol{\omega_n^2}$ and note that $\boldsymbol{M\omega_n^2 = K}$

Hence

$$H(\omega) = \frac{1}{K}\,\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + \mathbf{i}\,2\gamma\frac{\omega}{\omega_n}}$$

Magnitude can then be written as

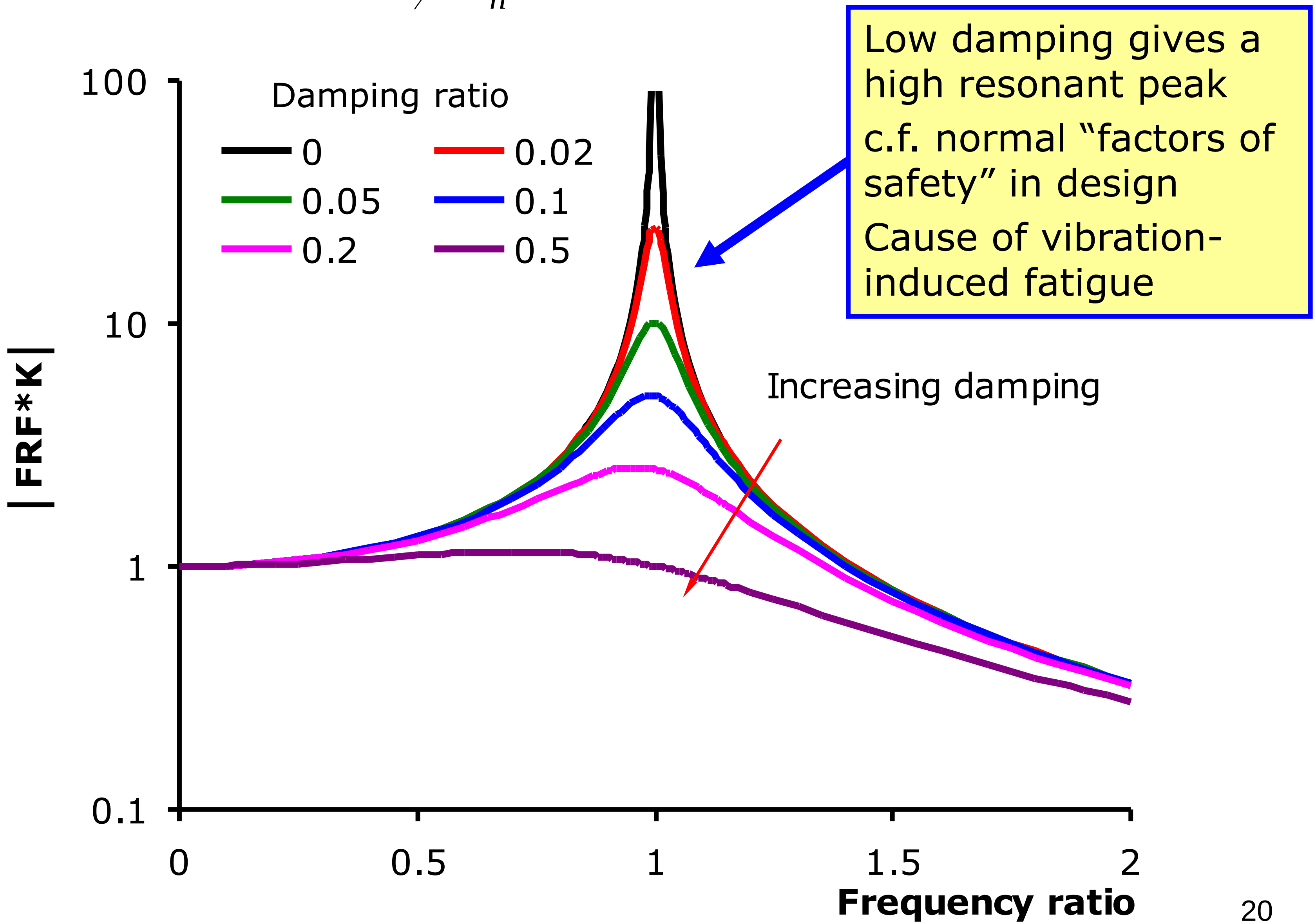
$$\left|H(\omega)\right| = \frac{|Z^*|}{F} = \frac{1}{K\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\gamma^2\frac{\omega^2}{\omega_n^2}}} = \frac{1}{\sqrt{\left(K - M\omega^2\right)^2 + \omega^2 C^2}}$$

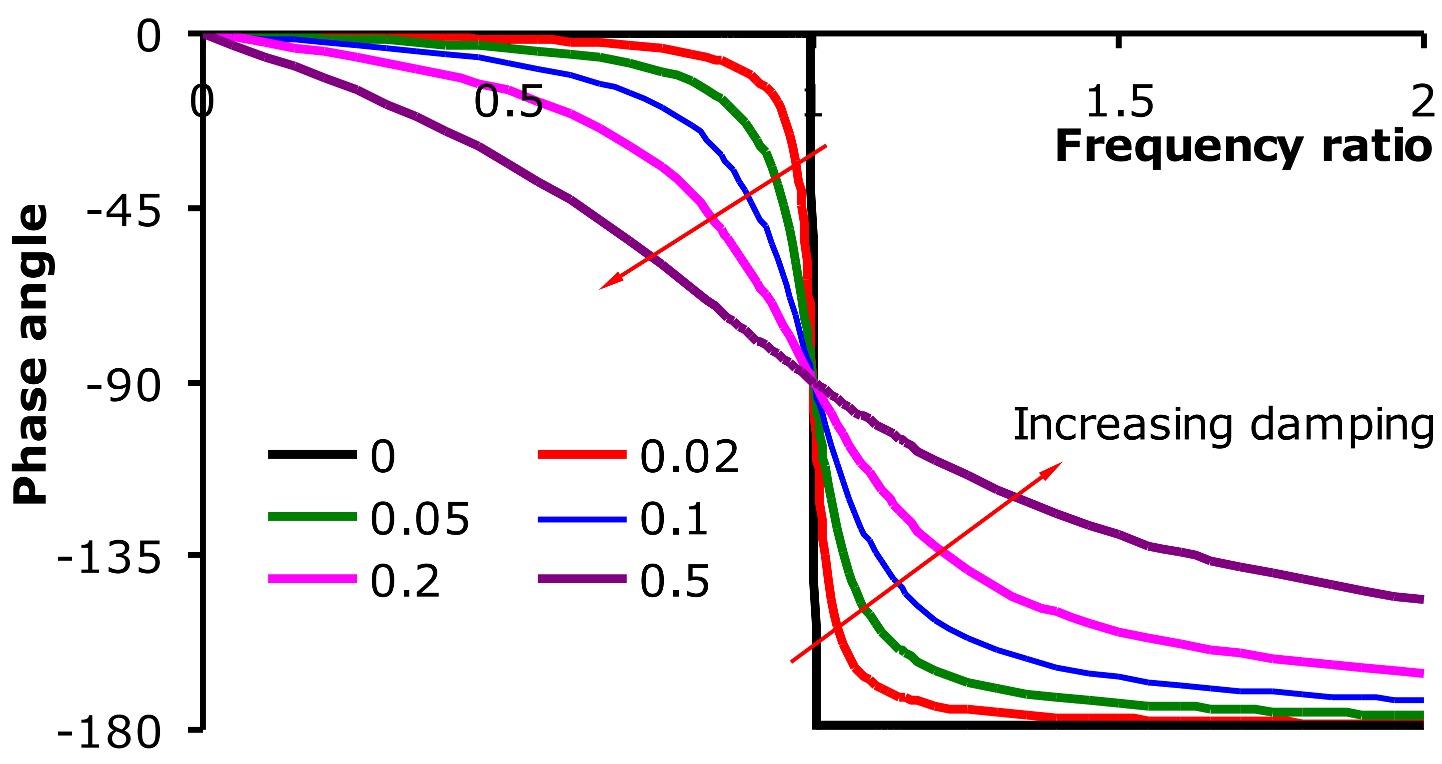
…and phase angle can then be written as

$$\alpha = \tan^{-1}\left(\frac{-2\gamma\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right) = \tan^{-1}\left(\frac{-\omega C}{K - M\omega^2}\right)$$

Note: These are equivalent to equations 2 and 3 found previously, just with different terms (and dividing equation 2 by *F*). You can prove this by substituting in equations from last week's lecture yourself.

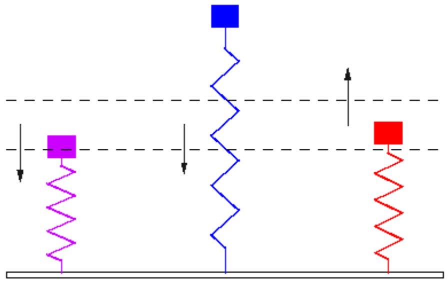
The response depends on ω / ω_n and on damping ratio γ





At low frequencies, the phase angle is approximately zero

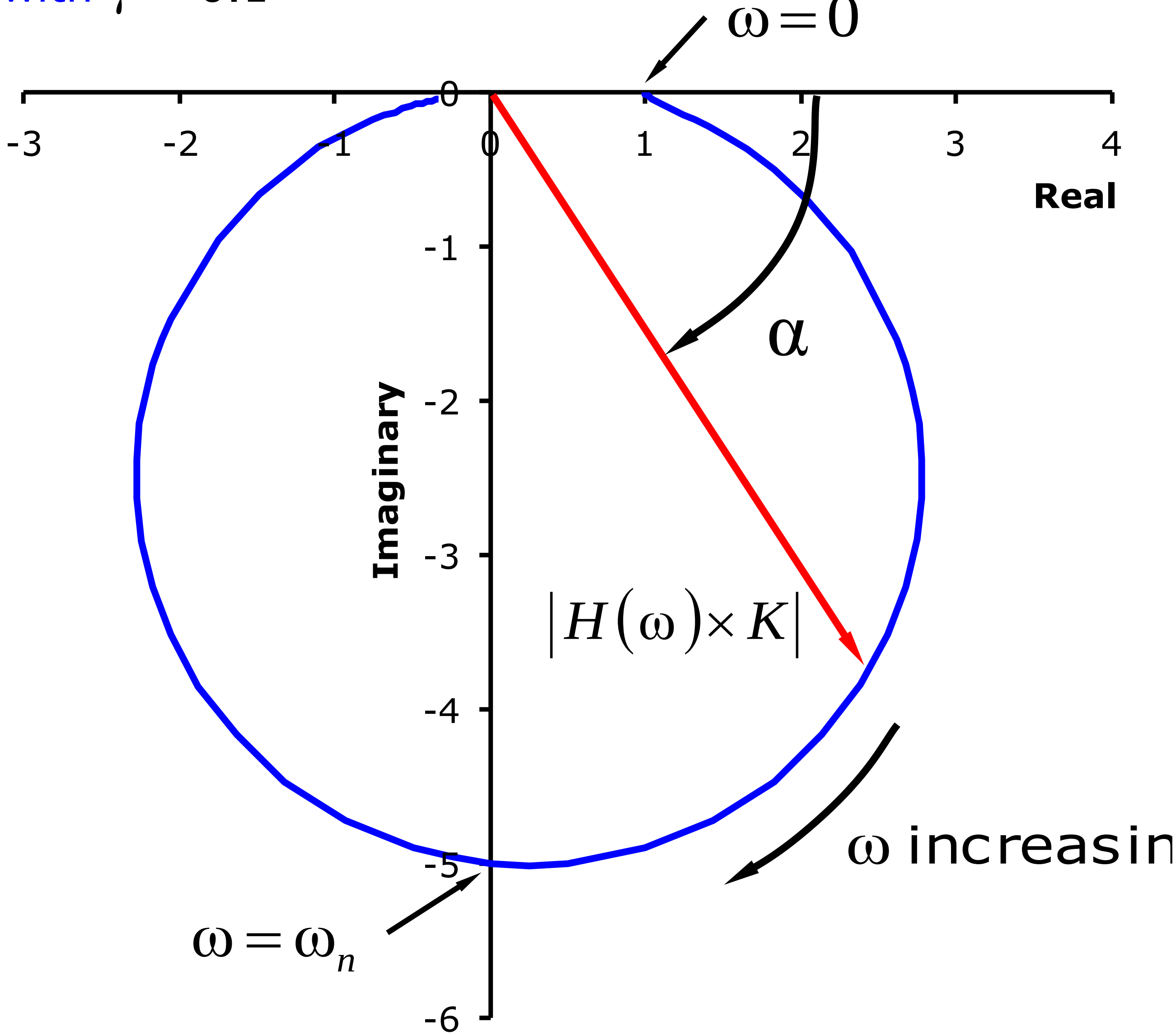
It drops by 180º as the system passes through resonance



See Web Links on MM2DYN Moodle site

www.kettering.edu/~drussell/Demos/SHO/mass-force.html

Example with $\gamma = 0.1$



https://www.youtube.com/watch?v=aZNnwQ8HJHU