3 Thermal Stress and Strain

Learning Summary

1. Recall that thermal strains arise when a change in temperature is applied to an unconstrained body (knowledge);
2. Recognise the cause of thermal strains and how 'thermal stresses' are caused by thermal strains (comprehension);
3. Solve problems involving both mechanical and thermal loading (application).

3.1 Introduction

Stresses and strains usually arise when mechanical loads are applied to a system. However, they can also exist when no mechanical loading is present. A typical example of this is when a temperature change occurs.

Changes of temperature in a body cause expansion/contraction. This phenomenon is quantified by the coefficient of thermal expansion, α. Some typical values of thermal expansion coefficient for some common engineering materials are presented in Table 1. For isotropic materials, α is the same for all directions.

Table 3.1

Material	Coefficient of Thermal Expansion, α, [°C^{-1}]
Concrete	10×10^{-6}
Steel	11×10^{-6}
Aluminium	23×10^{-6}
Nylon	144×10^{-6}
Rubber	162×10^{-6}

Uniform temperature change throughout an unrestrained body produces uniform strain but no stress, i.e. there is free expansion/contraction.

For a bar of length l, subjected to a temperature change ΔT, the change in length $\delta l_{thernal}$ due to the temperature change is given by:

$$\delta l_{thermal} = l\alpha\Delta T \tag{3.1}$$

The thermal strain due to this length change can be determined as follows:

$$\varepsilon_{thermal} = \frac{\delta l_{thermal}}{l} = \frac{l\alpha\Delta T}{l} = \alpha\Delta T \tag{3.2}$$

Using the principle of superposition, which states that:

$$\begin{bmatrix} The\ total\ effect\ of\ combined \\ loads\ applied\ to\ a\ body \end{bmatrix} = \sum \begin{bmatrix} The\ effects\ of\ the\ individual \\ loads\ applied\ spearately \end{bmatrix} \tag{3.3}$$

thermal extensions can simply be added to elastic (mechanical) extensions to give the total extension by:

$$\delta l_{total} = \delta l_{elastic} + \delta l_{thermal} \tag{3.4}$$

or, for an axial member:

$$\delta l_{total} = \frac{Fl}{AE} + l\alpha\Delta T \tag{3.5}$$

However, if the body is restrained, or the temperature is not uniform, thermal stresses are produced in the body.

3.2 Resistive Heating of a Bar

The bar shown in Figure 2.1 is subjected to a temperature rise of ΔT and restricted from expanding by constraints at each end.

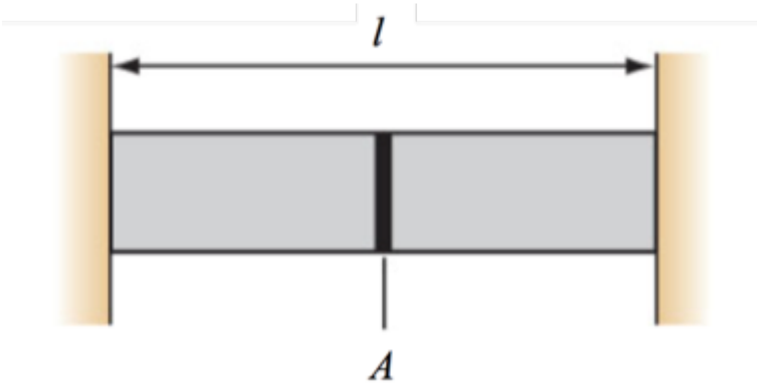

Figure 3.1

Since the bar cannot extend, applying Equation (3.4):

$$\delta l_{total} = \delta l_{elastic} + \delta l_{thermal} = 0 \tag{3.6}$$

or alternatively:

$$\delta l_{total} = \frac{Fl}{AE} + l\alpha\Delta T = 0 \tag{3.7}$$

Cancelling l in Equation (3.7) and rearranging to find the force F gives:

$$F = -AE\alpha\Delta T \tag{3.8}$$

and the stress in the bar, σ is:

$$\sigma = \frac{F}{A} = -E\alpha\Delta T \tag{3.9}$$

3.3 Compound Bar Assembly

A compound bar assembly consisting of one aluminium and one steel bar of the same dimensions between two rigid end plates which are able two slide without friction is shown in Figure 3.2.

If the whole assembly is subjected to a temperature change ΔT will the bars be in tension or compression?

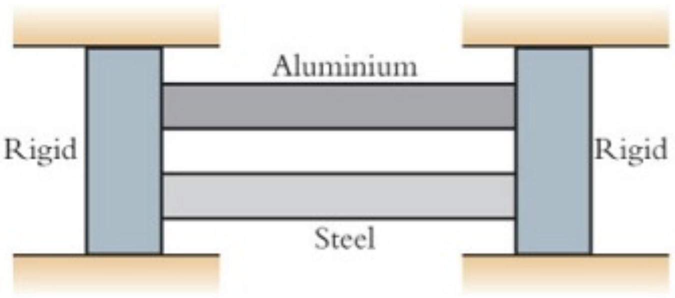

Figure 3.2

If we consider this 'intuitively', because of compatibility the extension of the bars must be identical, i.e.:

$$\delta l_{steel} = \delta l_{alu} \tag{3.10}$$

Referring to Table 3.1, we can see that $\alpha_{alu} > \alpha_{steel}$ therefore the aluminium bar will want to extend more than the steel bar but is constrained from doing so due to the rigid end blocks attached to the steel bar. This means that the **aluminium bar will be in compression**. The reverse is true of the steel bar, it wants to extend less than the aluminium bar but the rigid end blocks attached the aluminium bar forces it to extend further, therefore the **steel bar is in tension**. This is shown schematically in Figure 3.3, where the free expansion is compared to the constrained expansion.

We can consider an analytical solution to the same problem; again Equation (3.10) applies meaning that (from Equation (7)):

$$\frac{F_{steel}l}{A_{steel}E_{steel}} + l\alpha_{steel}\Delta T = \frac{F_{alu}l}{A_{alu}E_{alu}} + l\alpha_{alu}\Delta T \quad (3.11)$$

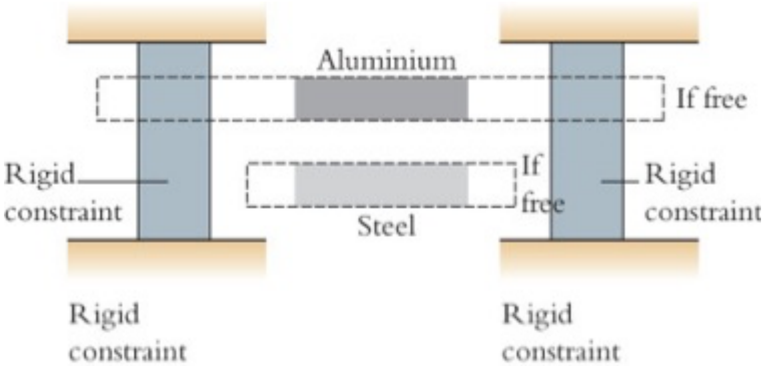

Figure 3.3

The equilibrium condition can be obtained from the FBD in Figure 3.4 as follows:

$$F_{steel} = -F_{alu} \quad (3.12)$$

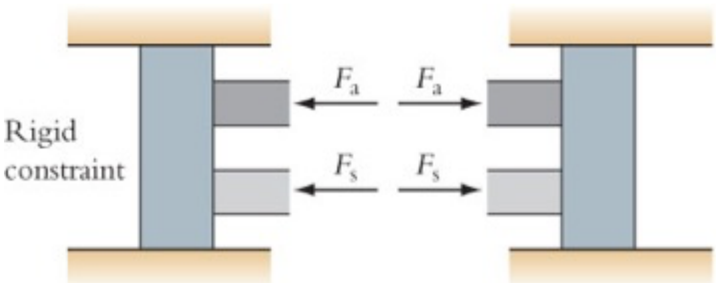

Figure 3.4

Substituting for F_{steel} from Equation (3.12) into Equation (3.11) gives:

$$l\Delta T(\alpha_{steel} - \alpha_{alu}) = F_{alu}l\left[\frac{1}{A_{alu}E_{alu}} + \frac{1}{A_{steel}E_{steel}}\right] \quad (3.13)$$

and therefore:

$$\sigma_{alu} = \frac{F_{alu}}{A_{alu}} = \frac{\Delta T(\alpha_{steel} - \alpha_{alu})}{\left[\frac{1}{E_{alu}} + \frac{A_{alu}}{A_{steel}E_{steel}}\right]} \tag{3.14}$$

As $\alpha_{steel} < \alpha_{alu}$ this means that $\sigma_{alu} < 0$ i.e. the **aluminium bar is in compression**.

If we consider that:

$$A_{alu}\sigma_{alu} = -A_{steel}\sigma_{steel} \tag{3.15}$$

Then:

$$\sigma_{steel} = -\frac{A_{alu}\sigma_{alu}}{A_{steel}} = -\frac{A_{alu}}{A_{steel}}\frac{\Delta T(\alpha_{alu} - \alpha_{steel})}{\left[\frac{1}{E_{steel}} + \frac{A_{steel}}{A_{alu}E_{alu}}\right]} \tag{3.16}$$

As $\alpha_{steel} < \alpha_{alu}$ this means that $\sigma_{steel} > 0$ i.e. the **steel bar is in tension**.

3.4 Generalised Hooke's Law in 3D

To incorporate thermal effects in 3D we add a thermal strain ($\alpha\Delta T$) term to the normal strains in Hooke's Law:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu[\sigma_y + \sigma_z]) + \alpha\Delta T \tag{3.17}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu[\sigma_x + \sigma_z]) + \alpha\Delta T \tag{3.18}$$

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \nu[\sigma_x + \sigma_y]) + \alpha\Delta T \tag{3.19}$$

$$\gamma_{xy} = \tau_{xy} / G \tag{3.20}$$

$$\gamma_{yz} = \tau_{yz} / G \tag{3.21}$$

$$\gamma_{zx} = \tau_{zx} / G \tag{3.22}$$

where ΔT is the temperature at a point relative to some datum. There is no change to the shear stress-strain relationship as for linearly elastic, isotropic materials; a temperature change produces only normal strains.

By introducing these thermal strains into the generalised Hooke's Law we can obtain solutions to thermal stress problems which are often very important in, for example, power and chemical plant, aeroengines and internal combustion engines (e.g. pistons and cylinder walls) etc.

3.5 Case 1: An initially straight uniform beam

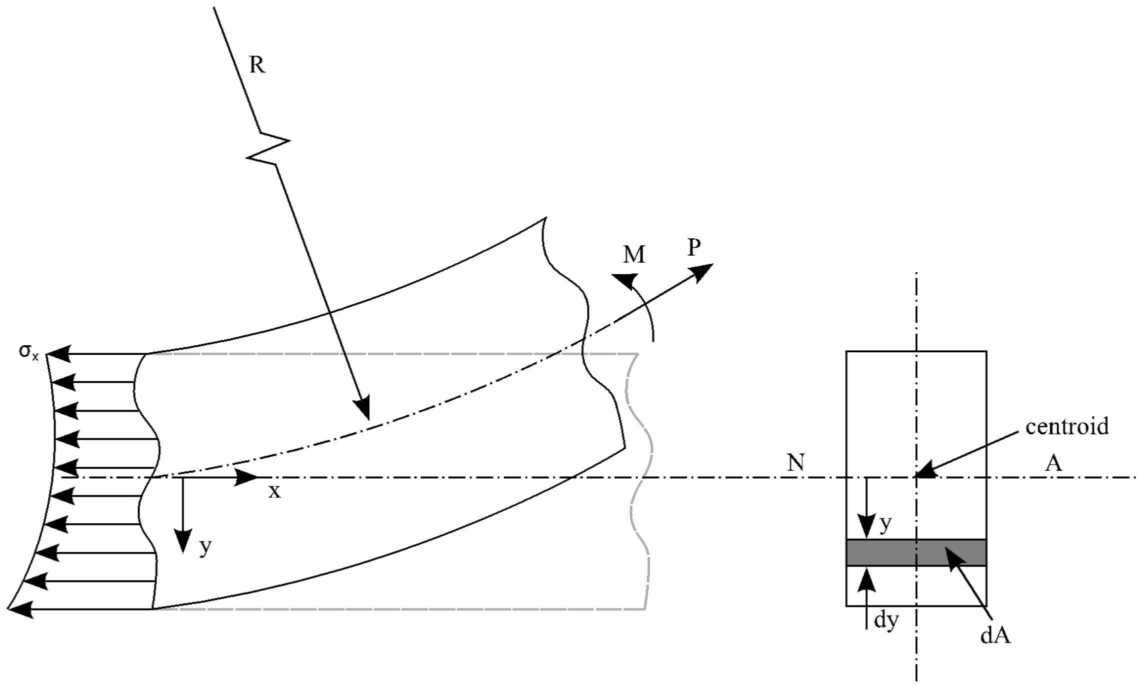

Figure 3.5

- Determine the deformations and stresses (small deformations)

The temperature variation is (assumed) purely a function of y, i.e. $\Delta T = \Delta T(y)$.

The coefficient of thermal expansion = α. Axial force P, and pure bending, about the z-z axis, M, are also applied.

$\sigma_z, \sigma_y, \tau_{xz}$, and τ_{yz}= 0 because the cross-sectional dimensions are small compared with the length.

Also τ_{yz} = 0, because M does not vary with x, $S = \dfrac{dM}{dx} = C$ (a constant value)

3.5.1 Compatibility

Remote from the ends, strain varies linearly with y,

$$\varepsilon_x = \bar{\varepsilon} + \frac{y}{R} \tag{3.23}$$

Where $\bar{\varepsilon}$ is the mean strain (at y = 0) and R is the radius of curvature.

3.5.2 Stress-strain

From the generalised Hooke's Law Equation (3.17) (as σ_y and σ_z are 0)

$$\varepsilon_x = \frac{\sigma_x}{E} + \alpha \Delta T \tag{3.24}$$

Substituting Equation (3.23) into Equation (3.24) and rearranging for σ_x gives:

$$\sigma_x = E\left(\bar{\varepsilon} + \frac{y}{R} - \alpha \Delta T\right) \tag{3.25}$$

3.5.3 Axial Equilibrium

$$P = \int_A \sigma_x dA \tag{3.26}$$

Substituting Equation (3.25) into Equation (3.26) gives:

$$P = E\int_A \left(\bar{\varepsilon} + \frac{y}{R} - \alpha \Delta T\right) dA \tag{3.27}$$

Multiplying out to give individual terms:

$$P = E\bar{\varepsilon}A + \frac{E}{R}\int_A y dA - E\alpha \int_A \Delta T dA \tag{3.28}$$

however, $\int_A \ ydA = 0$ because y is measured from an axis passing through the centroid, so Equation (3.28) reduces to:

$$P = E\bar{\varepsilon}A - E\alpha\int_A \Delta T dA \tag{3.29}$$

d

3.5.4 Moment Equilibrium

$$M = \int_A y\sigma_x dA \tag{3.30}$$

Substituting Equation (3.25) into Equation (3.30) gives

$$M = E\int_A \left(\bar{\varepsilon} + \frac{y}{R} - \alpha\Delta T\right) y dA \tag{3.31}$$

Multiplying out to give individual terms:

$$M = E\bar{\varepsilon}\int_A y dA + \frac{E}{R}\int_A y^2 dA - E\alpha\int_A \Delta T y dA \tag{3.32}$$

By definition $\int_A y^2 dA = I$ and $\int_A y dA = 0$ as before, therefore Equation (3.32) reduces to:

$$M = \frac{EI}{R} - E\alpha\int_A \Delta T y dA \tag{3.33}$$

3.5.5 Example 1

A rectangular beam, width b and depth d has a temperature variation given by:

$$\Delta T(y) = \Delta T_{max}\left(1 - \frac{4y^2}{d^2}\right) \tag{3.34}$$

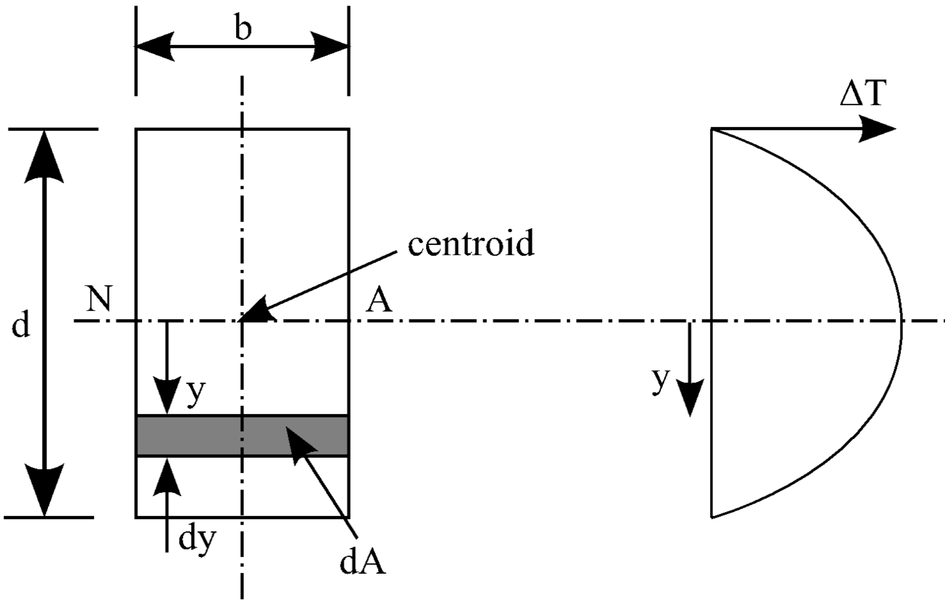

Figure 3.6

There is no restraint or applied loading (i.e. $P = M = 0$). Obtain the stress distribution.

Axial Force Equilibrium

Recalling Equation (3.29) and inserting the temperature variation, axial force and considering a rectangular cross-section from the problem gives:

$$0 = E\bar{\varepsilon}bd - E\alpha\int_{-\frac{d}{2}}^{\frac{d}{2}}\Delta T_{max}\left(1 - \frac{4y^2}{d^2}\right)bdy \tag{3.35}$$

Rearranging for the mean strain, $\bar{\varepsilon}$ gives:

$$\bar{\varepsilon} = \frac{\alpha}{d} \Delta T_{\max} \int_{-\frac{d}{2}}^{\frac{d}{2}} \left(1 - \frac{4y^2}{d^2} \right) dy \tag{3.36}$$

And evaluating the integral:

$$\bar{\varepsilon} = \frac{\alpha}{d} \Delta T_{\max} \left[y - \frac{4y^3}{3d^2} \right]_{-\frac{d}{2}}^{\frac{d}{2}} \tag{3.37}$$

Gives:

$$\bar{\varepsilon} = \frac{2}{3} \alpha \Delta T_{\max} \tag{3.38}$$

Moment Equilibrium

With M = *0* we can obtain 1/R from the moment equilibrium (Equation (3.33)) but from symmetry we can see that (1/R) = 0.

Stress Distribution

Using Equation (3.25) and substituting in the expression for mean strain (Equation (3.38)), 1/R and the temperature variation (Equation (3.34)) gives:

$$\sigma_x = E\left(\frac{2}{3} \alpha \Delta T_{\max} + 0 - \alpha \Delta T_{\max} \left(1 - \frac{4y^2}{d^2} \right) \right) \tag{3.39}$$

Which reduces to:

$$\sigma_x = E \alpha \Delta T_{\max} \left(\frac{4y^2}{d^2} - \frac{1}{3} \right) \tag{3.40}$$

Evaluate Stress Distribution

At y = 0,

$$\sigma_x = E\alpha\Delta T_{\max}\left(\frac{4\times 0^2}{d^2} - \frac{1}{3}\right) \tag{3.41}$$

Which gives:

$$\sigma_x = \frac{-E\alpha\Delta T_{\max}}{3} \tag{3.42}$$

At $y = \pm d/2$,

$$\sigma_x = E\alpha\Delta T_{\max}\left(\frac{4\left(\frac{d}{2}\right)^2}{d^2} - \frac{1}{3}\right) \tag{3.43}$$

Reduces to:

$$\sigma_x = E\alpha\Delta T_{\max}\left(1 - \frac{1}{3}\right) \tag{3.44}$$

And then:

$$\sigma_x = \frac{2E\alpha\Delta T_{\max}}{3} \tag{3.45}$$

We can also evaluate the point at which the stress, $\sigma_x = 0$, i.e. when $\frac{4y^2}{d^2} = \frac{1}{3}$, from Equation (3.40) which gives:

$$y = \sqrt{\frac{1}{12}d^2} \tag{3.46}$$

Or $y = \pm 0.287d$

This is the stress distribution away from the ends. At the ends, σ_x = 0 and there is a gradual transition between the two.

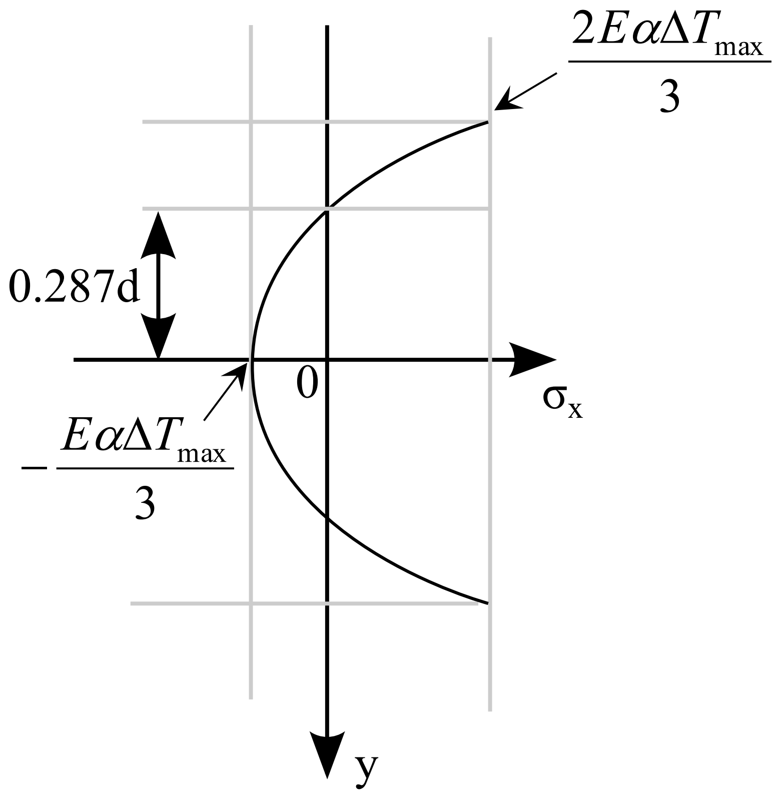

Figure 3.7

3.5.6 Example 2

A rectangular beam (again b x d), but with:

$$\Delta T(y) = \Delta T_{\max} \frac{2y}{d} \tag{3.47}$$

And the constrained so that ε = 0 and 1/R = 0.

Determine the stresses and restraints.

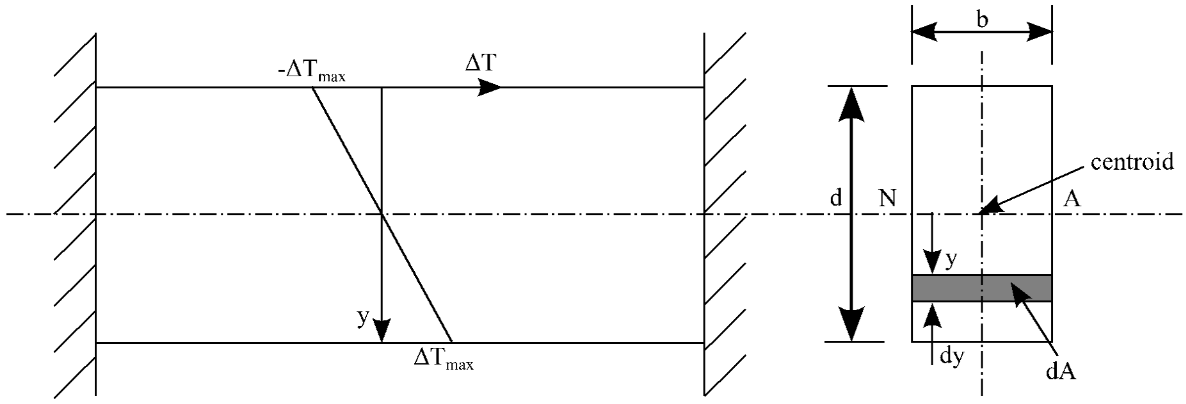

Figure 3.8

Axial Force Equilibrium

Recalling Equation (3.29) and substituting in for the temperature variation:

$$P = E\bar{\varepsilon}bd - E\alpha \int_{-\frac{d}{2}}^{\frac{d}{2}} \Delta T_{\max} \frac{2y}{d} b dy \tag{3.48}$$

Evaluate the integral:

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} \Delta T_{\max} \frac{2y}{d} b dy = \frac{2\Delta T_{\max} b}{d} \int_{-\frac{d}{2}}^{\frac{d}{2}} y dy = \frac{2\Delta T_{\max} b}{d} \left[\frac{y^2}{2} \right]_{-\frac{d}{2}}^{\frac{d}{2}} = 0 \tag{3.49}$$

Also,

$\bar{\varepsilon} = 0$ $\quad \therefore P$ = 0

Moment Equilibrium

Recalling Equation (3.31) and substituting in for the temperature variation and evaluating the integral:

$$\int_A \Delta T y dA = \frac{2\Delta T_{max} b}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}} y^2 dA = \frac{2\Delta T_{max} b}{d}\left[\frac{y^3}{3}\right]_{-\frac{d}{2}}^{\frac{d}{2}}$$
$$= \frac{2\Delta T_{max} b}{d}\left[\left(\frac{d^3}{24}\right) - \left(\frac{-d^3}{24}\right)\right] = \frac{\Delta T_{max} b d^2}{6} \tag{3.50}$$

Also as $1/R$ = 0, this gives

$$M = \frac{-E\alpha\Delta T_{max} b d^2}{6} \tag{3.51}$$

Stress Distribution

Using Equation (3.25) with $\varepsilon = 1/R = 0$

$$\sigma_x = -E\alpha\Delta T \tag{3.52}$$

Substituting in for the temperature variation:

$$\sigma_x = -E\alpha\Delta T_{max}\frac{2y}{d} \tag{3.53}$$

$$M = \frac{-E\alpha\Delta T_{max} b d^2}{6} \tag{3.54}$$

At $y = d/2$,

$$\sigma_x = -E\alpha\Delta T_{max} \tag{3.55}$$

At $y = -d/2$

$$\sigma_x = E\alpha\Delta T_{max} \tag{3.56}$$

Evaluate Stress Distribution

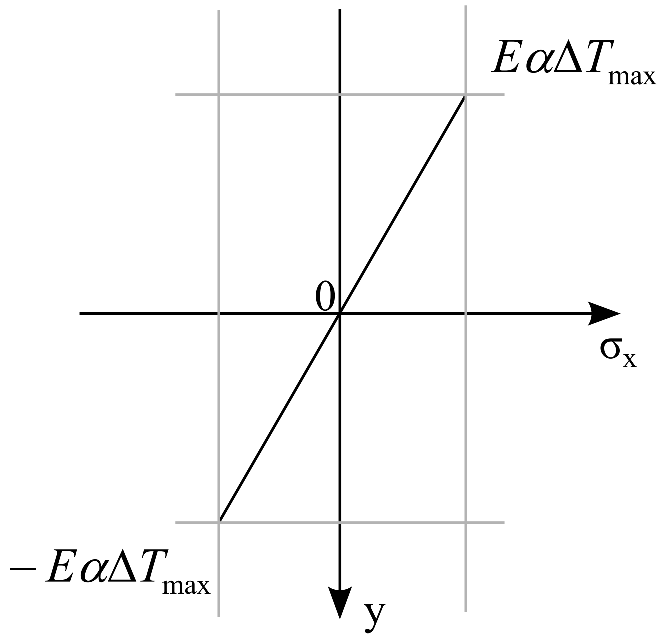

Figure 3.9

3.6 Case 2: Thin cylinders

Thin cylinders are in common use in power and chemical plant, e.g. pipes, pressure vessels, etc. Often temperature variations are approximately linear through the thickness. Considering positions remote from the ends, flanges, etc.

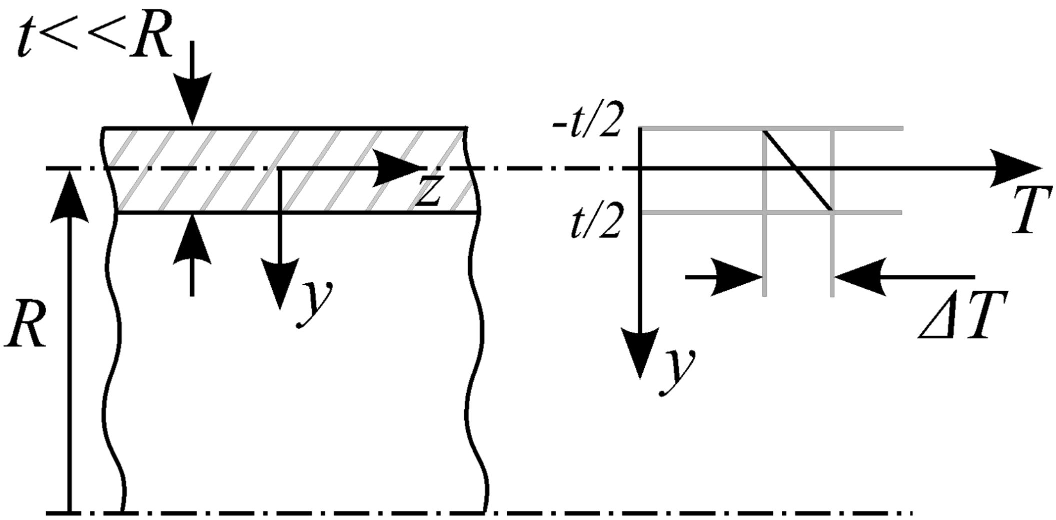

Figure 3.10

It is convenient to consider the effect of the uniform temperature change and the temperature gradient separately. If the cylinder is not restrained then the uniform temperature change causes overall dimensional changes, but no stress. The stresses due to axial restraint are easily calculated.

For the temperature gradient we have:

$$\Delta T = \Delta T_{wall} \frac{y}{t} \tag{3.57}$$

where ΔT_{wall} is the temperature difference across the wall.

For a thin cylinder:

$$\sigma_r \approx 0 \tag{3.58}$$

Now using a cylindrical coordinate system and substituting in for σ_r:

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_z) + \alpha\Delta T \tag{3.59}$$

And

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_\theta) + \alpha\Delta T \tag{3.60}$$

Remote from the ends of the cylinder sections remain plane and circular. Therefore, from compatibility considerations (with zero mean temperature change), the hoop and axial strains must both be zero. Therefore:

$$0 = \frac{1}{E}(\sigma_\theta - \nu\sigma_z) + \alpha\Delta T_{wall}\,\frac{y}{t} \tag{3.61}$$

And

$$0 = \frac{1}{E}(\sigma_z - \nu\sigma_\theta) + \alpha\Delta T_{wall}\,\frac{y}{t} \tag{3.62}$$

Solving, gives

$$\sigma_\theta = \sigma_z = \frac{-E\alpha\Delta T_{wall}}{(1-\nu)}\,\frac{y}{t} \tag{3.63}$$