Mechanics of Solids – Strain Energy Methods Notes

Learning Summary

1. Know the basic concept of strain energy stored in a material body under loading (knowledge);
2. Be able to calculate strain energy in an elastic body/structure arising from various types of loading, including tension/compression, bending and torsion (application);
3. Be able to apply Castigliano's theorem to linear elastic bodies in order to enable the deflection, change of slope and/or rotation of a body, or structure, to be calculated from strain energy expressions (application).

1. Introduction

We have seen in the Deflection of Beams topic how deflections and slopes of a beam can be determined by solving the differential equation of the elastic line, i.e., Macaulay's method. However, this method is not appropriate for more complex shaped structures or bodies where deflections occur. Here we introduce the concept of **strain energy**, which will enable us to calculate deflections of complex structures.

When a material is subject to loading, strain energy is stored within it. An Italian railway engineer, named Castigliano, derived a theorem and procedure for using this strain energy to determine deflections in structures or bodies. Castigliano's theorem is a powerful and flexible method for solving deflection problems.

2. Strain Energy Definition

The strain energy in a material body is equal to the work done on the body by the applied loads. Thus, if an elastic-plastic material body is subjected to a single axial load, P, as shown in Figure 1(a), causing a displacement, u, at the load application point, according to the behaviour shown in Figure 1(b), then the strain energy, U, is given as:

$$U = \int_0^u P \mathrm{d}u \tag{1}$$

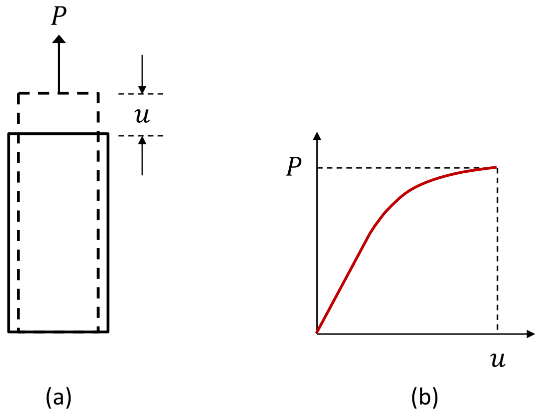


Figure 1

Similar expressions can be derived for bending of a beam and torsion of a bar, according to Figures 2 and 3, respectively, as:

$$U = \int_0^{\phi} M \mathrm{d}\phi \tag{2}$$

and

$$U = \int_0^{\theta} T \mathrm{d}\theta \tag{3}$$

where ϕ is the resulting change in slope in a beam due to applied bending moment, M, and θ is the resulting twist of a bar due to applied torque, T.

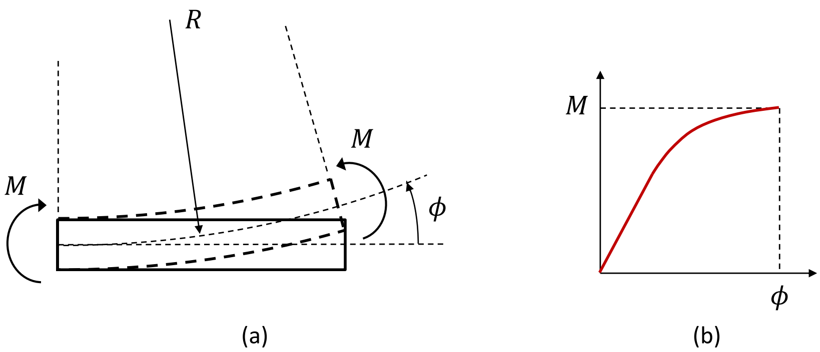


Figure 2

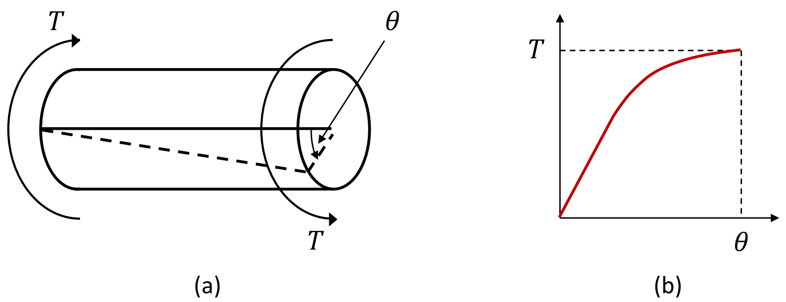


Figure 3

3. Application to Elastic Material Behaviour

Axially Loaded Beam

If the material body shown in Figure 1(a) behaves linear elastically, as shown in Figure 4, it can be seen that the work done, or strain energy, given by equation (1) can be simplified to:

$$U = \frac{1}{2}Pu$$

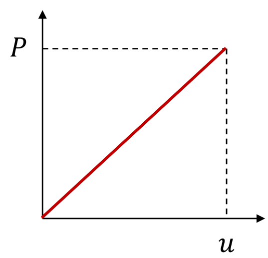


Figure 4

If this material body in Figure 1(a) represents an element, of length δs, of a larger beam, of length L, and the change in length of this element due to the applied load, P, is δu, then the strain energy within this element is:

$$\delta U = \frac{1}{2}P\delta u \tag{4}$$

It is important to note that there are transverse strains/displacements due to Poisson's effects but there are no transverse stresses/loads. Thus, there is no work done in the transverse direction.

Axial strain in the element is:

$$\varepsilon = \frac{\delta u}{\delta s} \tag{5}$$

and as the material behaves linear elastically, Hooke's law applies. Therefore:

$$\varepsilon = \frac{\sigma}{E} \tag{6}$$

Substituting equation (6) into equation (5) gives:

$$\frac{\delta u}{\delta s} = \frac{\sigma}{E} = \frac{P}{EA}$$

$$\therefore \delta u = \frac{P}{EA}\delta s \tag{7}$$

Substituting equation (7) into equation (4):

$$\delta U = \frac{P^2}{2EA}\delta s$$

As this is the expression for strain energy in the element of beam δs, integrating this expression over the full length of the beam, in order to give the total strain energy in the beam:

$$\boldsymbol{U = \int_0^L \frac{P^2}{2EA}\delta s} \tag{8}$$

Bending of a Beam

If the material body shown in Figure 2(a) behaves linear elastically, as shown in Figure 5, it can be seen that the work done, or strain energy, given by equation (2) can be simplified to:

$$U = \frac{1}{2}M\phi$$

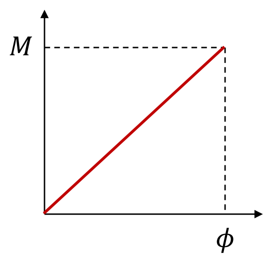


Figure 5

If this material body in Figure 2(a) represents an element, of length δs, of a larger beam, of length L, which bends to curvature R, giving a change in slope of this element due to the applied bending moment M, of $d\phi$, then strain energy within this element is:

$$\delta U = \frac{1}{2} M \delta \phi \tag{9}$$

From the elastic beam bending equation:

$$\frac{M}{I} = \frac{E}{R} \tag{10}$$

and as the angle subtended by the element is equal to the change in slope, the expression for the arc created by the element is:

$$\delta s = R \delta \phi \tag{11}$$

Rearranging equation (11) and substituting into equation (10) gives:

$$\frac{M}{I} = \frac{E}{\frac{\delta s}{\delta \phi}}$$

$$\therefore \delta \phi = \frac{M}{EI} \delta s \tag{12}$$

Substituting equation (12) into equation (9):

$$\delta U = \frac{M^2}{2EI} \delta s$$

As this is the expression for strain energy in the element of beam δs, integrating this expression over the full length of the beam, in order to give the total strain energy in the beam:

$$\boldsymbol{U = \int_0^L \frac{M^2}{2EI} \delta s} \tag{13}$$

Torsion of a Shaft

If the material body shown in Figure 3(a) behaves linear elastically, as shown in Figure 6, it can be seen that the work done, or strain energy, given by equation (3) can be simplified to:

$$U = \frac{1}{2} T \theta$$

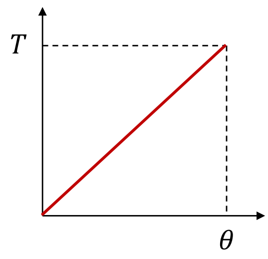


Figure 6

If this material body in Figure 3(a) represents an element, of length δs, of a larger beam, of length L, and the twist of this element due to the applied torque, T, is $\delta\theta$, then the strain energy within this element is:

$$\delta U = \frac{1}{2} T \delta\theta \tag{14}$$

From the elastic torsional equation:

$$\frac{T}{J} = \frac{G\theta}{L}$$

therefore, for the element, δs, this can be rewritten as:

$$\frac{T}{J} = \frac{G\delta\theta}{\delta s}$$

$$\therefore \delta\theta = \frac{T}{GJ}\delta s \tag{15}$$

Substituting equation (15) into equation (14) gives:

$$\delta U = \frac{T^2}{2GJ}\delta s$$

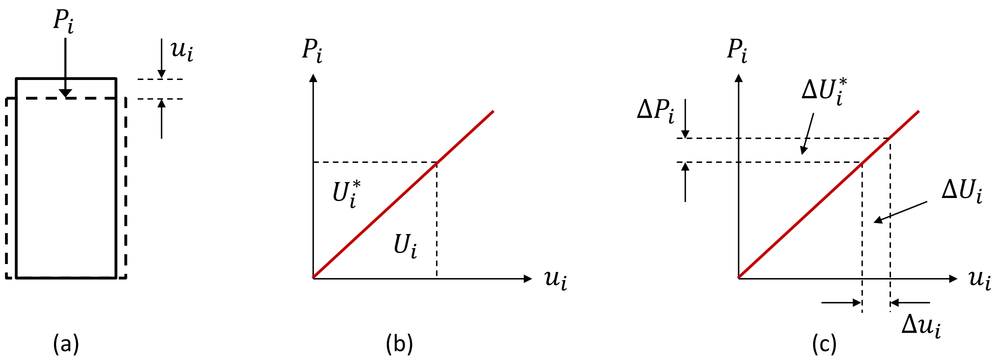
As this is the expression for strain energy in the element of beam δs, integrating this expression over the full length of the beam, in order to give the total strain energy in the beam:

$$\boldsymbol{U = \int_0^L \frac{T^2}{2GJ}\delta s} \tag{16}$$

Equations (8), (13) and (16) therefore summarise the strain energy expressions for elastic bodies for axial, bending and torsional loading types, respectively. In practical engineering structures, where members are relatively long and slender, strain energy due to axial loading can usually be neglected with bending usually being dominant. Strain energy due to shear deflections can also exist but, again, can normally be neglected.

4. Castigliano's Theorem

Consider a linear elastic body loaded by a force, P_i, as shown in Figure 7(a), noting that it may be the case that $i \neq 1$, i.e. there may be more than one force acting on the body. The corresponding displacement at the location, and in the direction of P_i, is u_i.



(a) (b) (c)

Figure 7

As discussed in section 2, the general expression for strain energy, shown as the area under the load-displacement plot shown in Figure 7(b), is:

$$U_i = \int_0^{u_i} P_i \mathrm{d}u_i$$

We can also define the complimentary strain energy, U_i^*, as:

$$U_i^* = \int_0^{P_i} u_i \mathrm{d}P_i$$

where U_i^* is the area above the load-displacement plot.

Now, consider a small increment of the load ΔP_i, while any other loads (assuming $i \neq 1$), remain constant. This causes an increment of the complementary strain energy, shown in Figure 7(c) as:

$$\Delta U_i^* = \int_{P_i}^{P_i+\Delta P_i} u_i \mathrm{d}P_i \approx u_i \Delta P_i$$

$$\therefore u_i = \frac{\Delta U_i^*}{\Delta P_i}$$

In the limit, as $\Delta P_i \to 0$, the above expression can be rewritten as:

$$\therefore u_i = \frac{\partial U_i^*}{\partial P_i} \tag{17}$$

Note that this is a partial derivative as, if $i \neq 1$, then all loads except P_i were kept constant in the derivation of equation (17), I.e., u_i is the deflection at the point of application of, and in the direction of, the load, P_i, and P_i is independent of other loads.

Equation (17) is the general form of **Castigliano's Theorem**, which, as stated in the introduction, is named after the Italian railway engineer, Carlo Alberto Castigliano (1847-1884), who developed the method. His theorem states that the deflection, u_i, at the location of a given load point, P_i, may be obtained by differentiating the complementary strain energy, U_i^*, with respect to the load, P_i, acting at that point.

Equation (17) applies to any elastic body (linear or non-linear). For linear elastic bodies specifically, however, the strain energy is equal to the complementary strain energy as shown in Figure 7(b). Thus,

$$U_i = U_i^*$$

And therefore,

$$u_i = \frac{\partial U_i}{\partial P_i} \tag{18}$$

Equation (18) is the well-known form of Castigliano's Theorem and states that the partial derivative of the strain energy, U_i, of a linear elastic system with respect to a specific independent force, P_i, is equal to the displacement of the structure at the point of application, and in the direction of, the P_i.

Castigliano's theorem may be also be used for the calculation of the change in slope of a beam, ϕ_i, at the location of a specific and independent bending moment, M_i, and to the calculation of the rotation of a bar, θ_i, subjected to a specific and independent torque, T_i, as follows,

$$\phi_i = \frac{\partial U_i}{\partial M_i}$$

and

$$\theta_i = \frac{\partial U_i}{\partial T_i}$$

5. Worked Example – Combined Strain Energy Example

The bent uniform bar, shown in Figure 8, has a circular cross-section of 40 mm diameter and is subjected to a vertical load, P, of 16 kN at one end and is clamped at the other.

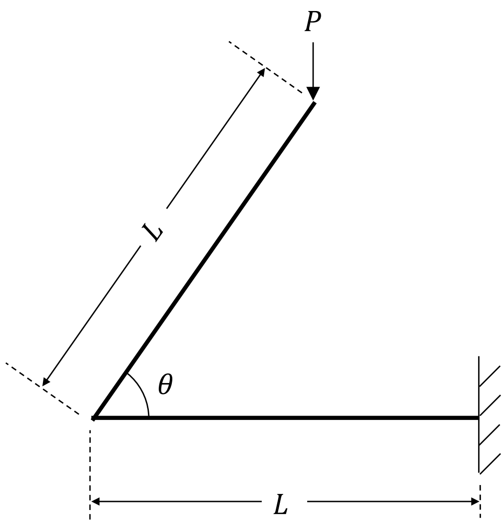
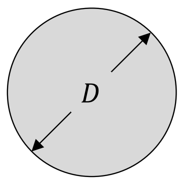


Figure 8

Use strain energy to determine the vertical deflection at the position of the applied load.

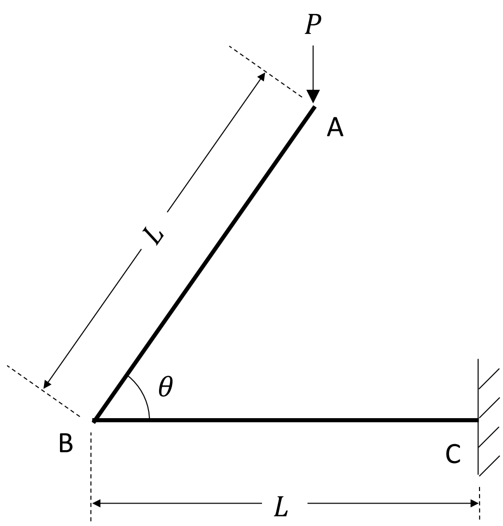
Assume $E = 225\ \text{GPa}$, $L = 0.75\ \text{m}$ and $\theta = 55°$.

Second Moment of Area, I, calculation:



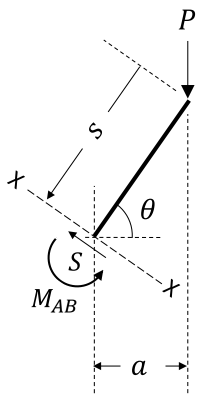
$$\therefore I = \frac{\pi D^4}{64} = \frac{\pi \times 40^4}{64} = 125{,}663.71\ \text{mm}^4$$

Labelling the structure:



Section AB (*bending only*)

Free Body Diagram:



Taking moments about X-X:

$$M_{AB} = Pa = Ps\cos\theta$$

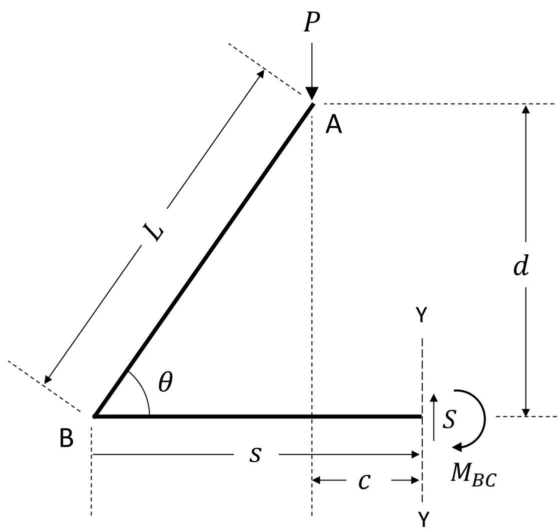
Substituting this into the equation for strain energy in a beam under bending gives,

$$U_{AB} = \int \frac{{M_{AB}}^2}{2EI} \mathrm{d}s = \int_0^L \frac{(Ps\cos\theta)^2}{2EI} \mathrm{d}s = \frac{(P\cos\theta)^2}{2EI} \int_0^L s^2 \mathrm{d}s = \frac{(P\cos\theta)^2}{2EI} \left[\frac{s^3}{3}\right]_0^L$$

$$\therefore U_{AB} = \frac{P^2 L^3}{6EI} \cos^2\theta$$

Section BC (*bending only*)

Free Body Diagram:



Taking moments about Y-Y:

$$M_{BC} = Pc = P(s - L\cos\theta)$$

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{BC} = \int \frac{{M_{BC}}^2}{2EI} \mathrm{d}s = \int_0^L \frac{\left(P(s - L\cos\theta)\right)^2}{2EI} \mathrm{d}s = \frac{P^2}{2EI} \int_0^L (s^2 - 2Ls\cos\theta + L^2\cos^2\theta) \mathrm{d}s$$

$$= \frac{P^2}{2EI} \left[\frac{s^3}{3} - Ls^2\cos\theta + L^2 s\cos^2\theta \right]_0^L$$

$$\therefore U_{BC} = \frac{P^2 L^3}{2EI} \left(\frac{1}{3} - \cos\theta + \cos^2\theta \right)$$

Total Strain Energy:

$$U = U_{AB} + U_{BC} = \frac{P^2 L^3}{6EI} \cos^2\theta + \frac{P^2 L^3}{2EI} \left(\frac{1}{3} - \cos\theta + \cos^2\theta \right)$$

$$\therefore U = \frac{P^2 L^3}{2EI} \left(\frac{4\cos^2\theta}{3} - \cos\theta + \frac{1}{3} \right)$$

Differentiating this with respect to the applied load, P, in order to calculate vertical deflection at position A, u_{v_A}:

$$u_{v_A} = \frac{\partial U}{\partial P} = \frac{PL^3}{EI} \left(\frac{4\cos^2\theta}{3} - \cos\theta + \frac{1}{3} \right)$$

Substituting values for P, L, E, I and θ into this gives:

$$\boldsymbol{u_{v_A} = 47.36\ \mathrm{mm}}$$

6. Dummy Loads

In order to determine the deflection, change of slope or rotation at a point in a structure where a load, bending moment or torque is not applied, a dummy load, bending moment or torque is added at the point of and in the direction that the deflection, change of slope or rotation is required. The expression for strain energy is then obtained as shown in the previous sections, but incorporating the dummy load, bending moment or torque, and differentiated with respect to this dummy load, bending moment or torque in order to obtain the required deflection, change of slope or rotation. The dummy load, bending moment or torque is then set to zero when numerically evaluating the deflection, change of slope or rotation (i.e., after the differentiation).

For example, to determine the horizontal deflection at the tip of the structure shown in Figure 8, it is necessary to add a horizontal dummy load, Q, at this position, as shown in Figure 9.

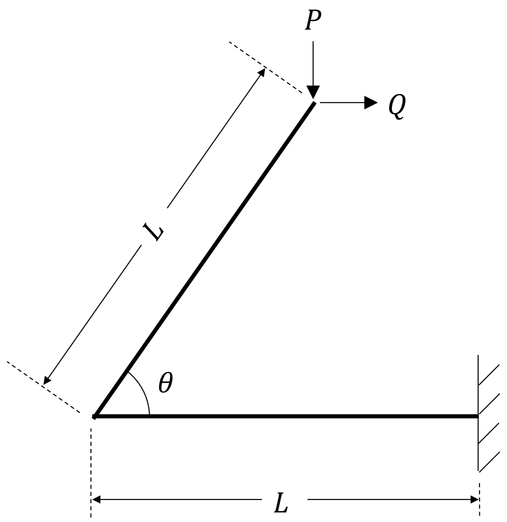


Figure 9

For this structure the total strain energy expression can be calculated as:

$$\therefore U = \frac{L^3}{6EI}(P^2\cos^2\theta + Q^2\sin^2\theta + PQ\cos\theta\sin\theta)$$
$$+\frac{L^3}{2EI}\left(\frac{P^2}{3} - P^2\cos\theta + P^2\cos^2\theta - PQ\sin\theta + Q^2\sin^2\theta + 2PQ\cos\theta\sin\theta\right) \quad (19)$$

The vertical deflection of the tip of the beam can be calculated by differentiating equation (19) with respect to the vertical applied load, P, as:

$$u_{v_A} = \frac{\partial U}{\partial P} = \frac{L^3}{6EI}(2P\cos^2\theta + Q\cos\theta\sin\theta) + \frac{L^3}{2EI}\left(\frac{2P}{3} - 2P\cos\theta + 2P\cos^2\theta - Q\sin\theta + 2Q\cos\theta\sin\theta\right)$$

Setting dummy load to zero and substituting values for P, L, E, I and θ into this gives:

$$\boldsymbol{u_{v_A} = 47.38\ \mathrm{mm}}$$

This is as calculated in section 5; hence it can be seen that the addition of the horizontal dummy load has not affected the calculated vertical deflection.

Differentiating equation (19) with respect to the dummy load, Q, in order to calculate the horizontal deflection at the tip of the beam, gives:

$$u_{h_A} = \frac{\partial U}{\partial Q} = \frac{L^3}{6EI}(2Q\sin^2\theta + P\cos\theta\sin\theta) + \frac{L^3}{2EI}(-P\sin\theta + 2Q\sin^2\theta + 2P\cos\theta\sin\theta)$$

Setting dummy load to zero and substituting values for P, L, E, I and θ into this gives:

$$\therefore \boldsymbol{u_{h_A} = 33.09\ \mathrm{mm}}$$

The tip of the beam therefore deflects downwards by $47.38\ \mathrm{mm}$ and to the right by $33.09\ \mathrm{mm}$, due to the applied load, P.