University of Nottingham
Department of Mechanical Engineering

MM2MS3 Mechanics of Solids 3
Exercise Sheet 4 – Deflection of Beams Solutions

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1. Derive expressions for the deflection and slope of the tip of a cantilever beam, length L, which carries:

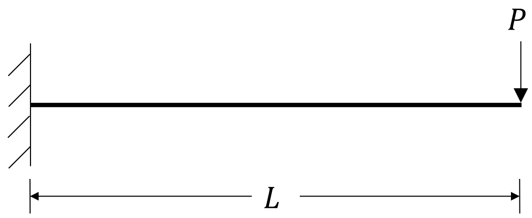
    (a) A point force, P, at the tip
    (b) A point couple, M_o, at the tip
    (c) A uniformly distributed load, w per unit, across its entire length

    The second moment of area of the cross-section is Im^4 and the Young's modulus of the material is $EMPa$.

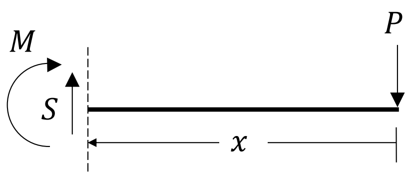
[Ans: a) $\frac{dy}{dx} = \frac{PL^2}{2EI}$ **&** $y = -\frac{PL^3}{3EI}$ ***b)*** $\frac{dy}{dx} = -\frac{M_oL}{EI}$ **&** $y = \frac{M_oL^2}{2EI}$ ***c)*** $\frac{dy}{dx} = \frac{wL^3}{6EI}$ **&** $y = -\frac{wL^4}{8EI}$***]***

Solution 1

(a)



Take origin from right hand side (from free edge – so that reaction force/moment at built-in end are not required) and section the beam as follows:



Taking moments about the section position gives:

$$M + Px = 0$$

$$\therefore M = -Px$$

Substituting this in the main deflection of beams equation gives:

$$EI\frac{d^2y}{dx^2} = M = -Px$$

Integrating gives:

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + A \tag{1}$$

Integrating again gives:

$$EIy = -\frac{Px^3}{6} + Ax + B \tag{2}$$

Boundary conditions:

(BC1) At $x = L$, $\frac{dy}{dx} = 0$, therefore from (1):

$$0 = -\frac{PL^2}{2} + A$$

$$\therefore A = \frac{PL^2}{2}$$

(BC2) At $x = L$, y = 0, therefore from (2):

$$0 = -\frac{PL^3}{6} + \left(\frac{PL^2}{2}\right)L + B$$

$$\therefore B = -\frac{PL^3}{3}$$

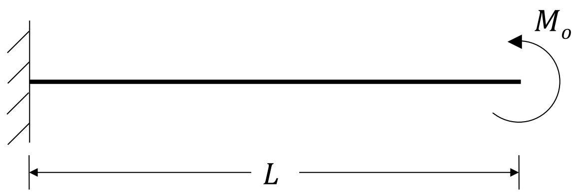
Therefore, from (1) and (2), at the tip of the beam (i.e, $x = 0$):

$$\boxed{\frac{dy}{dx} = \frac{\boldsymbol{PL^2}}{\boldsymbol{2EI}}}$$

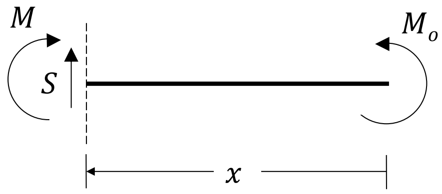
and

$$\boxed{y = -\frac{\boldsymbol{PL^3}}{\boldsymbol{3EI}}}$$

(b)



Take origin from right hand side (from free edge – so that reaction force/moment at built-in end are not required) and section the beam as follows:



Taking moments about the section position gives:

$$M = M_o$$

Substituting this in the main deflection of beams equation gives:

$$EI\frac{d^2y}{dx^2} = M = M_o$$

Integrating gives:

$$EI\frac{dy}{dx} = M_o x + A \tag{3}$$

Integrating again gives:

$$EIy = \frac{M_o x^2}{2} + Ax + B \tag{4}$$

Boundary conditions:

(BC1) At $x = L$, $\frac{dy}{dx} = 0$, therefore from (3):

$$A = -M_o L$$

(BC2) At $x = L$, y = 0, therefore from (4):

$$0 = \frac{M_o L^2}{2} + (-M_o L)L + B$$

$$\therefore B = \frac{M_o L^2}{2}$$

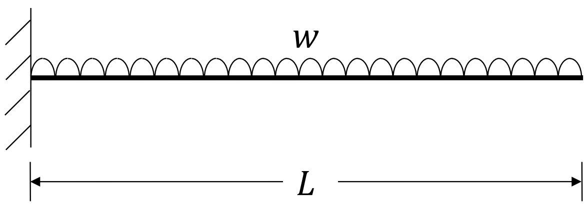
Therefore, from (3) and (4), at the tip of the beam (i.e. $x = 0$):

$$\boxed{\frac{dy}{dx} = -\frac{\boldsymbol{M_o L}}{\boldsymbol{EI}}}$$

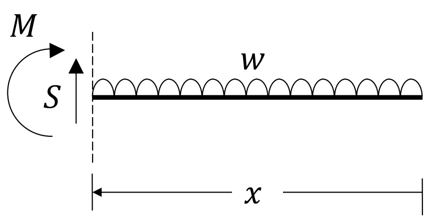
and,

$$\boxed{y = \frac{\boldsymbol{M_o L^2}}{\boldsymbol{2EI}}}$$

(c)



Take origin from right hand side (from free edge – so that reaction force/moment at built-in end are not required) and section the beam as follows:



Taking moments about the section position gives:

$$M + \frac{wx^2}{2} = 0$$

$$\therefore M = -\frac{wx^2}{2}$$

Substituting this in the main deflection of beams equation gives:

$$EI\frac{d^2y}{dx^2} = M = -\frac{wx^2}{2}$$

Integrating gives:

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + A \tag{5}$$

Integrating again gives:

$$EIy = -\frac{wx^4}{24} + Ax + B \tag{6}$$

Boundary conditions:

(BC1) At $x = L, \frac{dy}{dx} = 0$, therefore from (5):

$$A = \frac{wL^3}{6}$$

(BC2) At $x = L, y = 0$, therefore from (6):

$$0 = -\frac{wL^4}{24} + \left(\frac{wL^3}{6}\right)L + B$$

$$\therefore B = -\frac{wL^4}{8}$$

Therefore, from (5) and (6), at the tip of the beam (i.e. $x = 0$):

$$\boxed{\frac{dy}{dx} = \frac{\boldsymbol{wL^3}}{\boldsymbol{6EI}}}$$

and,

$$\boxed{y = -\frac{\boldsymbol{wL^4}}{\boldsymbol{8EI}}}$$

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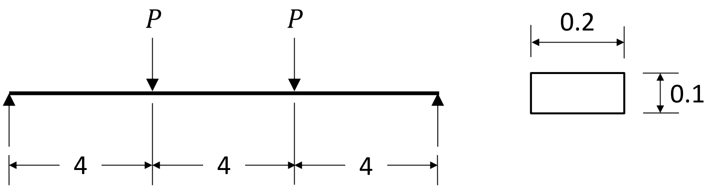
2. Figure Q2 shows a simply supported beam carrying two concentrated loads at the positions indicated. Given that the beam has a rectangular cross-section as shown, calculate (a) the deflection of the beam at a position 3m from the left hand end (b) and at a position 5m from the right hand end. Assume $E_{steel} = 200GPa$.



All dimensions in meters, P = 10kN

Fig Q2

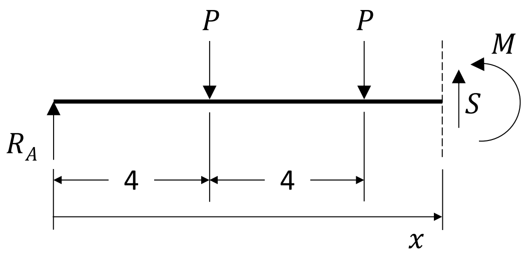
[Ans: a) -130.5mm, b) -178mm]

Solution 2

Symmetry gives:

$$R_A = R_B = P$$

Sectioning the beam after the last discontinuity (taking origin at left hand side of the beam) and drawing a Free Body Diagram of left hand side of section:



Taking moments about the section position:

$$M + P\langle x - 8\rangle + P\langle x - 4\rangle = R_A x$$

$$\therefore M = R_A x - P\langle x-8\rangle - P\langle x-4\rangle$$

Substituting this into the deflection of beams equation:

$$EI\frac{d^2y}{dx^2} = M = R_A x - P\langle x-8\rangle - P\langle x-4\rangle$$

Integrating:

$$EI\frac{dy}{dx} = \frac{R_A x^2}{2} - \frac{P\langle x-8\rangle^2}{2} - \frac{P\langle x-4\rangle^2}{2} + A$$

Integrating again:

$$EIy = \frac{R_A x^3}{6} - \frac{P\langle x-8\rangle^3}{6} - \frac{P\langle x-4\rangle^3}{6} + Ax + B \quad (1)$$

Applying boundary conditions:

(BC1) At $x = 0$, $y = 0$, therefore from (1):

$$B = 0$$

(BC2) At $x = 12$, $y = 0$, therefore from (1):

$$0 = \frac{R_A \times 12^3}{6} - \frac{P \times 4^3}{6} - \frac{P \times 8^3}{6} + 12A$$

$$\therefore A = -16P$$

Therefore from (1):

$$EIy = \frac{R_A x^3}{6} - \frac{P\langle x-8\rangle^3}{6} - \frac{P\langle x-4\rangle^3}{6} - 16Px$$

where,

$$I = \frac{bd^3}{12} = \frac{0.2 \times 0.1^3}{12} = 1.667 \times 10^{-5} m^4$$

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Therefore at $x = 3$:

$$200 \times 10^9 \times 1.667 \times 10^{-5} \times y = \frac{10000 \times 3^3}{6} - 0 - 0 - 16 \times 10000 \times 3$$

$$\boxed{\therefore y = -0.1305m = \boldsymbol{-130.5mm}}$$

And at $x = 7$:

$$200 \times 10^9 \times 1.667 \times 10^{-5} \times y = \frac{10000 \times 7^3}{6} - 0 - \frac{10000 \times 3^3}{6} - 16 \times 10000 \times 7$$

$$\boxed{\therefore y = -0.178m = \boldsymbol{-178mm}}$$

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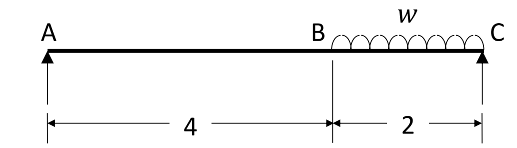
3. Find (a) the slope at point A and (b) the deflection at point B of the beam shown in Figure Q3. Assume a Flexural Rigidity, EI, of 4MNm^2.



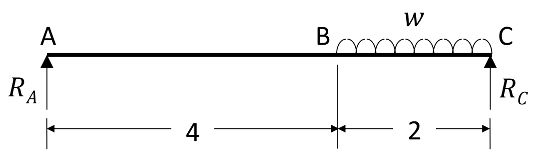
All dimensions in meters, w = 10kN/m

Fig Q3

[Ans: a) -4.72 x 10^{-3}rad, b) -10mm]

Solution 3

Reaction forces will be present at positions A and C, namely R_A and R_C as shown below.



Vertical equilibrium gives:

$$R_A + R_C = w \times 2 \tag{1}$$

Taking moments about position C:

$$R_A \times 6 = w \times 2 \times 1$$

$$\therefore R_A = 3333.33N$$

Substituting this into (1) gives:

$$R_C = 10000 \times 2 - 3333.33 = 16666.67N$$

Sectioning the beam after the last discontinuity (taking origin at left hand side of the beam) and drawing a Free Body Diagram of left hand side of section:

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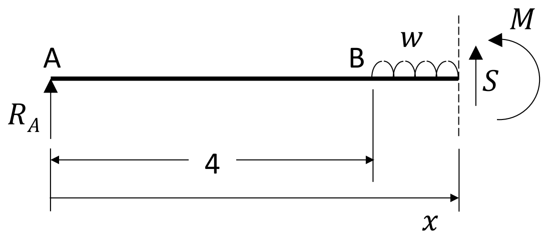


Taking moments about the section position:

$$M + \frac{w\langle x-4\rangle^2}{2} = R_A x$$

$$\therefore M = R_A x - \frac{w\langle x-4\rangle^2}{2}$$

Substituting this into the deflection of beams equation:

$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{w\langle x-4\rangle^2}{2}$$

Integrating:

$$EI\frac{dy}{dx} = \frac{R_A x^2}{2} - \frac{w\langle x-4\rangle^3}{6} + A \qquad (2)$$

Integrating again:

$$EIy = \frac{R_A x^3}{6} - \frac{w\langle x-4\rangle^4}{24} + Ax + B \qquad (3)$$

Applying boundary conditions:

(BC1) At $x = 0$, $y = 0$, therefore from (3):

$$B = 0$$

(BC2) At $x = 6$, $y = 0$, therefore from (3):

$$0 = \frac{3333.33 \times 6^3}{6} - \frac{10000 \times 2^4}{24} + 6A$$

$$\therefore A = -18888.87$$

Therefore from (2), at $x = 0$:

$$4 \times 10^6 \times \frac{dy}{dx} = 0 - 0 - 18888.87$$

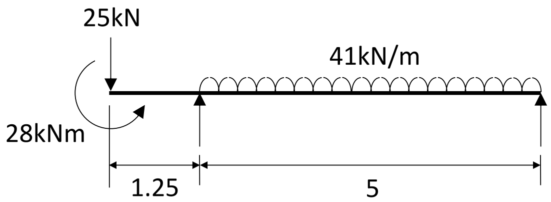
$$\boxed{\therefore \frac{dy}{dx} = \mathbf{-4.72 \times 10^{-3}} \boldsymbol{rad}}$$

And from (3), at $x = 4$:

$$4 \times 10^6 \times y = \frac{3333.33 \times 4^3}{6} - 0 - 18888.87 \times 4$$

$$\boxed{\therefore y = -0.01m = \mathbf{-10} \boldsymbol{mm}}$$

4. Determine (a) the slope and (b) the deflection at the left hand end of the beam shown in Figure Q4. Assume a Flexural Rigidity, EI, of 16.65MNm^2.



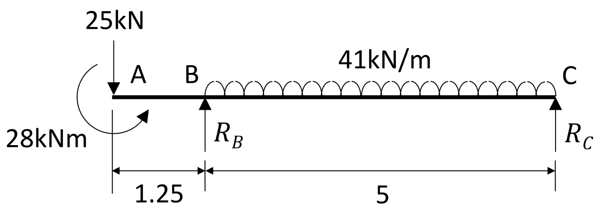
All dimensions in meters

Fig Q4

[Ans: a) $3.62 \times 10^{-3} rad$, b) 6.32mm upwards]

Solution 4

Reaction forces will be present at positions B and C, namely R_B and R_C as shown below.



Vertical equilibrium gives:

$$R_B + R_C = 25000 + 41000 \times 5 = 230000N \qquad (1)$$

Taking moments about position C:

$$28000 + 25000 \times 6.25 + 41000 \times 5 \times 2.5 = R_B \times 5$$

$$\therefore R_B = 139350N$$

Substituting this into (1) gives:

$$R_C = 230000 - 139350 = 90650N$$

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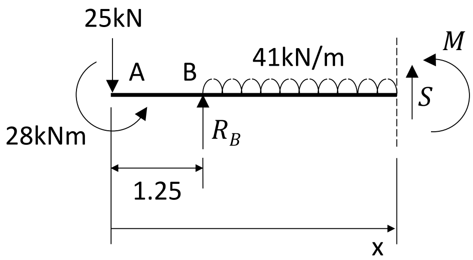
Sectioning the beam after the last discontinuity (taking origin at left hand side of the beam) and drawing a Free Body Diagram of left hand side of section:



Taking moments about the section position:

$$M + 28000 + 25000x + \frac{41000\langle x - 1.25\rangle^2}{2} = R_B\langle x - 1.25\rangle$$

$$\therefore M = R_B\langle x - 1.25\rangle - 28000 - 25000x - \frac{41000\langle x - 1.25\rangle^2}{2}$$

Substituting this into the deflection of beams equation:

$$EI\frac{d^2y}{dx^2} = M = R_B\langle x - 1.25\rangle - 28000 - 25000x - \frac{41000\langle x - 1.25\rangle^2}{2}$$

Integrating:

$$EI\frac{dy}{dx} = \frac{R_B\langle x - 1.25\rangle^2}{2} - 28000x - \frac{25000x^2}{2} - \frac{41000\langle x - 1.25\rangle^3}{6} + A \tag{2}$$

Integrating again:

$$EIy = \frac{R_B\langle x - 1.25\rangle^3}{6} - \frac{28000x^2}{2} - \frac{25000x^3}{6} - \frac{41000\langle x - 1.25\rangle^4}{24} + Ax + B \tag{3}$$

Applying boundary conditions:

(BC1) At $x = 1.25$, $y = 0$, therefore from (3):

$$0 = 0 - \frac{28000 \times 1.25^2}{2} - \frac{25000 \times 1.25^3}{6} - 0 + A \times 1.25 + B$$

$$\therefore 1.25A + B = 30013.02 \tag{4}$$

(BC2) At $x = 6.25$, $y = 0$, therefore from (3):

$$0 = \frac{139350 \times 5^3}{6} - \frac{28000 \times 6.25^2}{2} - \frac{25000 \times 6.25^3}{6} - \frac{41000 \times 5^4}{24} + A \times 6.25 + B$$

$$\therefore 6.25A + B = -271289.07 \tag{5}$$

Subtracting (4) from (5):

$$5A = -301302.09$$

$$\therefore A = -60260.42$$

Substituting this into (4) gives:

$$\therefore B = 105338.55$$

Therefore at $x = 0$, (2) and (3) give:

$$16.65 \times 10^6 \times \frac{dy}{dx} = 0 - 0 - 0 - 0 - 60260.42$$

$$\boxed{\therefore \frac{dy}{dx} = \boldsymbol{-3.61 \times 10^{-3} rad}}$$

and,

$$16.65 \times 10^6 \times y = 0 - 0 - 0 - 0 + 0 + 105338.55$$

$$\boxed{\therefore y = 0.00633m = \boldsymbol{6.33mm}\ (\boldsymbol{i.e. upwards})}$$