The University of Nottingham

Approximate Methods

MODULE: MMME2046 DYNAMICS & CONTROL

Often the lowest natural frequency of a structure is the most important. For example, if the 1st critical speed of a shaft is above its operating speed range, whirl is avoided. It is often the case that the 1st mode gives the largest displacement for a given excitation.

We will consider Rayleigh's Method for estimating the lowest natural frequency.

Section B then considers a method of creating single-degree-of-freedom approximations of more complex systems.

A. Rayleigh's Method

Lord Rayleigh observed that for an undamped system vibrating freely at one of its natural frequencies, energy is conserved so that

Maximum Kinetic Energy = Maximum Strain Energy

This is the basis for his method. The strain and kinetic energies can be found for any structure provided we know the deflected shape (i.e., the mode shape). Since we do not normally know the exact mode shape, it is necessary to make an estimate of it. This is an important step since the accuracy depends on making a good guess.

For systems with lumped mass and massless springs, the maximum kinetic for mass *i* is

$$\left(T_{\mathrm{MAX}}\right)_{\mathrm{mass}\ i} = \frac{1}{2}\omega^2 m_i X_i^2$$

where X_i is the amplitude of vibration for mass i based on the assumed mode shape.

The maximum strain energy in spring *j* is

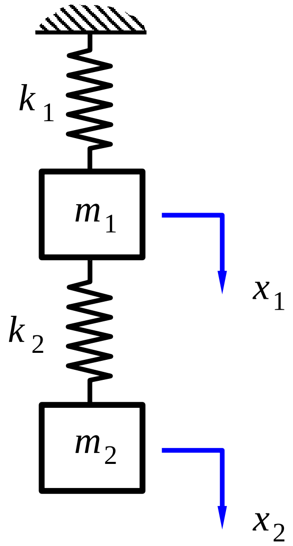
$$\left(U_{\mathrm{MAX}}\right)_{\mathrm{spring}\ j} = \frac{1}{2} k_j \left(\text{maximum change of length}\right)^2$$

The totals for the complete system are given by summing the contributions of all the masses and all the springs. Equating the strain and kinetic energies and re-arranging gives an expression for ω^2. The result can also be calculated from the mass and stiffness matrices for the system. That is

$$\omega^2 = \frac{\{\phi\}^T [K]\{\phi\}}{\{\phi\}^T [M]\{\phi\}}$$

where $\{\phi\}$ is the assumed mode shape vector.

Example 1 **Two-degree-of-freedom System**



The **instantaneous** kinetic energy of mass 1 is $\frac{1}{2} m_1 \dot{x}_1^2$

If $x_1(t) = X_1 \sin \omega t$ then $\dot{x}_1(t) = \omega X_1 \cos \omega t$

That is, the maximum velocity is ωX_1

Therefore, the **maximum** kinetic energy of mass 1 is

$$T_{\max} = \tfrac{1}{2} m_1 (\omega X_1)^2 = \tfrac{1}{2} m_1 \omega^2 X_1^2$$

For both masses,

$$T_{\max} = \tfrac{1}{2} m_1 \omega^2 X_1^2 + \tfrac{1}{2} m_2 \omega^2 X_2^2$$

$$= \tfrac{1}{2} \omega^2 \left(m_1 X_1^2 + m_2 X_2^2 \right)$$

The **instantaneous** strain energy of a spring is

$$\frac{1}{2} \text{Stiffness} \times \text{Change of length}^2$$

For the top spring, the instantaneous strain energy is

$$U = \tfrac{1}{2} k_1 x_1^2$$

The **maximum** strain energy is

$$U_{\max} = \tfrac{1}{2} k_1 X_1^2$$

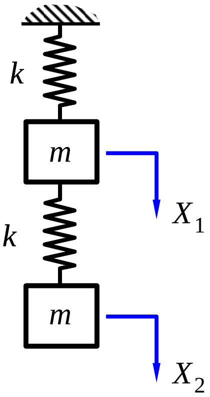
For both springs,

$$U_{\max} = \tfrac{1}{2} k_1 X_1^2 + \tfrac{1}{2} k_2 \left(X_1 - X_2 \right)^2$$

Equating $T_{\max}$ and $U_{\max}$, we get

$$\omega^2 = \frac{k_1 X_1^2 + k_2 \left(X_1 - X_2 \right)^2}{m_1 X_1^2 + m_2 X_2^2}$$

If we can estimate the mode shape, we will have values of X_1 and X_2 that can be substituted into this equation. Experience shows for the shape of the lowest mode of vibration will have all masses moving in phase with each other. This is generally true of other structures too. We therefore need to estimate the relative amplitude of the masses.



We analysed the two-mass problem previously, with m = 2 kg and k = 200 N/m. For the lower natural frequency, we found that the two masses vibrated in phase with each other and that

$$X_2 > X_1$$

A guess at values of of X_1 and X_2 that match this observation could be

$$X_1 = 1$$

$$X_2 = 2$$

This gives a value for ω_n of 1.007 Hz, which is an error of 2.3%; the exact value being 0.984 Hz.

The **static deflection shape** is invariably a good estimate for the lowest mode shape.

Here, noting that the bottom spring supports one mass, but the top spring carries the weight of both masses, the static deflection shape is

$$X_1 =$$

$$X_2 =$$

This gives ω_n = 0.987 Hz, which is an error of only 0.4%.

Try using the exact mode shape. You should find that it gives the exact answer.

Because Rayleigh's method imposes a deflection shape on the system, it effectively constrains it to vibrate in a different way to the true mode shape. As a result, Rayleigh's method will always give an over-estimate of the natural frequency (unless you happen to guess the exact mode shape).

$$\omega_{\text{Rayleigh}} \geq \omega_{\text{Exact}}$$

The technique to adopt is to try several possible mode shapes. The lowest of the predicted frequencies will be the most accurate.

Rayleigh's method for shafts and beams

Rayleigh's method can also be applied to shafts and beams. In this case, the expressions for the maximum kinetic and strain energies are

$$T_{\text{MAX}} = \tfrac{1}{2}\omega^2 \int_0^L \rho A \left[Y(x) \right]^2 \mathrm{d}x$$

$$U_{\text{MAX}} = \tfrac{1}{2}\int_0^L E I \left(\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} \right)^2 \mathrm{d}x$$

where $Y(x)$ is the mode shape function, which defines the amplitude of vibration of the shaft/beam along its length. Non-uniform cross-sections (where A and I are functions of x) can also be analysed, as can systems of interconnected beams and systems that include discrete masses and springs. In each case, we sum the contribution of each element to the overall strain and kinetic energies.

We need to estimate $Y(x)$ in order to evaluate the integrals.

Example 1: **Uniform Cantilever Beam**

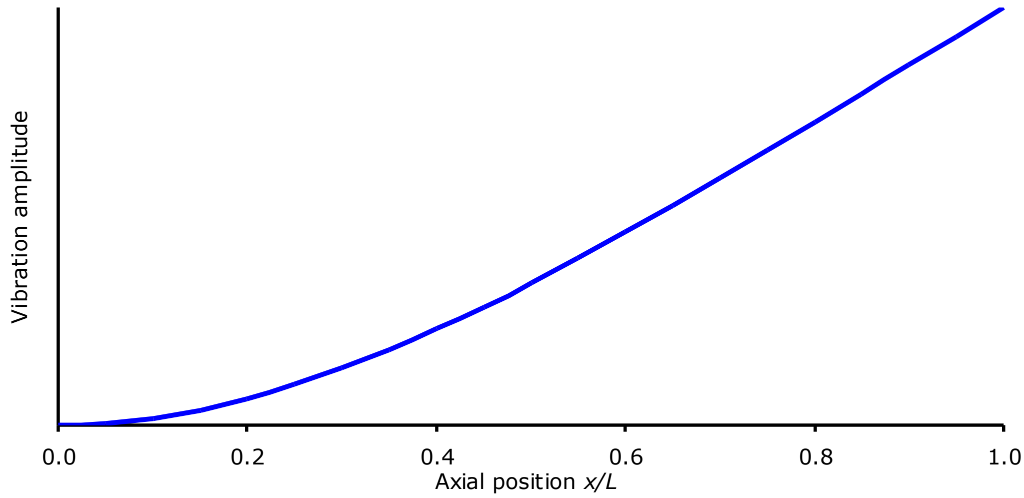
For this, we know that the exact answer is $\omega_n = \frac{3.52}{L^2}\sqrt{\frac{EI}{\rho A}}$

The main criterion for choosing the mode shape is to ensure that it satisfies the displacement and slope conditions at the ends of the beam/shaft.

For a cantilever beam, $Y = \frac{\mathrm{d}Y}{\mathrm{d}x} = 0$ at $x = 0$. The exact mode shape is:

$$Y_1(x) = \sin\lambda_1 x - \sinh\lambda_1 x - \frac{\sin\lambda_1 L + \sinh\lambda_1 L}{\cos\lambda_1 L + \cosh\lambda_1 L}\left(\cos\lambda_1 x - \cosh\lambda_1 x\right)$$

It is impossible to guess this function, particularly when we don't know the natural frequency (which is needed to work out the wavenumber, λ_1).



Mode shape for the lowest mode of a uniform cantilever beam

Choice 1: Quadratic function $Y(x) = C\,x^2$

This satisfies the displacement and slope conditions at the clamped end and gives a shape that is similar to the actual mode shape (above).

The maximum kinetic energy is given by

$$T_{\mathrm{MAX}} = \tfrac{1}{2}\omega^2 \int_0^L \rho A\left[C\,x^2\right]^2 \mathrm{d}x$$

$$= \tfrac{1}{2}\omega^2 \rho A \times \frac{C^2 L^5}{5}$$

$$= \omega^2 \frac{\rho A C^2 L^5}{10}$$

The maximum strain energy

$$U_{\mathrm{MAX}} = \frac{1}{2}\int_0^L E\,I\left(\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2}\right)^2 \mathrm{d}x$$

$$= \frac{1}{2}\int_0^L E\,I\left(2C\right)^2 \mathrm{d}x$$

$$= 2EIC^2L$$

Equating gives $\omega_n^2 = 20\,\dfrac{E\,I}{\rho A L^4}$

Prediction is $\omega_n = \dfrac{4.47}{L^2}\sqrt{\dfrac{E\,I}{\rho A}}$, which is significantly higher (27%) than the exact value.

Choice 2: Static deflection shape

$$Y(x) = C\left(x^4 - 4Lx^3 + 6L^2x^2\right)$$

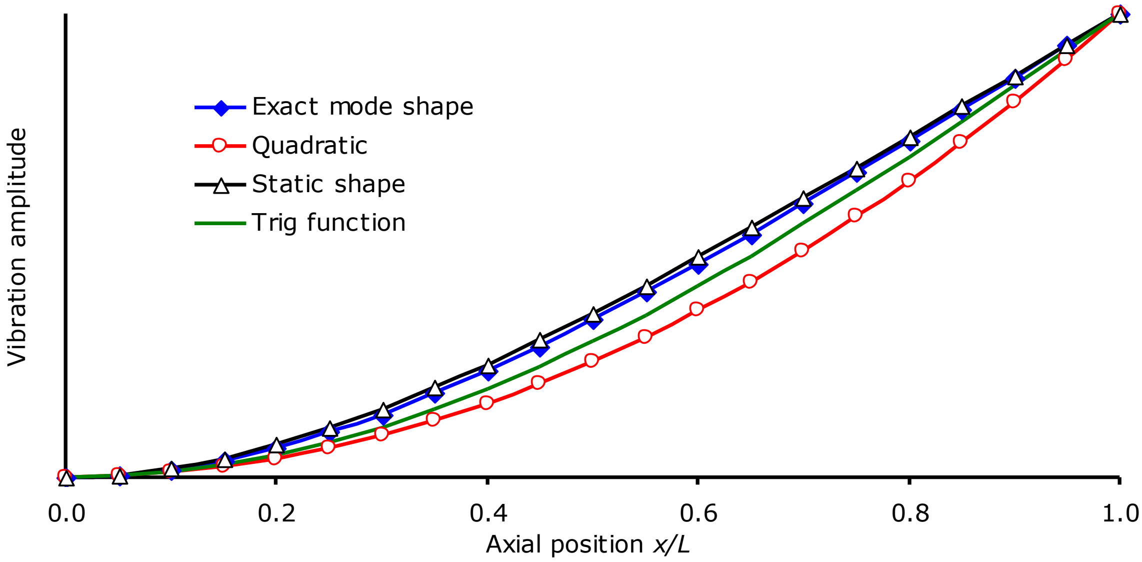
Prediction is $\omega_n = \dfrac{3.54}{L^2}\sqrt{\dfrac{E\,I}{\rho A}}$; an error of less than 1%.

Choice 3: A trigonometric function

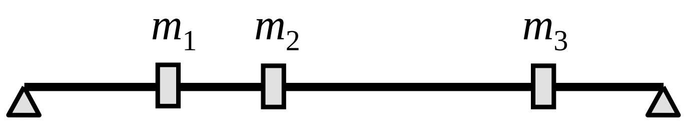
Using $Y(x) = 1 - \cos\left(\frac{\pi x}{2L}\right)$, the prediction is $\omega_n = \frac{3.66}{L^2}\sqrt{\frac{EI}{\rho A}}$, which is 4% high.

The trigonometric function is a good compromise.

- Much better than simple polynomial
- Not as good as the static deflection shape (which is generally the best estimate)
- Can be found for different beams by inspection



Example 2: **Beams and Shafts with Added Masses**



The added masses don't change the strain energy, but they add extra kinetic energy.

For mass at $x = x_r$ the maximum velocity is $\omega \cdot Y(x_r)$

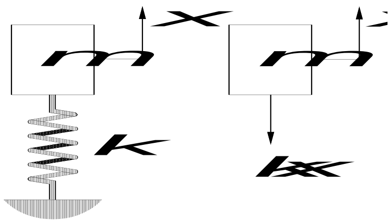
Contribution of mass r to the kinetic energy is $\frac{1}{2} m_r \omega^2 \left[Y(x_r) \right]^2$

These contributions need to be added to the kinetic energy of the shaft itself. Hence, the total kinetic energy becomes:

$$T_{\text{MAX}} = \frac{1}{2}\omega^2 \int_0^L \rho A \left[Y(x) \right]^2 dx + \sum_{\text{ALL MASSES}} \frac{1}{2} m_r \omega^2 \left[Y(x_r) \right]^2$$

Equations of Motion and Energy Methods

Consider the simple single-degree-of-freedom system below.



Using the approach based on Newton's 2nd Law, the equation of motion is

$$-k\,x = m\,\ddot{x}$$

$$\text{or} \quad m\,\ddot{x} + k\,x = 0 \qquad (1)$$

With no damping, energy is conserved so that at any instant, the sum of the kinetic and strain energies is constant. We can therefore write

$$\tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} k\,x^2 = \text{constant}$$

Differentiation with respect to time gives $m\,\dot{x}\,\dfrac{d\dot{x}}{dt} + k\,x\,\dfrac{dx}{dt} = 0$

or $m\,\dot{x}\,\ddot{x} + k\,x\,\dot{x} = 0$

Cancelling the velocity terms gives the original equation of motion [equation (1)].

This demonstrates that **the equation of motion reflects the rates of change of the kinetic and strain energies of the system**. The Italian-born mathematician Joseph Lagrange showed that the equations of motion of a system (including damping) could be derived from energy expressions. This approach is covered in detail in the *Advanced Dynamics of Machines* module.

B. Single-degree-of-freedom Dynamic Models of Complex Systems

The approach is based on the observation that if **the total strain and kinetic energies of two different dynamic systems are identical, an exact analogue relationship will exist between the two systems.** Such systems are called ***Dynamically Equivalent Systems***.

In situations where the real system has many degrees of freedom, but only one mode of vibration is of interest, the concept provides a powerful method for producing an approximate single-degree-of-freedom model to describe that mode of vibration. Not only are single-degree-of-freedom models easy to analyse, but examples in this module have confirmed that they can often give good predictions of the general behaviour of more complex systems. We will consider only undamped systems here.

The approximate model consists of a simple mass-spring system, in which the displacement of the mass represents the displacement of some chosen point on the real structure.

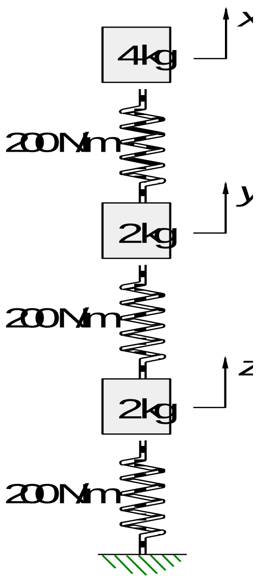
Using the concept of dynamically equivalent systems, the mass and spring stiffness of the approximate model are chosen so that the maximum strain and kinetic energies of real and model systems are the same.

To do this, we assume that the lowest mode of vibration is dominant and therefore defines the deformation pattern in the structure. This is a good assumption if the system is vibrating sinusoidally near its lowest mode of vibration, but can also work well in other cases. An example of the latter was shown in the section on periodic excitation where the response of a 5-degree-of-freedom system was compared with that of an equivalent single-degree-of-freedom model.

As with Rayleigh's Method, the accuracy of the model relies on having a good estimate of the real structure's mode shape.

Example 1: **Lumped mass system**

Objective: Find an approximate single-degree-of-freedom model to analyse the motion of the top mass of a 3-degree-of-freedom system.



The first step is to establish the link between the displacement of the mass in the approximate single-degree-of-freedom model and some chosen point on the real system. In this case, the obvious choice is to link the displacement of the approximate model to the displacement of the top mass, x, since it's the motion of this mass that we want to predict.

However, since we are assuming that the motion of the real system is given by our chosen mode shape, not only does the approximate model predict the behaviour of the coordinate used as the link with the real system, but we can also work out the motion of the other coordinates since they are all linked by the assumed mode shape. In this example, it means that once we've found $x(t)$ we can get $y(t)$ and $z(t)$, since each will be related to $x(t)$ in proportion to the mode shape $\{x : y : z\}$.

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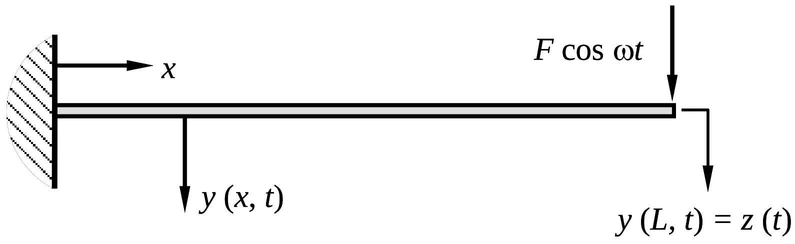
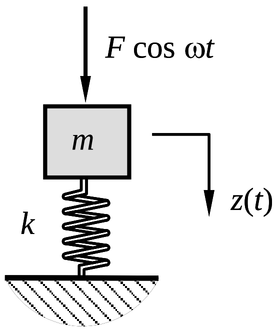
Example 2: **Forced response of a cantilever beam**



Objective: Use a single-degree-of-freedom model to estimate the steady-state response at the free end of a uniform cantilever beam due to a sinusoidal force with a frequency near the lowest natural frequency of the beam.



There are two stages. First, we set up the approximate model, then we use it to do the steady-state response calculation.

In this case, we choose to link the displacement of the mass in the approximate model to the displacement at the free end of the cantilever. In terms of the chosen displacement variables,

$$z(t) = y(L,t)$$

For steady-state, sinusoidal vibration, the link can be written as

$$Z \cos\omega t \;=\; Y(L)\cos\omega t \quad \text{or} \quad Z \;=\; Y(L)$$

To proceed, we need to choose an expression for $Y(x)$, the amplitude of vibration for the cantilever. Two possibilities are considered.

Choice #1: $Y(x) \;=\; C\,x^2$

Linking this with the approximate model, $Z \;=\; Y(L) \;=\; C\,L^2$

Hence, $C = \dfrac{Z}{L^2}$ so that $Y(x) \;=\; \dfrac{Z}{L^2}\,x^2$

This expression for $Y(x)$ is then used to calculate the maximum kinetic and strain energies in the beam. By equating these to the equivalent expressions for the single-degree-of-freedom model, we get the required mass and stiffness values.

For the kinetic energies, $T_{\mathrm{MAX}} \;=\; \dfrac{1}{2}\,\omega^2 \displaystyle\int_0^L \rho A\,[Y(x)]^2\,\mathrm{d}x \;=\dfrac{1}{2}\,m\,\omega^2\,Z^2$

This gives the mass for the approximate model to be $m = 0.2\,\rho AL$. Note that ρAL is the mass of the beam.

For the strain energies, $U_{\mathrm{MAX}} \;=\; \dfrac{1}{2}\displaystyle\int_0^L E\,I\left(\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \;=\; \dfrac{1}{2}\,k\,Z^2$

This gives the spring stiffness for the model to be $k \;=\; 4\,\dfrac{E\,I}{L^3}$

Applying the natural frequency test to the approximate model, we find that $\omega_1 = \frac{4.47}{L^2}\sqrt{\frac{EI}{\rho A}}$, which is the same (poor) result obtained with Rayleigh's Method using this choice for $Y(x)$.

Choice #2: Static deflection shape, $Y(x) = C(x^4 - 4Lx^3 + 6L^2x^2)$

Linking this expression at $x = L$ with the coordinate for the approximate model, we get

$$Z = Y(L) = C(L^4 - 4L^4 + 6L^4) = 3CL^4$$

Hence, $C = \frac{Z}{3L^4}$ and $Y(x) = \frac{Z}{3L^4}(x^4 - 4Lx^3 + 6L^2x^2)$.

Using this expression to calculate the mass and stiffness values for the model gives the following.

$$m = 0.257\,\rho AL \quad \text{and} \quad k = 3.20\frac{EI}{L^3}$$

The natural frequency test in this case gives $\omega_1 = \frac{3.53}{L^2}\sqrt{\frac{EI}{\rho A}}$, which is lower than the result from the first choice, confirming that the static deflection shape is the better approximation.

Forced response analysis

Having established the single-degree-of-freedom model, we can now use it to estimate the steady-state response at the free end of the beam due to a sinusoidal force. The equation of motion for the approximate model is:

$$m\ddot{z} + kz = F(t)$$

Making the standard substitutions, $F(t) = F\,\mathrm{e}^{\mathrm{i}\omega t}$ and $z(t) = Z^*\mathrm{e}^{\mathrm{i}\omega t}$, we get

$$Z^* = \frac{F}{(k - m\omega^2)}$$

Note that since there is no damping in this model, the expression for Z^* is real.

This meets the objective of finding the steady-state amplitude of the deflection at the free end of the cantilever. However, since we have assumed that the deflected shape of the beam can be defined by the mode shape (that is, the function $Y(x)$), the expression for Z^* can also tell us the vibration amplitude at **any** point along its length. In the case of Choice #2, this gives:

$$Y(x) = \frac{Z^*}{3L^4}(x^4 - 4Lx^3 + 6L^2x^2)$$

$$= \frac{F}{3L^4(k - m\omega^2)}(x^4 - 4Lx^3 + 6L^2x^2)$$