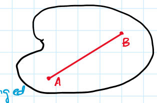
03.10.22 - Machine Dynamics

Monday, 3. October 2022 09:04

Analysis and design of rigid mechanisms and structures in motion

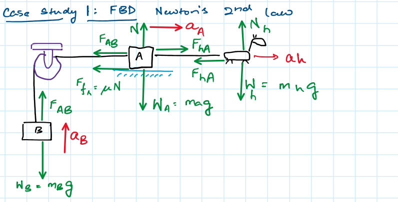
Rigid Body definition:

- System of particles
- Distances between particles remain unchanged



Particle → Rigid body → System of rigid Bodies

Case study 1: FBD Newton's 2nd law



Mass B

$\sum \vec{F_y} = m_B a_B$ $\quad F_{AB} - W_B = m_B a_B \Rightarrow F_B = m_B a_B + m_B g$ → eq ①

Mass A

$\sum \vec{F_x} = m_A a_A$ $\quad F_{hA} - F_{AB} - \mu N = m_A a_A$ → eq ②

$\sum \vec{F_y} = m_A a_{A_y}$ $\quad N - m_A g = 0 \Rightarrow N = m_A g$ → eq ③

combining eq ② & ③:

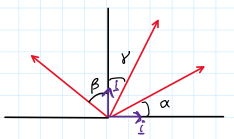
$F_{hA} = m_B a_B + m_B g + \mu m_A g + m_A a_A$ → eq ④

Mass - horse

However we assumed that $a_B = a_A = a_h = a$ → eq ⑤

$\therefore F_{Ah} = (m_B + m_A) a + m_B g + \mu m_A g$

Case study 2 - Vector resolution



$\underline{c} = \underline{a} + \underline{b}$

$a = 5 \quad \alpha = 20°$

$b = ? \quad \beta = 15°$

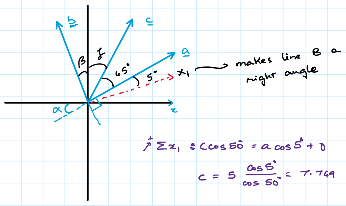
$c = ? \quad \gamma = 25°$

$\sum \vec{x}: \; c_x = a_x + b_x \rightarrow c \sin\gamma = a\cos\alpha - b\sin\beta$

$\sum \vec{y}: \; c_y = a_y + b_y \rightarrow c\cos\gamma = a\sin\alpha + b\cos\beta$

$$\begin{bmatrix} 0.2588 & 0.4226 \\ -0.9659 & 0.9063 \end{bmatrix} \begin{Bmatrix} b \\ c \end{Bmatrix} = \begin{bmatrix} 4.698 \\ 1.710 \end{bmatrix}$$

Method 2:



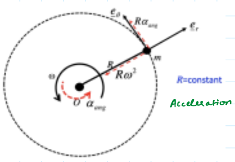
$\nearrow \sum x_1: \; c\cos 5 \bar{0} = a\cos 5° + 0$

$c = 5 \dfrac{\cos 5°}{\cos 50°} = 7.749$

$\nwarrow \sum y: \; 0 = -a\cos 45° + b\cos 50°$

$b = 5 \dfrac{\cos 45°}{\cos 50°} = 5.500$

Case study 3: Circular motion



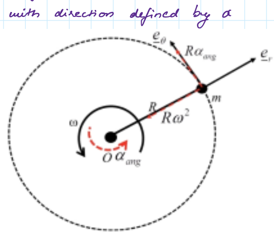
Position $\underline{r} = R\underline{e_r}$

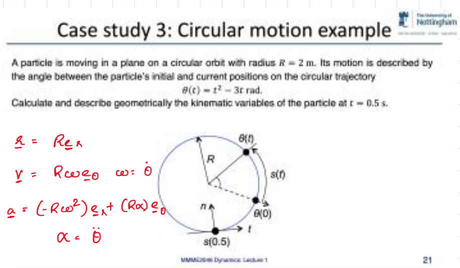
Velocity $\underline{v} = R\omega \underline{e_\theta}$

Acceleration $\underline{a} = (-R\omega^2)\underline{e_r} + (R\alpha_{ang})\underline{e_\theta}$

m has radial acceleration $R\omega^2$ towards O along e_r and tangential acceleration along e_θ

- when a point conducts circular motion:
  - The velocity is tangential to the circle with direction defined by ω.
  - There are two acceleration components (tangential + normal)
    - Normal component always has direction towards the centre of rotation.
    - Tangential component is tangential to the circle with direction defined by α





The particle position can be given by the arc length, a scalar function, in the following form

$$s(t) = R\theta(t) = 2(t^2 - 3t)\,m$$

The velocity magnitude and the tangent acceleration are given by:

$$v(t) = \dot{s}(t) = R\dot{\theta}(t) = 2(2t-3) = 4t - 6\,m/s$$

$$a_t(t) = \ddot{s}(t) = R\ddot{\theta}(t) = 4\,m/s^2$$

The normal (centripetal) acceleration is:

$$a_n(t) = R\omega^2 = R(2t-3)^2 = 8t^2 - 24t + 18\,ms^{-2}$$

Vectors in polar coordinates

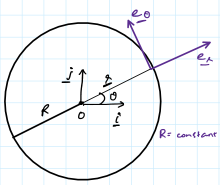
Position:

In cartesian coordinates:

$\underline{r} = x\underline{i} + y\underline{j}$

$x = R\cos\theta$

$y = R\sin\theta$



In polar coordinates:

$\underline{r} = R\underline{e_r}$

$\underline{e_r} = \cos\theta\, i + \sin\theta\, j$

$\underline{e_\theta} = -\sin\theta\, i + \cos\theta\, j$

$\dfrac{d\underline{e_r}}{d\theta} = (-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}) = \underline{e_\theta}$

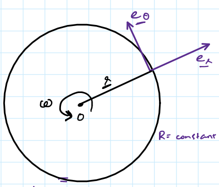
$\dfrac{d\underline{e_\theta}}{d\theta} = (-\cos\theta\, i - \sin\theta\, j) = -\underline{e_r}$

Circular motion - Velocity

Position: $\underline{r} = R\underline{e_r}$

Now $\theta = \theta(t)$

Angular velocity: $\dot{\theta} = \omega(t)$



Velocity

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d(R\underline{e_r})}{dt} = \frac{dR}{dt}\underline{e_r} + R\frac{d\underline{e_r}}{dt}$$

$$= \frac{dR}{dt}\underline{e_r} + R\frac{d\theta}{dt}\frac{d\underline{e_r}}{d\theta} = R\dot{\theta}\underline{e_\theta} = R\omega\underline{e_\theta}$$

(with $\frac{dR}{dt} = 0$)

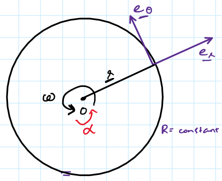
Acceleration

Position: $\underline{r} = R\underline{e_r}$

$\theta = \theta(t)$

$\dot{\theta} = \omega(t)$

$\ddot{\theta} = \dot{\omega} = \alpha(t)$



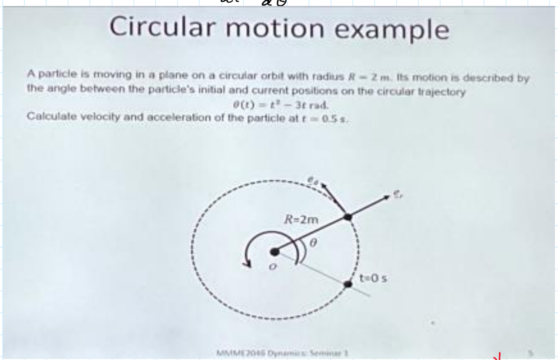
Velocity

$\underline{v} = R\omega\underline{e_\theta}$

Acceleration:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d(R\omega\underline{e_\theta})}{dt} = R\frac{d(\omega e_\theta)}{dt} = R\dot{\omega}e_\theta + R\omega\frac{de_\theta}{dt}$$

$$= R\dot{\omega}e_\theta + R\omega\frac{d\theta}{dt}\frac{de_\theta}{d\theta} = R\alpha e_\theta - R\omega^2 e_r$$



Velocity magnitude:

$v(t) = R\omega = R\dot{\theta}(t) = 2(2t-3)$

$= \underline{4t - 6\, ms^{-1}}$

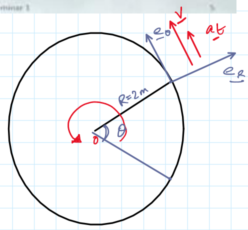
Tangential acceleration is:

$a_t(t) = R\alpha = R\ddot{\theta}(t) = \underline{4ms^{-2}}$

Normal centripetal a is:

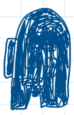
$a_n(t) = R\omega^2 = R(2t-3)^2$

$= \underline{8t^2 - 24t + 18ms^{-2}}$



$\dot{\theta} = 2t - 3 = \omega$

$\ddot{\theta} = \underline{2\,rad/s^2}$



← Zack is suspiciously hot and french

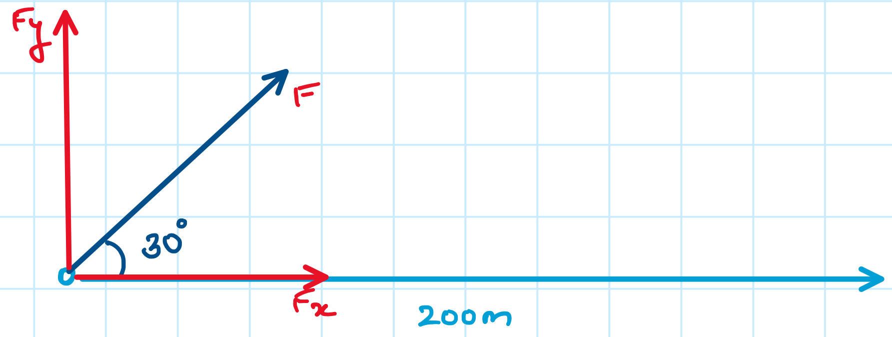
pls cx / sus

Questions - Problem sheet 1

Tuesday, 4. October 2022 22:41

1. A projectile is fired at a target 200 m away horizontally at an angle of 30° to the horizontal. Calculate the initial velocity required to hit the target and the time taken to reach the target.

Answer: [47.57 m/s, 4.855 s]



⇒ Horizontally:

$$s = ut + \frac{1}{2}at^2$$

But $a = 0$

$$200 = ut = F_x t = F\cos 30 \times t$$

$$\Rightarrow t = \frac{200}{F\cos 30}$$

⇒ Vertically:

$$S = ut + \frac{1}{2}at^2$$

$$\Rightarrow 0 = F\sin 30 \times t + \frac{1}{2}(9.81)t^2$$

$$\Rightarrow t = \frac{2F\sin 30}{9.81}$$

$$\therefore \frac{200}{F\cos 30} = \frac{2F\sin 30}{9.81}$$

$$\Rightarrow 981 = F^2 \sin 30 \times \cos 30$$

$$\Rightarrow F = 47.59\ ms^{-1}$$

$\Rightarrow \quad F = \underline{\underline{47.59\ ms^{-1}}}$

$\Rightarrow \quad t = \underline{\underline{4.85s}}$

2. A particle moves on a circular path at a constant radius R about a fixed point O with a fixed angular velocity ω. Using a polar coordinate system, find the expressions for the velocity and acceleration of the particle. What is the magnitude of velocity and acceleration if R=10 m, and ω=2 rad/s?

Answer: [$\underline{v} = R\omega\underline{e}_\theta$; $\underline{a} = -R\omega^2\underline{e}_r$; v=20 m/s; a=40 m/s^2]

$\Rightarrow$ ω - fixed angular velocity

$\underline{V} = R\omega\underline{e}_\theta \qquad \underline{a} = -R\omega^2\underline{e}_r$

$\Rightarrow \quad V = 10 \times 2 = \underline{\underline{20ms^{-1}}}$

$a = 10 \times (2)^2 = \underline{\underline{40\ ms^{-2}}}$

3. The position vector of a particle at time t is r=(3t+1)i+2t^2j (r measured in metres) with i and j the unit vectors in the horizontal and vertical directions. Find the initial position vector and show that the acceleration is constant.

$\Rightarrow \quad r = (3t+1)i + 2t^2 j$

$\Rightarrow$ differentiating it we get:

$V = 3i + 4tj$

differentiating it further, we get acceleration

$a = \underline{\underline{4j}}$

4. A particle moves such that at time t:

$$\dot{r} = 4ti + 5t^2 j$$

4. A particle moves such that at time *t:*

$$\dot{r} = 4ti + 5t^2 j$$

At time t=0 the particle has a position vector r=5*i* – 6*j*. Find the position vector at the general case of time *t.*

Answer: $r = (2t^2 + 5)i + \left(\frac{5}{3}t^3 - 6\right)j$

$\Rightarrow \quad \dot{r} = 4ti + 5t^2 j \Longrightarrow r(t) = \left(\frac{4t^2}{2}\right)i + \left(\frac{5t^3}{3}\right)j$

$\Rightarrow \quad r(t) = (2t^2)i + \left(\frac{5t^3}{3}\right)j$

$\Rightarrow$ General case position includes position at t = 0

$\therefore \quad r = (2t^2 + 5)i + \left(\frac{5}{3}t^3 - 6\right)j$

5. A remote control car is being tested in a horizontal playground. At time t seconds, the position vector, r (in metres), of the car relative to a fixed point O is given by

$$r = \frac{9}{2}t^2 i + \frac{8}{5}t^{\frac{5}{2}} j$$

At the instant when t = 4s,

a) Show that the car is moving with velocity (36*i*+32*j*)ms^{-1} .

b) Find the magnitude of the acceleration of the car.

Answer: [b) 15 ms^{-2}]

(a) $\quad v = \frac{dr}{dt} = 9ti + 4t^{3/2} j$

$\Rightarrow \quad v(4) = 9(4)i + 4(4)^{3/2} j = (36i + 32j)\ ms^{-1}$

(b) $\quad a = \frac{dv}{dt} = 9i + 6t^{1/2} j$

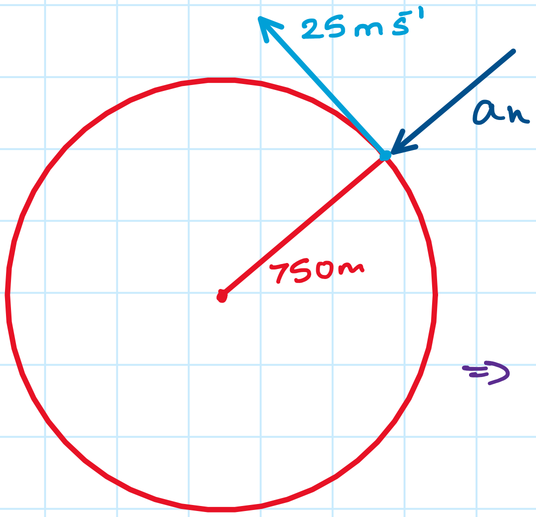
$a(4) = 9i + 6(4)^{1/2} j = 9i + 12j$

$$a(4) = 9i + 6(4)^{1/2} j = 9i + 12j$$

$$a = \sqrt{81 + 144} = 15 \, m s^{-2}$$

6. A motorist is traveling on a curved section of highway of radius r=750m with the speed of 90 km/h. The motorist suddenly applies the brakes, causing the automobile to slow down at a constant rate. Knowing that after t=8s the speed has been reduced to 72 km/h, determine the magnitude of the acceleration of the automobile immediately after the brakes have been applied. Tip: Work in polar coordinates.

Answer: [1.041 m/s^2]



$$V_1 = 90 \times \frac{5}{18} = 25 \, m s^{-1}$$

$$V_2 = 72 \times \frac{5}{18} = 20 \, m s^{-1}$$

$$\Rightarrow V = R\omega$$

$$\omega_1 = \frac{V_1}{R} = \frac{25}{750} = 0.033 \, rad \, s^{-1}$$

$$\omega_2 = \frac{V_2}{R} = \frac{20}{750} = 0.0267 \, rad \, s^{-1}$$

$$\Rightarrow \omega_2 = \omega_1 + \alpha t$$

$$\Rightarrow 0.0267 = 0.0333 + \alpha(8)$$

$$\Rightarrow \alpha = -\frac{1}{1200} \, rad \, s^{-2}$$

$$\Rightarrow \quad a = R\alpha = 750 \times \left(-\frac{1}{1200}\right)$$

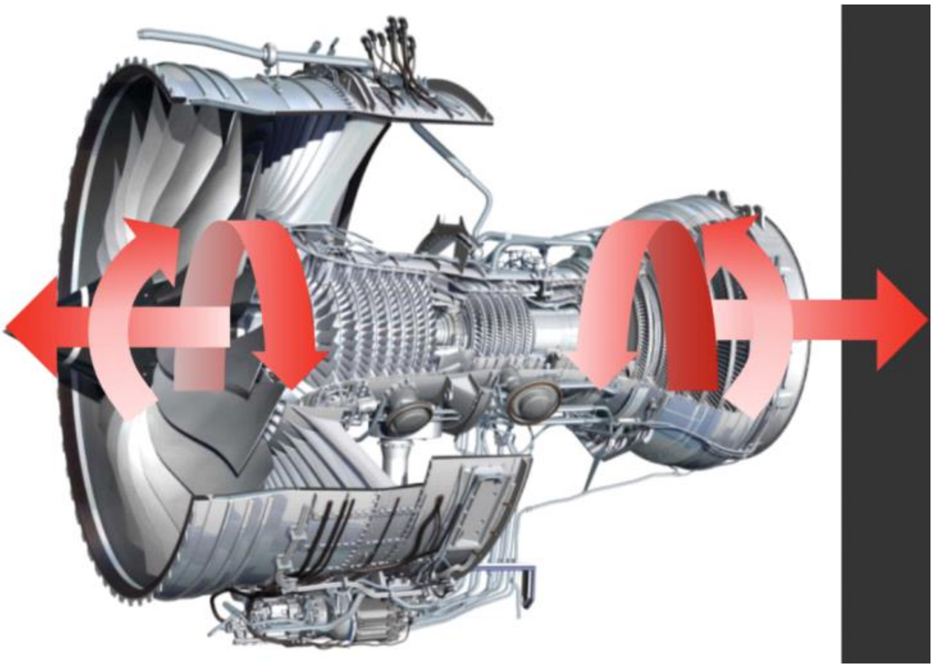
$$a_a = -\frac{5}{8}\ ms^{-2}$$

$$\Rightarrow \quad a_n = R\omega_1^2 = 750 \times (0.033)^2$$

$$= \frac{5}{6}\ ms^{-2}$$

$$\Rightarrow \quad a = \sqrt{a_a^2 + a_n^2} = \sqrt{\left(-\frac{5}{8}\right)^2 + \left(\frac{5}{6}\right)^2}$$

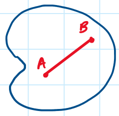
$$= 1.106\ ms^{-2}$$



10.10.22

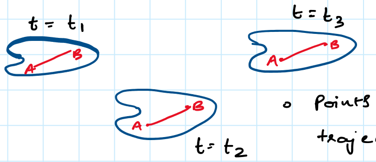
Monday, 10. October 2022 08:59

Rigid body



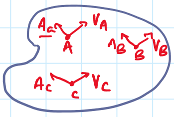
- System of particles
- Distances between particles remain unchanged
- Deformations are neglected

Particle - Rigid body - System of Rigid bodies



t = t_1 t = t_2 t = t_3

- Line segments maintain orientation.
- Points move on "parallel" trajectories

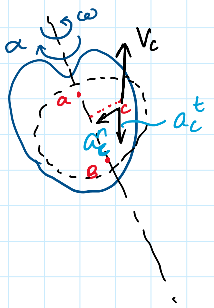


At any instant of time:

$V_{object} = V_A = V_B$

$A_{object} = a_A = a_B$

Rigid body motion: Rotation about fixed axis



Kinematics:

$\theta(t)$ angle of rotation

$\dot{\theta}(t) = \omega(t)$ angular velocity

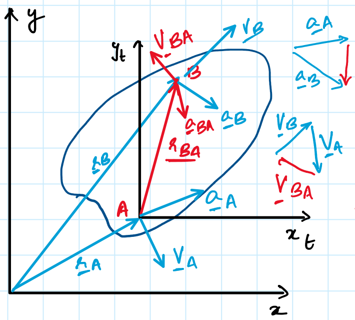
$\ddot{\theta}(t) = \alpha(t)$ angular acceleration

For point C:

$V_c = \omega d$ velocity magnitude

$a_c^n = \omega^2 d$, $a_c^t = \alpha d$ } acceleration components

Relative motion:

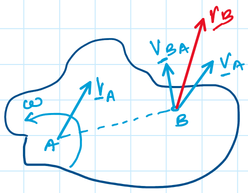


a_{BA}: $r_B = r_A + r_{BA}$

$V_B = V_A + V_{BA}$

$a_B = a_A + a_{BA}$

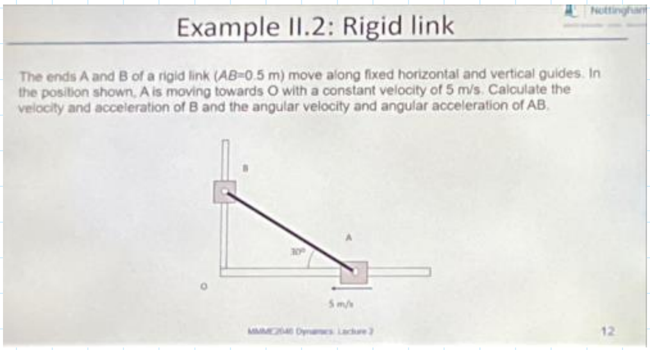
A: reference

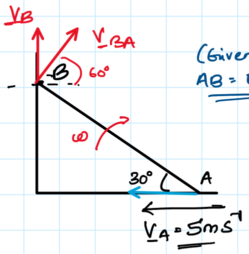


$V_B = V_A + V_{BA}$

Relative motion of B is circular around A:

- magnitude: $V_{BA} = \omega AB$





(Given) AB = 0.5m

Velocity of B is calculated using

$V_B = V_A + V_{BA}$

$V_{BA} = \omega AB$

$\rightarrow^+$: $0 = -V_A + V_{BA}\cos 60°$

$= -V_A + \omega AB \cos 60°$

$\omega = \frac{V_A}{AB\cos 60°} = \frac{5}{0.5 \times 0.5}$

$= 20\ rad\,s^{-1}$

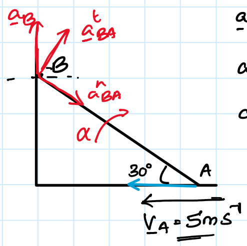
$\uparrow^+$ $V_B = 0 + V_{BA}\sin 60°$

$= 20 \times 0.5 \sin 60°$

$= 8.66\ m s^{-1}$

$V_{BA} = \omega AB = 20 \times 0.5 = 10\ m s^{-1}$

Acceleration analysis



$a_B = a_A + a_{BA}^n + a_{BA}^t$

$a_{BA}^n = \omega^2 AB = 200\ m s^{-2}$

$a_{BA}^t = \alpha AB$

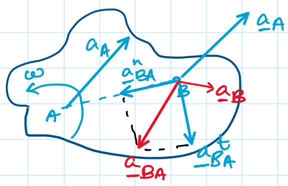
$\rightarrow^+$: $0 = 0 + a_{BA}^n \cos 30 + \alpha AB \cos 60°$

$\alpha = -\frac{a_{BA}^n \cos 30°}{AB \cos 60°} = -692.8\ rad\,s^{-2}$

$\uparrow^+$ $a_B = 0 - a_{BA}^n \sin 30 + \alpha AB \sin 60$

$= -400\ m s^{-2}$

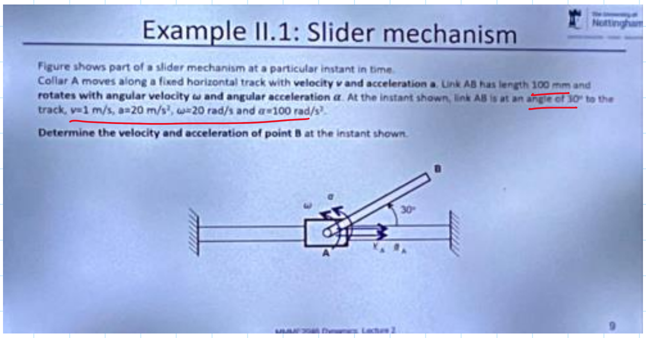
- direction : perpendicular to AB
- sense : governed by the angular velocity



$$\underline{a}_B = \underline{a}_A + \underline{a}_{BA} = \underline{a}_A + \underline{a}^n_{BA} + \underline{a}^t_{BA}$$

magnitude: $a^n_{BA} = \omega^2 AB$

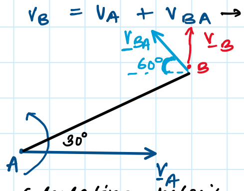
$a^t_{BA} = \alpha AB$



<u>Velocity analysis:</u>

Velocity of B is calculated using

$$v_B = v_A + v_{BA} \rightarrow (1)$$



$AB = 0.1\,m$

$v_A = 1\,m/s$

$\omega = 20\,rad\,s^{-1}$

$v_{BA} = \omega AB = 20 \times 0.1 = \underline{2\,ms^{-1}}$

Calculating velocity B by resolving eq. (1) in horizontal and vertical directions:

$\xrightarrow{+}$: $V_{Bx} = V_A - V_{BA}\cos 60° = 1 - 2\cos 60° = \underline{0}$

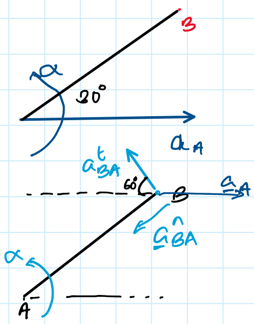
$\uparrow+$: $V_{By} = 0 + V_{BA}\sin 60° = 2\sin 60° = \underline{1.732\,ms^{-1}}$

velocity of B is vertically upwards

<u>Acceleration analysis</u>

Acceleration of B is calculated using

$$a_B = a_A + a^n_{BA} + a^t_{BA}$$



$AB = 0.1\,m$ $\quad \omega = 20\,rad\,s$

$a_A = 20\,ms^{-2}$ $\quad \alpha = 100\,rad\,s^{-2}$

$a^n_{BA} = \omega^2 AB = 20^2 \times 0.1 = 40\,ms^{-2}$

$a^t_{BA} = \alpha AB = 100 \times 0.1 = 10\,ms^{-2}$

calculate acceleration of B,

Σx: $a_{Bx} = a_A - a^n_{BA}\cos 30° - a^t_{BA}\cos 60°$

$= 20 - 40\cos 30° - 10\cos 60°$

$= \underline{-19.64\,ms^{-2}}$

$\overset{\uparrow+}{\Sigma} Y$ $\quad a_{By} = 0 + a^t_{BA}\sin 60° - a^n_{BA}\sin 30°$

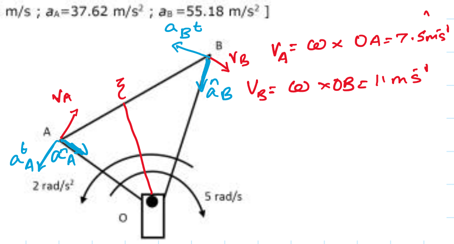
$= \underline{-11.34\,ms^{-2}}$

Questions

Wednesday, 12. October 2022 15:58

1. The triangle below is rotating about O. Find the magnitude of the velocity and acceleration at A and B at the instant shown, and show a sketch with the velocity and acceleration vectors directions. OA=1.5 m, OB=2.2 m.

[v_A=7.5 m/s ; v_B=11 m/s ; a_A=37.62 m/s^2 ; a_B =55.18 m/s^2]



$\Rightarrow a^n_A = \omega^2 OA = 37.5 m/s^2$

$a^n_B = \omega^2 OB = 55 m/s^2$

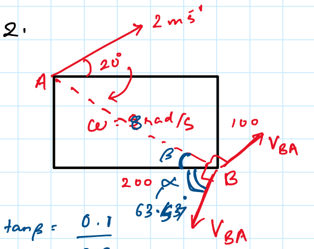
$a^t_A = \alpha \times OA = 3 ms^{-2}$

$a^t_B = \alpha \times OB = 4.4 ms^{-2}$

$\Rightarrow a_A = \sqrt{a_{An}^2 + a_{At}^2} = 37.62 ms^{-2}$

$a_B = \sqrt{a_{Bn}^2 + a_{Bt}^2} = 55.18 ms^{-2}$

2.



$V_{BA} = ?$

$AB = \sqrt{0.1^2 + 0.2^2}$

$= 0.22 m$

$\tan\beta = \frac{0.1}{0.2}$

$\beta = 26.57°$

$\alpha = 90 - 26.57 = 63.43°$

$V_{BA} = \omega \times (AB)$

$= 8 \times 0.22 = 1.76 ms^{-1}$

$\Rightarrow \underline{V}_B = \underline{V}_A + \underline{V}_{BA}$

$\overset{+}{\rightarrow} V_{Bx} = V_A \cos(20°) - V_{BA} \cos(63.43°) = 1.092 ms^{-1}$

$V_{By} = V_A \sin(20°) - V_{BA} \cos(63.43°) = -0.103 ms^{-1}$

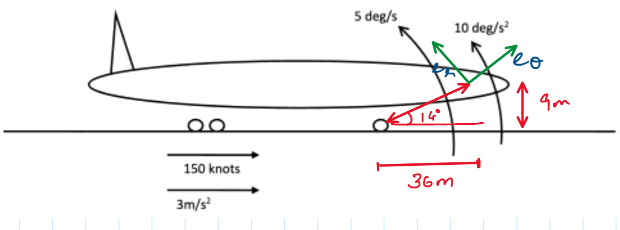
$\Rightarrow V_B = \sqrt{V_{Bx}^2 + V_{By}^2} = \sqrt{1.092^2 + 0.103^2} =$

$\theta = \frac{V_{By}}{V_{Bx}} =$

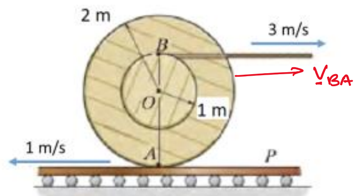
3.

3. Find the velocity and acceleration experienced by a pilot as the plane below "rotates" at take off speed. Assume the pilot sits 9 m above and 36 m in front of the rear landing gear. There is 0.5144 m/s in a knot.

[76.43 m/s; 6.32 m/s^2]



4. Determine the angular velocity of the spool shown below. The cable wraps around the inner core, and the spool does not slip on the platform P. Radius of the spool is 2 m, radius of the inner core is 1 m. Velocity of the platform is 1 m/s, velocity of the cable is 3 m/s.
[4/3 rad/s]



$\Rightarrow V_{cable} = 3 ms^{-1}$ (V_b) $\quad V_{platform} = 1 ms^{-1}$ (V_a)

$\Rightarrow \omega = ?$

$\underline{V}_B = \underline{V}_A + \underline{V}_{BA}$

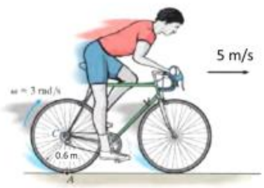
$\overset{+}{\rightarrow}: V_B = -V_A + V_{BA}$

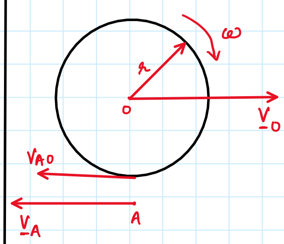
$V_{BA} = \omega AB$

$V_B + V_A = \omega AB$

$\omega = \frac{V_A + V_B}{AB} = 4/3 \text{ rad } s^{-1}$

5. The bicycle has a velocity v=5 m/s, and at the same instant the rear wheel has a clockwise angular velocity ω = 3 rad/s, which causes it to slip at its contact point A. Radius of the rear wheel is 0.6 m. Determine the velocity of point A.
[3.2 m/s]





$V_O = 5 ms^{-1}$

$\omega = 3 rad s^{-1}$

$R = 0.6 m$

$\underline{V}_A = \underline{V}_O + \underline{V}_{AO}$

$V_{AO} = \omega \cdot AO$

$\overset{+}{\rightarrow}: -V_A = V_O - V_{AO}$

$V_A = V_{AO} - V_O$

$= \omega AO - V_O$

$= 3 \times 0.6 - 5$

$= 3.2 ms^{-1}$

$\Rightarrow$ $V_a = 150 \times 0.5144 = \underline{\underline{77.16\ ms^{-1}}}$ $\quad a_a = 3ms^{-2}$

$\omega_{BA} = 5 \times \frac{\pi}{180} = \underline{\underline{0.0872\ rad\,s^{-1}}}$

$\alpha_{BA} = 10 \times \frac{\pi}{180} = \underline{\underline{0.174\ rad\,s^{-2}}}$

$\Rightarrow$ Position relative to the front landing gear

$R = \sqrt{9^2 + 36^2} = \underline{\underline{37.10\ m}}$

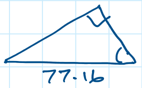
$\Rightarrow$ $V_{BA} = R \times \omega = 37.10 \times 0.0872 = \underline{\underline{3.235\ ms^{-1}}}$

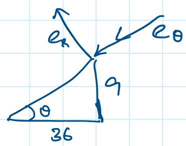
$a_{BA} = R\,\omega^2 e_r + R\,\alpha\,e_\theta$

$= 37.11 \times (0.0872)^2 e_r + 37.11 \times 0.174$

$= \underline{\underline{0.282}} e_r + \underline{\underline{6.457}} e_\theta$

$\Rightarrow$ $\underline{V_B}$:



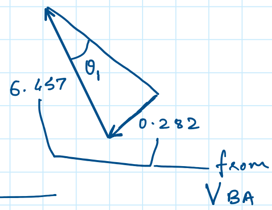


$\theta = \tan^{-1}\left(\frac{9}{36}\right) = \underline{\underline{14.04}}$

$V_B = \sqrt{V_a^2 + V_{BA}^2} = \sqrt{77.16^2 + 3.235^2}$

$= \underline{\underline{77.23}}\ ms^{-1}$

$\Rightarrow$ $\underline{a_B}$:



$a = \sqrt{6.457^2 + 0.282}$

$= 6.46\ ms^{-1}$ $\qquad \theta_2 = \tan^{-1}\left(\frac{0.282}{6.457}\right)$

$= \underline{\underline{2.5^\circ}}$

$\theta_1 + \theta_2 = 14.04^\circ + 2.5^\circ = \underline{\underline{16.54^\circ}}$

$90 - 16.54 = \underline{\underline{73.46^\circ}}$

$a_B^2 = 3^2 + 6.486^2 - (2 \times 3 \times 6.486 \times \cos 73.46)$

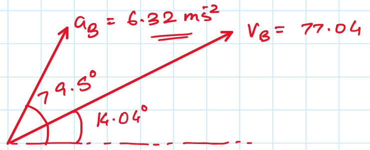
$a_B^2 = 39.98$ $\qquad \theta_3 = \sin^{-1}\left(\frac{\sin(73.46)}{6.325} \times 6.486\right)$

$a_B = \underline{\underline{6.32\ ms^{-2}}}$

$= \underline{\underline{79.5^\circ}}$

$a_B = 6.32\ ms^{-2}$

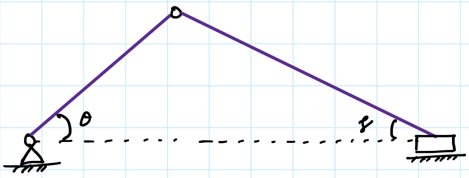
$= 79.5°$



17.10.22

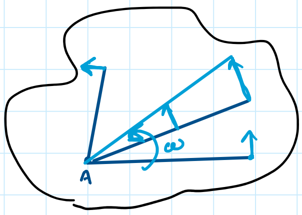
Monday, 17. October 2022 08:59

- Linkage is a mechanism that consists of a rigid links and one of the links is rigidly attached to a base
- Constraints are imposed on the rigid body motion.



Instantaneous centre of rotation:

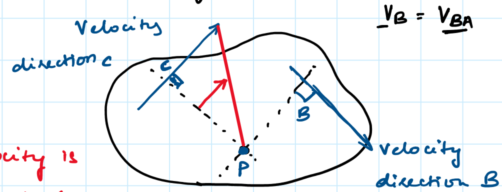
This is a point with zero velocity at a particular moment



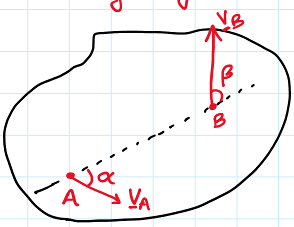
Point A is stationary at instant (instant centre or velocity pole)

For any other points: $\underline{V}_A = 0$

$\underline{V}_B = \underline{V_{BA}}$



Angular velocity is independent of the choice of origin!



known: $\underline{V}_B = \underline{V}_A + \underline{V}_{BA}$

then: $\underline{V}_B \| AB = \underline{V}_A \| AB + \underline{V}_B \| AB$

but: $\underline{V}_{BA} \| AB = 0$ (since $\underline{V}_{BA} \perp AB$)

$\therefore \underline{V}_B \| AB = \underline{V}_B \| AB$

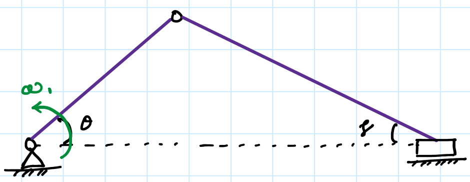
Velocity components on the axis joining the points are equal.

$V_B \cos\beta = V_A \cos\alpha$

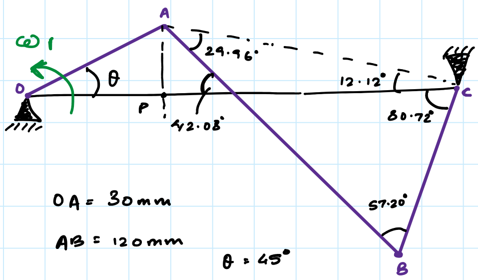
Example:

$\omega_1 = 100 \text{ rad s}^{-1}$ = const. AB = 80mm BC: 240 mm

BG = 120mm $\theta = 45°$



⇒ Geometry: (using sin rule)



OA = 30mm

AB = 120mm $\theta = 45°$

BC = 60mm

OC = 120mm $\omega_1 = 30 \text{ rad s}^{-1}$

$OP = AP = OA \cos 45° = 21.21 \text{ mm}$

$PC = OC - OP = 98.79 \text{ mm}$

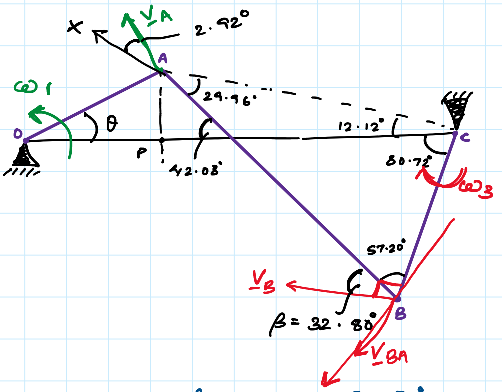
$AC = \sqrt{PC^2 + AP^2} = 101.0 \text{ mm}$

$\Rightarrow \angle PCA = \tan^{-1}\dfrac{AP}{PC} = 12.12°$

$\angle ABC = \sin^{-1}\left(\dfrac{AC}{BC}\sin 29.96°\right) = 57.20°$

$\angle BAC = \cos^{-1}\left(AB^2 + AC^2\right.$

$\Rightarrow V_A = \omega_1 OA = 30 \times 0.03 = 0.9 \text{ ms}^{-1}$



$V_A \cos 2.92° = V_B \cos 32.80°$

$V_B = \dfrac{0.9 \cos 2.92}{\cos 32.80} = 1.069 \text{ ms}^{-1}$

$\Rightarrow \omega_3 = \dfrac{V_B}{BC} = 17.82 \text{ rad s}^{-1}$

$\Rightarrow \underline{V}_B = \underline{V}_A + \underline{V}_{BA}$

$(\perp AB)$: $V_B \sin 32.80° = -V_A \sin 2.92° + \omega_2 AB$

⇒ Geometry: (using sin rule)

$$\frac{\sin \gamma}{AB} = \frac{\sin \theta}{BC}$$

$$\Rightarrow \gamma = \sin^{-1}\left(\frac{\sin 45 \times 80}{240}\right)$$

$$= 13.63^\circ$$

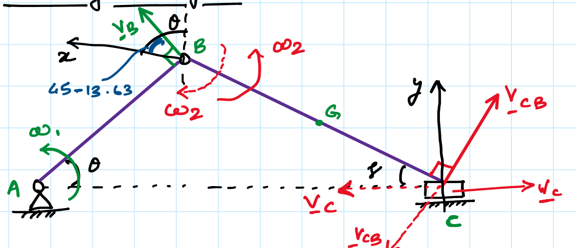
⇒ Using sin rule again:

$$AC = \frac{BC \sin 121.4^\circ}{\sin 45^\circ} = 0.2897\,m$$

check:

$$AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cos 121.4^\circ$$

Velocity analysis:



$$v_B = \omega_1 AB = 100 \times 0.08 = 8\,ms^{-1}$$

$$v_C = v_B + v_{CB}$$

$$v_{CB} = \omega_2 BC$$

$$\uparrow^+ \quad 0 = v_B \cos 45^\circ + v_{CB} \cos 13.63^\circ \quad (v_{CB} = \omega_2 BC)$$

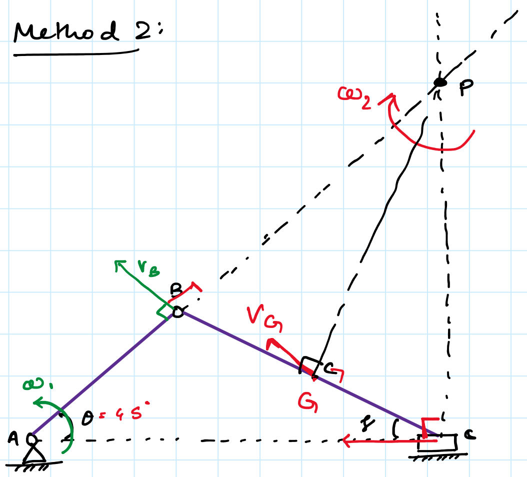
$$\omega_2 = -\frac{v_B \cos 45}{BC \cos 13.63^\circ} = -24.25\,rad\,s^{-1}$$

$$\nwarrow^+ (x): \quad -v_C \cos 13.63^\circ = v_B \cos 31.37^\circ + 0$$

$$\Rightarrow v_C = -\frac{v_B \cos 31.37}{\cos 13.63}$$

$$= -7.029\,ms^{-1}$$

Method 2:



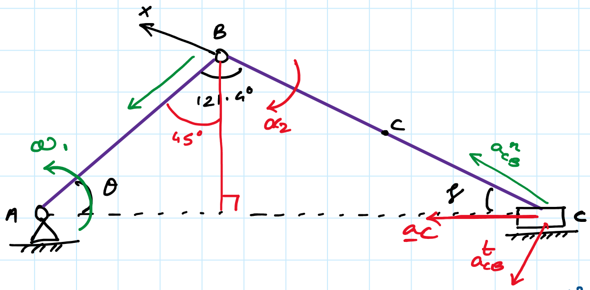
$$v_B = \omega_1 AB = 100 \times 0.08 = 8\,ms^{-1}$$

$$\swarrow^+ (\perp AB): \quad v_B \sin 32.80^\circ = -v_A \sin 2.92^\circ + \omega_2 AB$$

$$\omega_2 = \frac{v_B \sin 32.80^\circ + v_A \sin 2.92}{AB}$$

$$= 5.21\,rad\,s^{-1}$$

Acceleration analysis:



$$a_B = a^n_B = \omega_1^2 AB = 100^2 \times 0.08 = 800\,ms^{-2}$$

$$a_C = a_B + a^n_{CB} + a^t_{CB}$$

$$a^n_{CB} = \omega_2^2 BC = 24.25 \times 0.24 = 141.1\,ms^{-2}$$

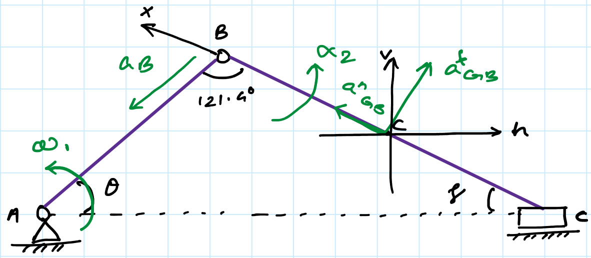
$$a^t_{CB} = \alpha_2 BC = 0.24\alpha_2$$

$$\uparrow^+ \Sigma x: \quad a_C \cos 13.63^\circ = a^n_B \cos 58.63 + a^n_{CB} + 0$$

$$\longrightarrow a_C = 573.7\,ms^{-2}$$

$$\uparrow^+ \Sigma y: \quad 0 = -a^n_B \cos 45^\circ + a^n_{CB} \sin 13.63^\circ - a^t_{CB} \cos 13.63^\circ$$

$$\alpha_2 = -2283\,rad\,s^{-2}$$



$$a^n_{GB} = \omega_2^2 \cdot BG = 24.25^2 \times 0.12 = 70.55\,ms^{-2}$$

$$a^t_{GB} = \alpha_2 \cdot BG = 2283 \times 0.12 =$$

$V_B = \omega_1 \cdot AB = \ldots$

$$PA = \frac{AC}{\cos 45} = \underline{\underline{0.4097\,m}}$$

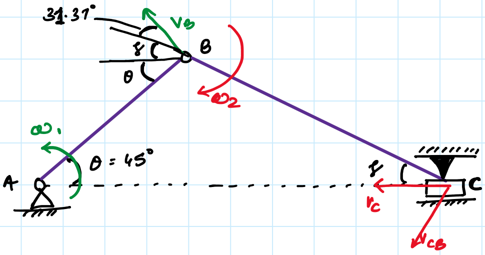
$$PB = 0.3297\,m$$

$$\omega_2 = \frac{V_B}{PB} = \underline{\underline{24.26\ rad\,s^{-1}}}$$

$$PC = \frac{AC}{\tan 45^\circ} = 0.2897\,m$$

$$V_C = \omega_2\,PC = 24.26 \times 0.2897$$
$$= \underline{\underline{7.028\,ms^{-1}}}$$

Method 3



$$V_C \cos\gamma = V_B \cos(90^\circ - \gamma - \theta)$$

$$V_C \cos 13.63^\circ = V_B \cos 31.37^\circ$$

$$V_C = \frac{V_B \cos 31.37^\circ}{\cos 13.63} = \underline{\underline{7.029\,ms^{-1}}}$$

$\Rightarrow \quad \underline{V}_C = \underline{V}_B + \underline{V}_{CB}$

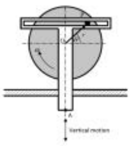
$\swarrow^{+} (\perp BC): \quad V_C \sin 13.63^\circ = -V_B \sin 31.37^\circ + \omega_2 BC$

$$\omega_2 = \underline{\underline{24.25\ rad\,s^{-1}}}$$

Questions:

Wednesday, 19. October 2022 19:30

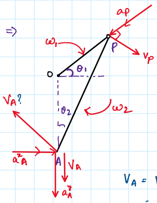
1. The figure below shows a slotted link (Scotch Yoke) mechanism used to convert rotational motion into vertical translational motion – this is an example of a reciprocating motion mechanism. The mechanism consists of a circular disc and a sliding yoke with a slot. The disc rotates about a fixed centre through O and has a pin (P) rigidly attached to it at distance r from the centre of the disc. As the disc rotates, contact between the pin and the slot causes the yoke to move vertically as shown.



The pin is located at a radius r = 2 cm and the wheel has constant angular velocity ω = 6 rad/s, and at the instant shown θ = 30°.

Calculate the magnitude and direction of the velocity and acceleration of point A.

[0.104 m/s downwards; 0.36 m/s² downwards]



$\theta_1 = 30°$ $\omega_1 = 6\ rad\ s^{-1}$

$r = 0.02m$

$V_a = (\downarrow) V_p$

$V_p = r \times \omega_1$

$= 0.02 \times 6$

$= 0.12\ ms^{-1}$

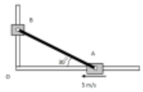
$V_A = V_p \cos(60°) = 0.12 \times \cos 30°$

$= 0.1039 = 0.109\ ms^{-1}$

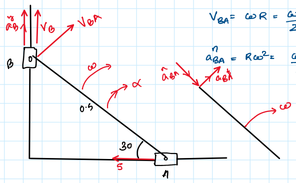
$a_p = R\omega_1^2 = 0.02 \times 6^2 = 0.72\ ms^{-2}$

$a_A^t = 0.72 \times \sin(30) = 0.36\ ms^{-2}$

2. The ends A and B of a rigid link (AB=0.5 m) move along fixed horizontal and vertical guides. In the position shown, A is moving towards O with a constant velocity of 5 m/s. Calculate the velocity and acceleration of B and the angular velocity and angular acceleration of AB.



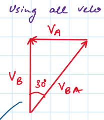
[8.660 m/s; 400 m/s²; 20 rad/s (CW); 692.8 rad/s² (ACW)]



$V_{BA} = \omega R = \frac{\omega}{2}$

$a_{BA}^n = R\omega^2 = \frac{\omega^2}{2}$

⇒ Using all velocities, we get:



$V_A = V_{BA} \sin 30°$

⇒ $5 = V_{BA} \sin 30°$

⇒ $V_{BA} = 10\ ms^{-1}$

⇒ $\omega = \frac{V_{BA}}{R_{BA}} = \frac{10}{0.5} = 20\ rad\ s^{-1}$

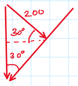
⇒ ∴ $a_{BA}^n = \frac{(20)^2}{2} = 200\ m\ s^{-2}$

⇒ $V_A = V_B \tan\theta$ ⇒ $V_B = \frac{5}{\tan 30}$

$V_B = 8.66\ ms^{-1}$

$\alpha = 200\ rad\ s^{-1}$

⇒ the acceleration triangle:-



$a^t = \frac{200}{\tan 30}$

$= -346.4\ ms^{-1}$

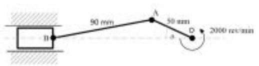
⇒ $a_{BA} = \sqrt{200^2 + (346.4)^2} = 399.99$

$= 400\ ms^{-1}$

⇒ $a_{BA}^t = R\alpha$

⇒ $\alpha = \frac{346.4}{0.5} = 692.8\ rad\ s^{-2}$

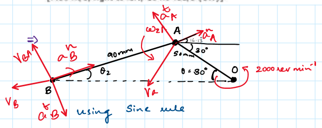
3. The figure shows a slider-crank mechanism consisting of a 50 mm radius crank (OA) which rotates at 2000 rev/min, and a connecting rod (AB) having length 90 mm.



At the instant shown, angle θ = 30°.

Calculate the magnitude and direction of the velocity of piston B relative to crank centre O, and the angular velocity of connecting rod AB.

[7.86 m/s, right to left; 104.9 rad/s (CW)]



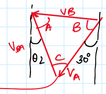
⇒ using Sine rule

$\frac{\sin\theta_2}{50} = \frac{\sin 30}{90}$

⇒ $\theta_2 = 16.13°$

⇒ $\omega_1 = 2000\ rev\ min^{-1} = 209\ rad\ s^{-1}$

⇒ $\omega_2 = \frac{V_{AB}}{0.09}$



⇒ $V_B = V_A + V_{BA}$

⇒ $V_A = R\omega = 0.05 \times 209.46$

$= 10.472\ ms^{-1}$

⇒ $A = 90° - 16.13° = 73.87°$

$B = 90° - 30° = 60°$

$C = 180° - (73.87 + 30) = 46.13°$

⇒ Applying Sine rule:

$\frac{\sin 73.87}{10.472} = \frac{\sin 60}{V_{BA}} = \frac{\sin 46.13°}{V_B}$

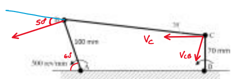
⇒ $V_{BA} = 9.44\ ms^{-1}$

⇒ $V_B = 7.86\ ms^{-1}$

⇒ $\omega_2 = \frac{V_{BA}}{0.09} = \frac{9.44}{0.09}$

$= 104.89\ rad\ s^{-1}$

4. The figure shows a 4 bar chain ABCD consisting of input crank AB having length 100 mm and output crank DC having length 70 mm. At the instant shown, the input crank has angular velocity 500 rev/min and orientation 60°, the connecting rod has orientation 20°, and the output crank is vertical.



Calculate the angular velocity of the output crank.

[489 rev/min]

⇒ angular velocity of crank AB is

$\omega = 500\left(\frac{2\pi}{60}\right) = 52.4\ rad\ s^{-1}$

⇒ Velocity of B is:

$V_B = AB \times \omega = 0.1 \times 52.4$

$= 5.24\ ms^{-1}$

⇒ Angular velocity of the output crank can be obtained by noting that the output tangential to the crank is in pure rotation about D. This means V_{CD} tangential to crank

$V_{CD} = CD \times \omega_{out}$

ω_{out} - angular velocity of output crank

⇒ Calculating velocity C relative to A:

$\underline{V_C} = \underline{V_B} + \underline{V_{CB}}$

⇒ Noting ω_{BC} produces tangential velocity V_{CB} & is not known. Necessary to resolve above eq. parallel to BC to get an eq. in terms of V_C. No need to consider to V_{CB}.

$\nwarrow$ $\underline{V_C} = \underline{V_A} + \underline{V_{CB}}$

$V_C \cos 20 = V_B \cos 50 + 0$

⇒ $V_C = \frac{5.24 \cos 50}{\cos 20} = 3.58\ ms^{-1}$

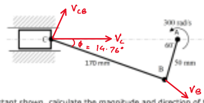
⇒ We know:

$V_C = CD \times \omega_{out}$

∴ $\omega_{out} = \frac{3.58}{CD} = \frac{3.58}{0.07}$

$= 51.2\ rad\ s^{-1} = 489\ rev/min$

5. A piston, connecting rod and crank mechanism is shown in the below figure. The crank rotates at a constant angular velocity of 300 rad/s.



At the instant shown, calculate the magnitude and direction of the acceleration of piston C and the angular acceleration of connecting rod BC.

[1589 m/ s², left to right, 23158 rad/s² (ACW)]

⇒ $AB = 0.05m$ $BC = 0.17m$

⇒ Using sine rule

$\frac{\sin\phi}{AB} = \frac{\sin 60}{CB}$

⇒ $\frac{\sin\phi}{0.05} = \frac{\sin 60}{0.17}$

⇒ $\phi = 14.76°$

⇒ $\omega = 300\ rad\ s^{-1}$

⇒ The tangential velocity of B is:

$V_B = AB \times \omega = 0.05 \times 300$

$= 15\ ms^{-1}$

⇒ Angular velocity of C is needed for acceleration calculations and can be calculated from:

$\underline{V_C} = \underline{V_B} + \underline{V_{CB}}$

⇒ Angular velocity of BC, calc by considering the velocity components vertically.

$\uparrow$ $\underline{V_C} = \underline{V_B} + \underline{V_{CB}}$

$0 = -V_B \cos 60 + V_{CB} \cos 14.76$

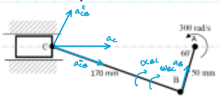
⇒ Re-arranging:

$V_{CB} = 15 \frac{\cos 60}{\cos 14.757} = 7.756\ ms^{-1}$

⇒ Noting that $V_{CB} = 0.17\,\omega_{BC}$, then

$\omega_{CB} = \frac{7.756}{0.17} = 45.62\ rad\ s^{-1}$

Acceleration analysis:



⇒ where $a_B = AB\,\omega^2 = 0.05 \times 300^2$

$= 4500\ ms^{-2}$

$a_{CB}^n = CB\,\omega_{BC}^2 = 0.17 \times 45.62^2$

$= 353.8\ ms^{-2}$

$a_{CB}^t = CB\,\alpha_{BC} = 0.17\,\alpha_{BC}$

⇒ The acceleration of C is calculated using acceleration vector equation

$\underline{a_C} = \underline{a_B} + \underline{a_{CB}^n} + \underline{a_{CB}^t}$ (2)

⇒ Resolving eq (2) in the vertical direction gives:

$0 = a_B \sin 60 - CB\,\omega_{BC}^2 \sin\phi + CB\alpha_{BC} \cos\phi$

$0 = 3897.11 - 90.1378 + \alpha_{BC}\ 14.82$

$\alpha_{BC} = -23793.58\ rad\ s^{-2}$

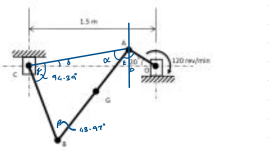
⇒ Resolving eq (2) parallel link BC (C to B +ve) gives:

$a_C \cos\phi = a_B \cos(60 + \phi) + CB\,\omega_{BC}^2$

$a_C \cos(14.76) = 4500 \cos(60 + 14.76) + 0.17 \times (45.62)^2$

⇒ $a_C = 1589.12\ ms^{-2}$

6. In the four-bar linkage OABC shown, crank OA is driven clockwise at a constant speed of 120 rev/min. The centre of mass for link AB is located at G, the midpoint of link AB. In the position shown, determine the linear acceleration of G and the angular acceleration of AB. The link lengths are as follows: OA=0.5m, AB=1.5m, BC=1m.



[52.44 m/s²; 71.61 rad/s² (CW)]

⇒ $OA = 0.5m$ $AB = 1.5m$ $BC = 1.0m$ $CO = 1.5m$

$AD = OA \sin 20° = 0.1710m$

$OD = OA \cos 20° = 0.4698m$

$CD = OC - OD = 1.5 - 0.4698 = 1.0302m$

⇒ $AC = \sqrt{AD^2 + CD^2} = \sqrt{0.1710^2 + 1.0302^2} = 1.044m$

Using cosine rule:

⇒ $AC^2 = BC^2 + BA^2 - (2 \times BA \times BC \times \cos\beta)$

⇒ $(1.044)^2 = 1^2 + 1.5^2 - (2 \times 1.5 \times 1 \times \cos\beta)$

⇒ $-2.159 = -3\cos\beta$

⇒ $\beta = \cos^{-1}\left(\frac{2.159}{3}\right) = 43.97°$

⇒ Similarly:

$AB^2 = BC^2 + AC^2 - (2 \times BC \times AC \times \cos\gamma)$

$1.5^2 = 1^2 + (1.044)^2 - (2 \times 1 \times 1.044 \times \cos\gamma)$

⇒ $0.16 = -2.088 \times \cos\gamma$

⇒ $\gamma = \cos^{-1}\left(\frac{0.16}{2.088}\right) = 94.39°$

⇒ Using sine rule:

$\frac{BC}{\sin\alpha} = \frac{AC}{\sin\beta}$

⇒ $\frac{1}{\sin\alpha} = \frac{1.044}{\sin 43.97}$

⇒ $\alpha = 41.68°$

⇒ checking if $\alpha + \beta + \gamma = 180$

$41.68 + 43.97 + 94.39 = 180°$

⇒ Now calculating for δ:

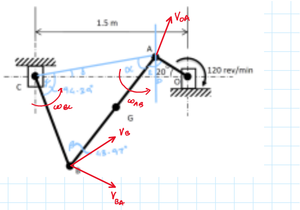
$\tan\delta = \frac{AD}{CD} = \frac{0.1710}{1.0302}$

$\delta = 9.426°$

⇒ $\varepsilon = (180 - 20 - 90) + (180 - 90 - \delta - \alpha)$

$= 108.896°$

To calculate the acceleration we need the acceleration of A and B. The acceleration of A can be calculated because OA rotates at constant angular speed, but to calculate the acceleration of B we need to calculate the angular velocity of link BA – it is also helpful to calculate the angular velocity of link BC. We start by considering velocity of B to calculate the angular velocity of link BA and the angular velocity of link BC.



angular velocity of the crank is $\omega_{OA} = 120(2\pi/60)$

$\omega_{OA} = 120\left(\frac{2\pi}{60}\right) = 12.57\ rad\ s^{-1}$

⇒ Tangential velocity of A is

$V_A = OA \cdot \omega_{OA} = 0.5 \times 12.57$

$= 6.285\ ms^{-1}$

⇒ Velocity of B can be determined using relative equation:

$\underline{V_B} = \underline{V_A} + \underline{V_{BA}}$

⇒ Resolving this equation along AB gives:

$\nearrow$ $\underline{V_B} = \underline{V_A} + \underline{V_{BA}}$

i.e. $V_A \cos(\varepsilon - 90°) = V_B \cos(90 - \beta)$

⇒ $V_B = \frac{6.285 \cos(108.91 - 90)}{\cos(90 - 43.94)}$

$V_B = 8.568\ ms^{-1}$

⇒ Noting that CB has fixed length & rotates point C, tangential $V_B = \omega_{BC}\,BC$, so:

$\omega_{BC} = \frac{V_B}{BC} = \frac{8.569}{1.0} = 8.569$

f A
ul
about fixed

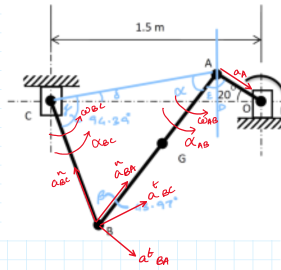
⇒ The calculate angular velocity ω_{AB}, resolve
vector equation perpendicular to AB, i.e,

$$\underline{V}_B = \underline{V}_A + \underline{V}_{BA}$$

i.e, $V_B \sin(90-\beta) = -V_A \sin(\varepsilon - 90) +$

⇒ $8.569 \sin(90 - 43.94) = -6.285 \sin$

⇒ $\omega_{AB} = 5.471 \text{ rad s}^{-1}$



⇒ The acceleration of B is calculating

$$\underline{a}_B = \underline{a}_A + \underline{a}^n_{BA} + \underline{a}^t_{BA}$$

⇒ Noting that link BC is in pure rota

$$\underline{a}_B = \underline{a}^n_{BC} + \underline{a}^t_{BC}$$

⇒ Combining these equations gives:

$$\underline{a}^n_{BC} + \underline{a}^t_{BC} = \underline{a}_A + \underline{a}^n_{BA} + \underline{a}^t_{BA}$$

⇒ where:

$$a^n_{BC} = BC\,\omega^2_{BC} = 73.43 \text{ m s}^{-2}$$

$$a^t_{BC} = BC\,\alpha_{BC} = 1 \times \alpha_{BC} = \alpha$$

$$a_A = OA \times \omega^2_{OA} = 79.00 \text{ m s}^{-2}$$

$$a^n_{BA} = AB\,\omega^2_{BA} = 44.80 \text{ m s}^{-2}$$

$$a^t_{BA} = AB\,\alpha_{BA} = 1.5\,\alpha_{BA}$$

⇒ Resolving the acceleration vector equa
AB

$$a^n_{BC} + a^t_{BC} = \underline{a}_A + \underline{a}^n_{BA} +$$

i.e, $a^n_{BC} \cos\beta + a^t_{BC} \sin\beta = a_{OA} \cos$

⇒ Substituting the values:

$$73.43 \cos 43.94 + \alpha_{BC} \sin 43.9$$

$$= 79.00 \cos$$

⇒ $\alpha_{BC} = 25.38 \text{ rad s}^{-2}$

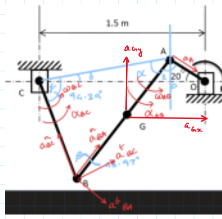
⇒ Resolving the vector equation para

$$\underline{a}^n_{BC} + \underline{a}^t_{BC} = \underline{a}_A$$

i.e $a^n_{BC} + 0 = -a_A \cos(\varepsilon - \beta) +$

⇒ $73.43 = -79.00 \cos(64.96) +$
$- 1.5\alpha_{AB} \sin$

$$\alpha_{AB} = -71.61 \text{ rad s}^{-2}$$



⇒ The acceleration of G is calcula
Vector equation

$$a_G = a_A + \underline{a}^n_{GA} + \underline{a}^t_{GA}$$

⇒ where:-

$$a^n_{GA} = 0.5\,a^n_{BA} = 22.45$$

$$a^t_{GA} = 0.5\,a^t_{BA} = 53.71$$

• Resolving the vector equation in th
direction gives:

$$\underline{a}_G = \underline{a}_A + \underline{a}^n_{GA} + \underline{a}^t_{GA}$$

$a_{Gx} = 79.00 \cos 20 + 22.45 \cos(180$
$- 53.71 \sin(180 - \varepsilon - 20)$

the velocity

AB ω_{AB}

(108.91 - 90)
+ 1.5 ω_{AB}

using vector eq.:-

sion then:

2

BC

sion parallel to

- a^t_{BA}
s (180 - θ) + a^n_{BA}

4
71.1 + 44.9

llel to BC give:
+ a^n_{BA} + a^t_{BA}
a^n_{BA} cos β - a^t_{BA} sin β
44.90 cos 43.94
n 43.94

[redacted]

ted using

ms^{-2}
ms^{-2}

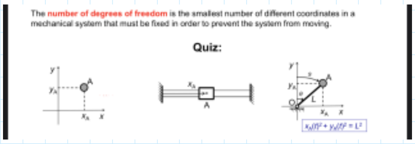
horizontal

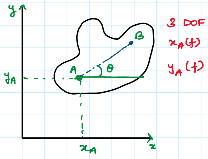
- θ - 20)
= 46.54ms^{-2}

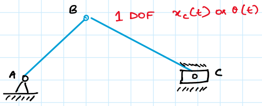
Planar Dynamics of Rigid Bodies

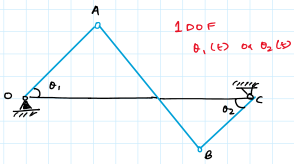
Monday, 24. October 2022 09:01

DOF of a mechanical system in motion are the independent coordinates needed to uniquely specify the position of the system.

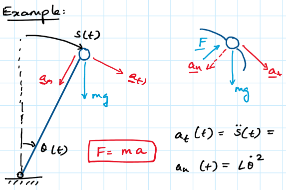








Example:



$a_t(t) = \ddot{s}(t) = L\ddot{\theta}$ $[\ddot{\theta} = \alpha]$

$a_n(t) = L\dot{\theta}^2$ $[\dot{\theta} = \omega]$

$F = ma$

$\Rightarrow \sum F_t = ma_t : \quad mg\sin\theta = mL\ddot{\theta}$

$\ddot{\theta} = \frac{g}{L}\sin\theta$

$\sum F_n = ma_n : \quad mg\cos\theta - F = mL\dot{\theta}^2$

$\Rightarrow \ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\times\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$

$\rightarrow \dot{\theta}\,d\dot{\theta} = \ddot{\theta}\,d\theta = \frac{g}{L}\sin\theta\,d\theta$

$\Rightarrow$ Integrating and assuming $\dot{\theta}(0) = 0$:

$\int x\,dx = \frac{x^2}{2} + C$

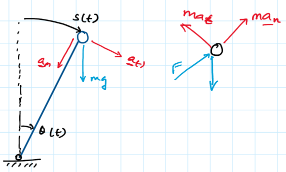
$\dot{\theta}^2 = \frac{2g}{L}(1-\cos\theta)$ ← $\frac{\dot{\theta}^2}{2} + C_1 = -\frac{g}{L}\cos\theta + C_2$

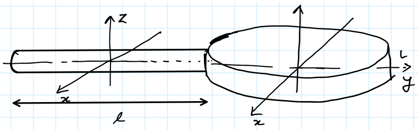
$\frac{\dot{\theta}^2}{2} = C - \frac{g}{L}\cos\theta$; $[\theta = 0]$ $0 = C - \frac{g}{L}$

$F = mg\cos\theta - mL\dot{\theta}^2 = mg(3\cos\theta - 2)$

d'Alembert's principle:

transforms a dynamic system into an equivalent static system.





$\ell = 50$ cm $\quad r = 2$ cm

$L = 2$ cm $\quad R = 10$ cm

$M = J\alpha$ → Torque

J = moment of inertia

α = angular acceleration

Steel rod: $m_R = \pi r^2 \ell \rho_{st} = \pi\times 0.02^2\times 0.5\times 7800$

$= 4.901$ kg

$J_{GR,y} = \frac{1}{2}m_R r^2 = \frac{1}{2}\times 4.901\times 0.02^2$

$= 9.802\times 10^{-4}$ kg·m²

$J_{GR,x} = J_{GR,z} = \frac{1}{12}m_R\ell^2 = \frac{1}{12}4.901\times 0.5^2$

$= 0.1021$ kg·m²

Aluminium Disk: $m_D = \pi R^2 L \rho_{Al} = \pi\times 0.1^2\times 0.02\times 2700$

$= 1.696$ kg

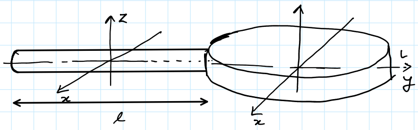
$J_{GD,x} = J_{GD,y} = \frac{1}{12}m_D(3R^2 + L^2)$

$= \frac{1}{12}\times 1.696\times(3\times 0.1^2 + 0.02^2)$

$= 4.297\times 10^{-3}$ kg·m²

$J_{GD,z} = \frac{1}{2}m_D R^2 = \frac{1}{2}\times 1.696\times 0.1^2$

$= 8.480\times 10^{-3}$ kg·m²



$\ell = 50$ cm $\quad r = 2$ cm

$L = 2$ cm $\quad R = 10$ cm

Parallel axis theorem:

$J_{x1} = J_{GR,x} + m_R\left(\frac{\ell}{2}\right)^2 + J_{GD,x} + m_D(\ell + R)^2$

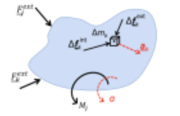
$= 0.1021 + (4.901\times 0.25^2) + (4.297\times 10^{-3})$

$+ (1.696\times 0.6^2) = 1.023$ kgm²

$J_y = J_{GR,y} + J_{GD,y} = 9.802\times 10^{-4} + 4.297\times 10^{-3} = 5.277\times 10^{-3}$ kgm²

$J_{z1} =$

Fundamental Laws of Rigid Body Motion



Newton's 2nd law

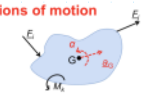
$F = ma_G$ (1)

F : resultant of the external forces
a_G : acceleration of mass centre

MMME1028 result: $M_G = J_G\alpha$ (2)

M_G : resultant of the applied moments about the axis of rotation
J_G : mass moment of inertia about the axis of rotation
α : angular acceleration of the rigid body

Equations of motion

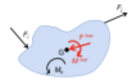


$\rightarrow^+: \sum F_x = m_G a_{Gx}$

$\uparrow^+: \sum F_y = m_G a_{Gy}$

$\circlearrowleft^+: \sum M_G = J_G\alpha$

D'Alembert's principle



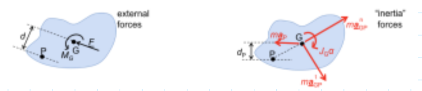
$\rightarrow^+: \sum F_x - F_x^{inertia} = \sum F_x - m_G a_{Gx} = 0$

$\uparrow^+: \sum F_y - F_y^{inertia} = \sum F_y - m_G a_{Gy}$

$\circlearrowleft^+: \sum M_G - M^{inertia} = \sum M_G - J_G\alpha = 0$

D'Alembert's principle

Given: arbitrary point P with known a_P



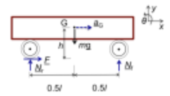
$a_G = a_P + a_{GP}^n + a_{GP}^t$

$\circlearrowleft^+: M_P' = M_P + Fd - J_G\alpha - mPG^2\alpha + ma_P d_P = 0$

$M_P + ma_P d_P = J_P\alpha$

Translation

Example 3: Rear-Wheel Drive Car, a_{max} = ? m = 1500 kg l = 2.5 m h = 0.5 m μ = 1.0



$F = ma_G$ (1)

$M_G = 0$ since $\alpha = 0$ & $\omega = 0$ (2)

$\rightarrow^+: \sum F_x = ma_{Gx} \rightarrow F = ma_{Gx}$

$\uparrow^+: \sum F_y = 0 \rightarrow N_f + N_r - mg = 0$

$F \leq F_{lim} = \mu N$

$\circlearrowleft^+: \sum M_G = 0 \rightarrow N_f\frac{\ell}{2} - N_r\frac{\ell}{2} + Fh = 0$

$F = \mu N_r$

$N_r = \frac{mg\ell}{2(\ell - \mu h)} = \frac{1500\times 9.81\times 2.5}{2(2.5 - (1.0\times 0.5))} = 9188$ N

$N_f = N_g - N_r$; $N_r \geq N_f$

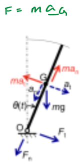
$a_{max} = \frac{\mu g\ell}{2(\ell - \mu h)} = \frac{1.0\times 2.5}{2(2.5 - (1.0\times 0.5))}g = 0.625g$

Rotation

Example 4: Pendulum Motion in a Vertical Plane

$F = ma_G$ (1) $\quad M_o = J_o\alpha$ (5) point O on the axis of rotation

$\alpha = \ddot{\theta}$ and $a_o = 0$; $a_n = \dot{\theta}^2\frac{L}{2}$; $a_t = \ddot{\theta}\frac{L}{2}$



$mg\frac{L}{2}\sin\theta = J_o\alpha$; $mg\frac{L}{2}\sin\theta = \frac{1}{3}mL^2\ddot{\theta}$

$\Rightarrow J_o = J_G + m\left(\frac{L}{2}\right)^2 = \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 = \frac{1}{3}mL^2$

$\Rightarrow \ddot{\theta} = \frac{3}{2}\frac{g}{L}\sin\theta$; $\dot{\theta}^2 = \frac{3g}{L}(1-\cos\theta)$

$\Rightarrow a_t = \frac{L}{2}\ddot{\theta} = \frac{3}{4}\frac{g}{L}\sin\theta$; $a_n = \frac{L}{2}\dot{\theta}^2 = \frac{3g}{L}(1-\cos\theta)$

$\Rightarrow \sum F_t' = 0: \quad F_t - ma_t + mg\sin\theta = 0$

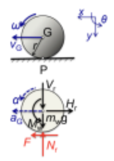
$\sum F_n' = 0: \quad F_n - ma_n + mg\cos\theta = 0$

General planar motion

Example 5: Rear-Wheel Drive Car (cont'd) m_w = 20 kg r = 0.3 m

Torque required to achieve a_{max} = ? J_G = 1.35 kg.m²

$x_G(t) = \theta(t)r, \quad \dot{x}_G(t) = \dot{\theta}(t)r, \quad \ddot{x}_G(t) = \ddot{\theta}(t)r$



$\leftarrow^+: F - H_r = m_w\ddot{x}_G$

$\uparrow^+: V_r + m_w g - N_r = m_w\ddot{y}_G = 0$

$\circlearrowleft^+: M_w - F_r = J_G\ddot{\theta}$

$V_r = N_r - m_w g = 4594 - 20\times 9.8 = 4398$

$H_r =$

$M_W =$

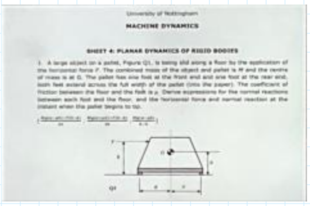
For one tyre:

N_r = 4594 N and F = 4594 N

MM2DYN Dynamics: Lecture 4

Questions:

Thursday, 27. October 2022 17:01



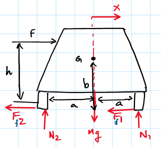
University of Nottingham

MACHINE DYNAMICS

SHEET 4: PLANAR DYNAMICS OF RIGID BODIES

1. A large object on a pallet, Figure Q1, is being slid along a floor by the application of the horizontal force F. The combined mass of the object and pallet is M and the centre of mass is at G. The pallet has one foot at the front end and one foot at the rear end; both feet extend across the full width of the pallet (into the paper). The coefficient of friction between the floor and the feet is μ. Derive expressions for the normal reactions between each foot and the floor, and the horizontal force and normal reaction at the instant when the pallet begins to tip.

Q1



$\sum F = M\underline{a}$

$\rightarrow^+$: $F - F_{f1} - F_{f2} = M\ddot{x}$ ①

$\uparrow^+$: $N_1 + N_2 - Mg = 0$ ②

$\curvearrowleft^+$: $\sum M_G = I\alpha$

$-F(h-b) - F_{f2}b - N_2 a - F_{f1}b + N_1 a = 0$ ③

② $N_1 = Mg - N_2$

③ $-F(h-b) - \mu N_2 b - N_2 a - \mu(Mg - N_2)b + (Mg - N_2)a = 0$

$N_2(-\mu b - a - \mu b - a) - F(h-b) - \mu Mgb + Mga = 0$

$\Rightarrow 2N_2(-\mu b - a) - F(h-b) + Mg(a - \mu b) = 0$

③ → $-F(h-b) - N_2 a + N_1 a - (F_{f1} + F_{f2})b = 0$

$\mu N_1 + \mu N_2 = \mu(N_1 + N_2) = \mu Mg$

$\Rightarrow -F(h-b) - N_2 a + N_1 a - \mu Mgb = 0$

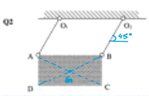
$$N_1 - N_2 = \frac{\mu Mgb + F(h-b)}{a}$$

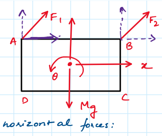
$$N_1 + N_2 = Mg$$

2. An object, in the form of a heavy uniform rectangular plate ABCD is used for impact testing of vehicles. It is hinged at its upper corners to two parallel rods (O_1A, O_2B) each 2 m long. These are hinged to a rigid support at their upper ends and have negligible mass. The plate can swing in the vertical plane as a bifilar pendulum such that edge AB remains horizontal. Find the forces in the supporting rods immediately after the plate is released from rest with the rods at 45° to the vertical. The mass of the plate is 160 kg and it has dimensions AB=2 m and BC=1 m.

[277.3 N ; 831.7 N]

Q2





⇒ Applying horizontal forces:

$\rightarrow^+$: $M\ddot{x} = F_1\cos 45 + F_2\cos 45$

$= (F_1 + F_2)\frac{\sqrt{2}}{2}$ ⟶ ①

⇒ Vertically

$\uparrow^+$: $M\ddot{y}$: $F_1 \sin 45 + F_2 \sin 45 = mg$

⇒ $M\ddot{y}$: $(F_1 + F_2)\frac{\sqrt{2}}{2} - mg$ ⟶ ②

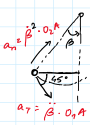
⇒ For moments:

$\curvearrowleft^+$: $I\ddot{\theta} = (F_2 \sin 45 - F_1 \sin 45)\frac{AB}{2}$

$- (F_1 \cos 45 + F_2 \cos 45)\frac{BC}{2}$

$= (F_2 - F_1)\frac{AB\sqrt{2}}{4} - (F_1 + F_2)\frac{BC\sqrt{2}}{4}$ ⟶ ③

⇒ For translation $\theta = 0$, $\dot{\theta} = 0$ & $\ddot{\theta} = 0$! The trajectories of points on ABCD are circular



Note: β and θ are not related

* At $t = 0$s, $\dot{\beta}(0) = 0$ so $a_n = 0$
* $a_{Ax} = a_t \cos 45°$ and $a_{Ay} = -a_t \sin 45$
* $\underline{a}_G = \underline{a}_A$ for translation

⇒ From $a_{Ax} = -a_{Ay}$ ⟶ $\ddot{x} = -\ddot{y}$

⇒ Combining (1) & (2) $F_1 + F_2 = Mg\frac{\sqrt{2}}{2}$ ④

⇒ From ③

$(F_2 - F_1)AB = (F_1 + F_2)BC$

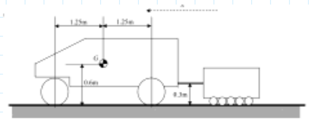
④: $F_1 + F_2 = 1109.87$

③: $2F_2 - 2F_1 = F_1 + F_2$

$3F_1 - F_2 = 0$

⇒ $F_1 = \underline{\underline{277.47}}$ N $F_2 = \underline{\underline{832.4}}$ N

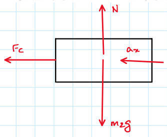
3.



Q3

3. Figure Q3 shows a *rear wheel drive* vehicle of mass 1500 kg with centre of mass at G. The vehicle is connected by a taut inextensible cable to a load of mass 500 kg that runs on rollers with negligible mass and friction. All the relevant dimensions are indicated in Figure Q3. If the coefficient of friction between the rear wheels and the ground is 0.5, determine the maximum acceleration that the vehicle and load can achieve and the corresponding normal reaction between each of the tyres and the ground. Neglect the mass of the wheels and assume that the front wheels are free to roll. Assume that changes in vehicle geometry due to suspension displacements are negligible.

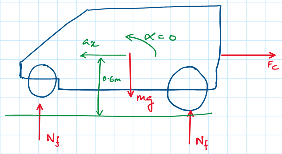
[2.053 m/s², 4105 N at each rear tyre, 3245 N at each front tyre]



$\rightarrow^+$ $\sum F_x = m\ddot{x}_G \rightarrow m_2 a_x = F_c$

$\uparrow^+$: $\sum F_y = m\ddot{y} = 0 \rightarrow 0 = N - m_2 g$

~~$\sum M_G = 0$~~



⇒ $F = \mu N_f$ (maximum acceleration conditions)

$\rightarrow^+$ $\sum F_x = m\ddot{x}_G \rightarrow F_c - F = -m_1 a_x = m_2 a_x - \mu N_f$ (1)

$\uparrow^+$ $\sum F_y = m\ddot{y}_G = 0 \rightarrow m_1 a_y = N_f + N_r - m_1 g = 0$ ②

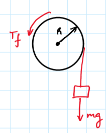
$\curvearrowleft^+$ $\sum M_G = 0 \longrightarrow 0 = 1.25N_r - 1.25N_f - 0.6\mu N_r + 0.3m_2 a_x$ ③

⇒ Solving ② & ③ for N_r & N_f & then sub in ① gives:-

$N_r = 8210$ N, $N_f = 6490$ N, $a_x = \underline{2.053}$ m s^{-2}

4. In a test to find the moment of inertia of the armature and shaft of a small electric motor, a mass of 2.5 kg attached to a cord wound round the 100 mm diameter shaft was found to be just sufficient to overcome the friction of the bearings. An additional mass of 3 kg was attached to the cord and allowed to fall freely from rest (with the cord initially just taut). At the end of 10.2 s the attached overall mass had fallen a distance of 2 m. If the bearing friction is assumed to remain constant, find the moment of inertia of the armature and shaft.

[1.898 kg.m²]

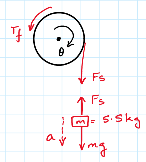


The bearing friction force T_f

$\curvearrowleft^+$: $T_f - mgr = 0$

⇒ $T_f = mgr = 2.5 \times 9.81 \times \frac{0.1}{2}$

⇒ $T_f = \underline{\underline{1.23}}$ Nm



$\ddot{\theta} = \frac{a}{r}$

$\downarrow^+$: $Ma = Mg - F_s$ ⟶ ①

$\curvearrowright^+$: $I\ddot{\theta} = F_s \cdot r - T_f$

⇒ $\frac{I\ddot{\theta}}{r} = F_s - \frac{T_f}{r}$

⇒ $\frac{Ia}{r^2} = F_s - \frac{T_f}{r}$ ⟶ ②

⇒ $h = \frac{at^2}{2}$ ⟶ $a = \frac{2h}{t^2} = \frac{2 \times 2}{(10.2)^2} = \underline{\underline{0.03845}}$ m s^{-2}

Adding ① & ②, we get:

⇒ $Ma + \frac{Ia}{r^2} = Mg - \frac{T_f}{r}$

⇒ $a\left(M + \frac{I}{r^2}\right) = Mg - \frac{T_f}{r}$

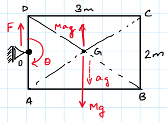
⇒ $0.03848\left(5.5 + \frac{I}{0.05^2}\right) = (5.5 \times 9.81) - \frac{1.23}{0.05}$

⇒ $5.5 + \frac{I}{0.05^2} = 762.864$

⇒ $I = \underline{\underline{1.893}}$ kg·m²

5. A homogeneous rectangular plate measuring 3m x 2m with a mass of 100 kg can rotate in a vertical plane about a horizontal, frictionless hinge O attached to the midpoint of the shorter side. If it is released from rest with OG horizontal (G is the centre of mass of the plate) find the angular acceleration of the plate and the reaction force at the hinge immediately after release.

[4.410 rad/s²; 318.5 N (vertically upwards)]



⇒ $\curvearrowright^+$: $I_O\ddot{\theta} = Mg \times 1.5$

$J_O = I_G + M\left(\frac{3}{2}\right)^2$

$J_O = \frac{1}{12} \times 100 \times (3^2 + 2^2) + 100\left(\frac{3}{2}\right)^2$

$J_O = \underline{\underline{333.3}}$ kg·m²

⇒ $333.3 \times \ddot{\theta} = 100 \times 9.81 \times 1.5$

⇒ $\ddot{\theta} = \frac{100 \times 9.81 \times 1.5}{333.3} = \underline{\underline{4.4145}}$ rad s^{-2}

At $t = 0$s, $\dot{\theta} = 0 \longrightarrow a^n_G = 0$!

$a_G = a^t_G = \ddot{\theta} \times 1.5 = 4.4145 \times 1.5$

$a_G = \underline{\underline{6.622}}$ m s^{-2}

⇒ From D'Alembert:

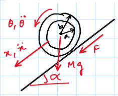
$\uparrow^+$: $F - Mg + Ma_G = 0$

$F = M(g - a_G) = 100(9.81 - 6.622)$

$F = \underline{\underline{318.8}}$ N

6. A uniform cylindrical tube of mass *M* starts to roll without sliding from rest down a slope inclined at an angle *α*. The inner and outer radii of the tube are *a* and *b*. Find the time *T* taken for the tube to roll a distance *L* down the plane. (The moment of inertia for objects can be found in the MM1DMS notes).

$[T = \sqrt{\frac{L(a^2+3b^2)}{gb^2\sin\alpha}}]$



$x = \theta b$, $\dot{x} = \dot{\theta}b$, $\ddot{x} = \ddot{\theta}b$ [rolling not skidding]

$\swarrow^+$: $M\ddot{x} = Mg\sin\alpha + F$ ⟶ ①

$\curvearrowleft^+$: $I_G\ddot{\theta} = -Fb$ ⟶ ②

⇒ ① × b^2: $M\ddot{x}b^2 = b^2 Mg\sin\alpha + Fb^2$ ⟶ ③

② × b: $I_G(\ddot{\theta}b) = -Fb^2$ ⟶ ④

⇒ ③ + ④: $I_G\ddot{x} + M\ddot{x}b^2 = b^2 Mg\sin\alpha$

$\ddot{x}(I_G + Mb^2) = b^2 Mg\sin\alpha$

⇒ $\ddot{x} = \frac{b^2 Mg\sin\alpha}{I_G + Mb^2}$

⇒ But we know that:

$I_G = \frac{1}{2}M(b^2 + a^2)$

⇒ We know that:

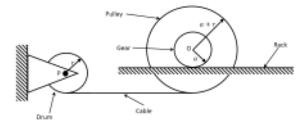
$L = \frac{\ddot{x}T^2}{2}$ ⟹ $T = \sqrt{\frac{2L}{\ddot{x}}}$

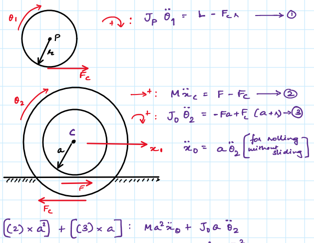
⇒ $T = \sqrt{2L \cdot \frac{I_G + Mb^2}{b^2 Mg\sin\alpha}} = \sqrt{2L \cdot \frac{\frac{1}{2}(Mb^2 + Ma^2) + Mb^2}{b^2 Mg\sin\alpha}}$

$T = \sqrt{2L \cdot \frac{1}{2}\frac{3Mb^2 + Ma^2}{b^2 Mg\sin\alpha}} = \sqrt{\frac{L(3b^2 + a^2)}{b^2 g\sin\alpha}}$

7. A drum of radius r rotating about fixed point P is used to wind up the cable that is wrapped around the pulley of radius $a + r$, centred at O. A gear of radius a, also centred at O, is integrally attached to the pulley and rolls on a fixed toothed rack as shown. The drum has moment of inertia J_P about its axis. The pulley and gear have mass M and moment of inertia J_0 about the axis through O. If a torque L is applied to the drum, find the horizontal acceleration of the gear centre O.

[$\frac{aL}{J_0+J_P+Ma^2}$]





$\circlearrowright +$: $J_P \ddot{\theta}_1 = L - F_c r \longrightarrow ①$

$\xrightarrow{+}$: $M\ddot{x}_c = F - F_c \longrightarrow ②$

$\circlearrowright +$: $J_0 \ddot{\theta}_2 = -Fa + F_c(a+r) \longrightarrow ③$

$\ddot{x}_0 = a\ddot{\theta}_2$ [for rolling without sliding]

$[(2)\times a] + [(3)\times a]$: $Ma^2\ddot{x}_0 + J_0 a\ddot{\theta}_2$

$= Fa^2 + F_c a^2$

$- Fa^2 + F_c a(a+r)$

$\Rightarrow \ddot{x}_0 (Ma^2 + J_0) = Fa^2 + F_c a^2 - Fa^2 + F_c a^2 + F_c a r$

$\Rightarrow \ddot{x}_0 (Ma^2 + J_0) = F_c a r \quad (\ast)$

⇒ In order to express F_c from (1), we have:

$\theta_1 r = \theta_2 (a+r) - \theta_2 a$ [from cable & pulley motion]

$\Rightarrow \theta_1 r = \theta_2 a + \theta_2 r - \theta_2 a$

$\Rightarrow \boxed{\theta_1 = \theta_2}$

⇒ In eq ①: ($\ddot{\theta}_1 = \ddot{\theta}_2$)

$J_P \ddot{\theta}_1 = L - F_c r$

⇒ ① × a: $F_c r a = La - J_P \ddot{x}_0$ ($= \ddot{\theta}_0 a$)

⇒ From eq $(\ast)$, we get

$\ddot{x}_0 (Ma^2 + J) = La - J_P \ddot{x}_0$

$\Rightarrow \ddot{x}_0 Ma^2 + J_0 \ddot{x}_0 = La - J_P \ddot{x}_0$

$\Rightarrow \ddot{x}_0 Ma^2 + J_0 \ddot{x}_0 + J_P \ddot{x}_0 = La$

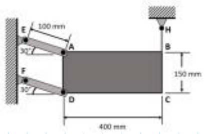
$\Rightarrow \ddot{x}_0 (Ma^2 + J_0 + J_P) = aL$

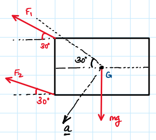
$$\Rightarrow \ddot{x}_0 = \frac{aL}{J_0 + J_P + Ma^2}$$

9. The thin homogenous plate ABCD of mass m=5 kg is held in place as shown below by the wire HB. Neglecting the mass of the links EA and FD, determine immediately after the wire has been cut:

(a) The acceleration of the centre of mass of the plate. [8.496 m/s2]

(b) The force carried by each one of the links EA and FD. [16.52 N, -6.62 N]





$m = 5\,kg$

Part (a):

$m|\underline{a}| = mg\cos 30$

$\Rightarrow |a| = 9.81\cos 30$

$= \underline{\underline{8.4957\,ms^{-2}}}$

Part (b):

$\circlearrowright$: $I\ddot{\theta} = 0 = -F_1\sin(30°)(0.2) + F_1\cos(30°)(0.075)$

$- F_2\sin(30°)(0.2) - F_2\cos(30°)(0.075)$

$= -0.1F_1 + \frac{3\sqrt{3}}{80}F_1 - 0.1F_2 - \frac{3\sqrt{3}}{80}F_2$

$\Rightarrow \left(\frac{3\sqrt{3}}{80} - 0.1\right)F_1 = \left(\frac{3\sqrt{3}}{80} + 0.1\right)F_2 \longrightarrow ①$

$\nwarrow +$: $0 = F_1 + F_2 - mg\sin(30°) \longrightarrow ②$

⇒ From ①, we get:

$$F_1 = \frac{\left(\frac{3\sqrt{3}}{80} + 0.1\right)}{\left(\frac{3\sqrt{3}}{80} - 0.1\right)} F_2$$

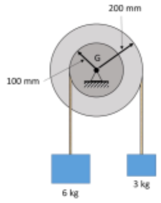
⇒ In ②:

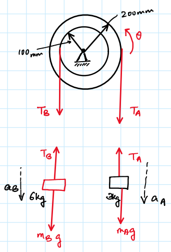
$$0 = \left[\frac{\left(\frac{3\sqrt{3}}{80} + 0.1\right)}{\left(\frac{3\sqrt{3}}{80} - 0.1\right)} + 1\right] F_2 - mg\sin(30°)$$

$\Rightarrow F_2 = \underline{\underline{-6.617\,N}}$

$F_1 = \underline{\underline{31.14\,N}}$

10. The pulley below is connected to two blocks as shown. The total moment of inertia of the system around G is equal to I_G=0.3 kgm². Assuming a frictionless system, determine the angular acceleration of the pulley and the acceleration of each block. [0 m/s2, 0 m/s2].





$l_1 = 100\,mm$

$l_2 = 200\,mm$

$m_A = 3\,kg$

$m_B = 6\,kg$

$I_G = 0.3\,kg\,m^2$

Equations of motion

$\circlearrowleft +$: $I_G\ddot{\theta} = T_B r_2 - T_A r_1 \quad ①$

$+\downarrow$: $m_A a_A = m_A g - T_A \quad ②$

$+\downarrow$: $m_B a_B = m_B g - T_B \quad ③$

⇒ From ② & ③:

$T_A = m_A g - m_A a_A$

$T_B = m_B g - m_B a_B$

⇒ In ①:-

$I_G\ddot{\theta} = r_1 \cdot m_B g - m_B a_B r_1 - (m_A g r_2 - m_A a_A r_2)$

$= r_1 m_B g - m_B a_B r_1 - m_A g r_2 + m_A a_A r_2$

$= m_B g r_1 - m_B(r_1\ddot{\theta})r_1 - m_A g r_2 + m_A(r_2\ddot{\theta})r_2$

$\Rightarrow (I_G + m_B r_1^2 - m_A r_2^2)\ddot{\theta} = m_B g r_1 - m_A g r_2$

$$\Rightarrow \ddot{\theta} = \frac{m_B g r_1 - m_A g r_2}{(I_G + m_B r_1^2 - m_A r_2^2)}$$

$$\ddot{\theta} = \frac{6(9.81)(0.1) - 3(9.81)(0.2)}{0.3 + 6(0.1)^2 - 3(0.2)^2}$$

$\ddot{\theta} = \underline{\underline{0\,rad\,s^{-2}}}$

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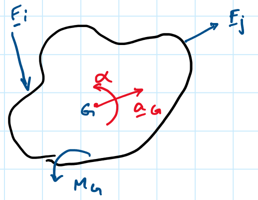
Wednesday, 2. November 2022 12:56

Planar Dynamics of Rigid Bodies:

Monday, 31. October 2022 09:01

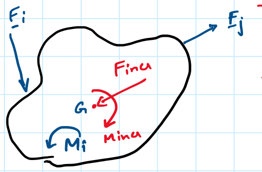


$\xrightarrow{+}$: $\sum F_x = m_G a_{Gx}$

$\uparrow +$: $\sum F_y = m_G a_{Gy}$

$\circlearrowleft +$: $\sum M_G = J_G \alpha$

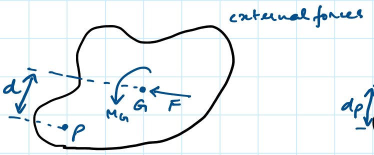
D'Alembert's principle

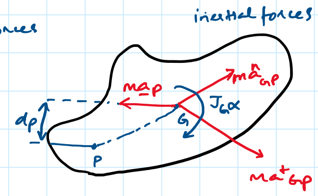


$\xrightarrow{+}$: $\sum F_x - F_x^{inertia} = \sum F_x - m_G a_{Gx} = 0$

$\uparrow +$: $\sum F_y - F_y^{inertia} = \sum F_y - m_G a_{Gy} = 0$

$\circlearrowleft +$: $\sum M_G - M^{inertia} = \sum M_G - J_G \alpha = 0$





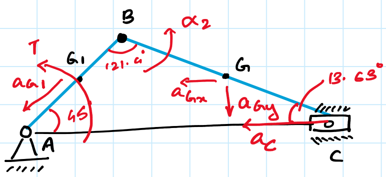
$$a_G = a_P + a^n_{GP} + a^t_{GP}$$

$$\circlearrowleft +: \; M'_P = M_G + Fd - J_G\alpha - mPG^2\alpha + m a_P d_P = 0$$

Parallel axis theorem

$$M_P + m a_P d_P = J_P \alpha$$

Worked example:



$\omega_1 = 100 \text{ rad s}^{-1}$ AB = 80 mm

BC = 240 mm BG = 120 mm

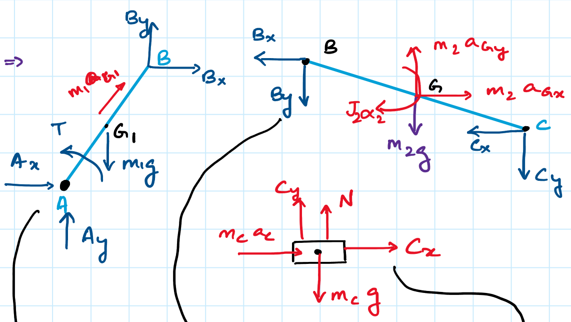
$\theta = 45°$ $m_1 = 0.02$ kg $m_2 = 0.06$ kg

$m_c = 0.2$ kg

$a_{G_1} = 400 \text{ m s}^{-2}$ $a_{Gx} = 579.7 \text{ m s}^{-2}$ $a_{Gy} = 282.8 \text{ m s}^{-2}$

$a_c = 573.7 \text{ m s}^{-2}$ $\alpha = 2283 \text{ rad s}^{-2}$

=>



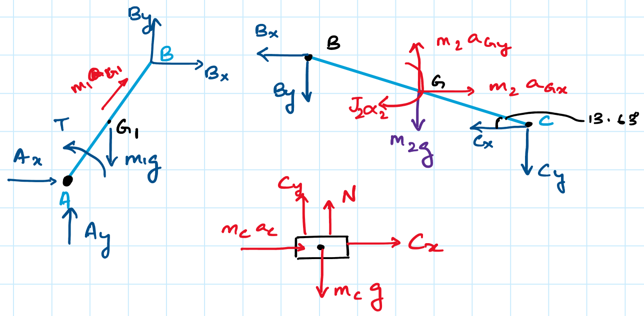
$m_1 g = 0.02 \times 9.8 = 0.196$ N

$m_2 g = 0.06 \times 9.8 = 0.588$ N

$m_c g = 0.2 \times 9.8 = 1.96$ N

$m_2 a_{Gx} = 0.06 \times 569.7$

$m_c a_c = 0.2 \times 573.7$



For BC:

$\xrightarrow{+}$ $\sum F_x^{(BC)} = 0$:

$-B_x - C_x + m_2 a_{Gx} = 0$

$B_x = m_2 a_{Gx} - C_x = 148.9$ N

$\uparrow +$ $\sum F_y^{(BC)} = 0$: $-C_y - m_2 g + m_2 a_{Gy} - B_y = 0$

$B_y = -21.0$ N

For AB:

$\xrightarrow{+}$ $F_x^{(AB)} = 0$: $B_x + A_x + m_1 a_G \cos 45° = 0$

$\uparrow +$ $F_y^{(AB)} = 0$: $B_y + A_y - m_1 g + m_1 a_G \sin 45 = 0$

$A_x = -B_x - m_1 a_G \cos 45° = -154.55$ N

$A_y = 15.53$ N

$m_1 g = 0.02 \times 9.8$

$= 0.196\,N$

$m_1 a_{G_1} = 0.02 \times 400$

$= 8\,N$

$= 0.588\,N$

$m_2 a_{Gx} = 0.06 \times 569.7$

$= 34.18\,N$

$m_G a_{Gy} = 0.06 \times 282.8$

$= 16.97\,N$

$= 196\,N$

$m_C a_C = 0.2 \times 573.7$

$= 114.7\,N$

$$J_2 \alpha_2 = \left(\frac{1}{12} m_2 BC^2\right) = \frac{1}{12} \times 0.06 \times 0.24^2 \times 2283$$

$$= 0.6575\,Nm$$

For the point C:

$\xrightarrow{+}$: $\sum F_x^C = 0$: $C_x + m_C a_C = 0$

$\hookrightarrow C_x = -m_C a_C = -114.7\,N$

$\curvearrowleft^{+}$: $\sum M_B^{BC} = 0$:

$- C_y BC \cos 13.63^\circ - C_x BC \sin 13.63^\circ$

$+ m_2 a_{Gx} BG \cos 13.6^\circ + m_2 a_{Gy} BG \sin 13.63^\circ$

$- J_2 \alpha_2 = 0 \longrightarrow$

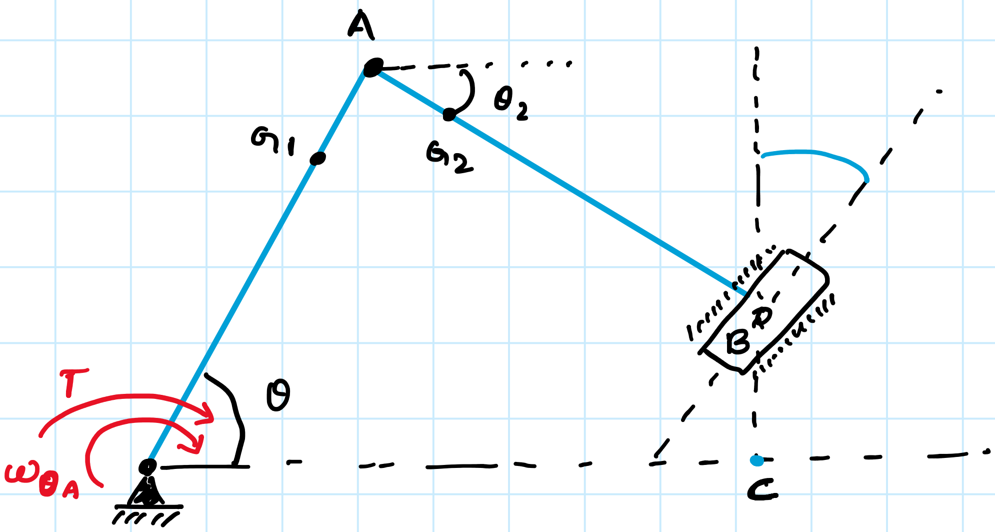
$\boxed{C_y = 37.34\,N}$

$\uparrow^{+} \sum F_y^G = 0$: $C_y + N + m_C g = 0$

$\hookrightarrow N = -m_C g - C_y = -35.38\,N$

Coursework

Monday, 31. October 2022 10:31

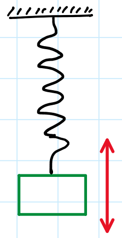


Vibrations:

Monday, 7. November 2022 09:02

Single degree of freedom:

- One mass — one direction of movement



- Works well for structures with one resonance or where one resonance dominates the vibration behaviour.

- Used as first approximation for more complicated structures.

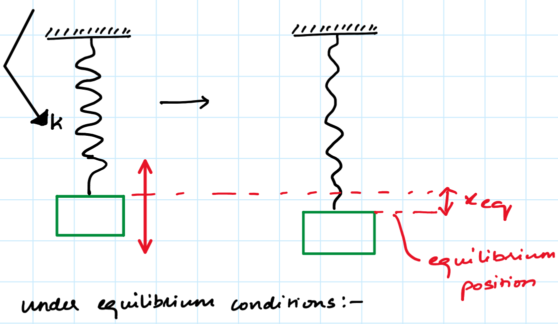
Analysis - 3 steps:

- Convert the physical structure into a dynamic mass model.

- Draw a free body diagram.
  - Create a free body by removing any restraining springs
  - Select a motion coordinate & mark it
  - Apply a positive deflection in the chosen motion coordinate, identify the forces that result and draw them on the diagram.

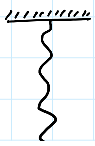
- Apply the appropriate form of Newton's 2nd law of motion to give the

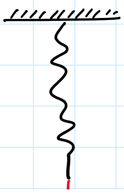
Example:



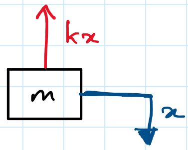
under equilibrium conditions:-

$mg = kx_{eq}$



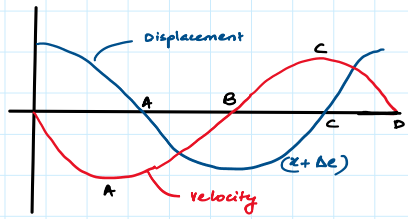


$\longrightarrow \quad -kx = m\ddot{x}$



$\Rightarrow \quad \boxed{m\ddot{x} + kx = 0}$

Equation of motion



$x = x \cos \omega t$

$\dot{x} = -\omega X \sin \omega t$

$\ddot{x} = -\omega^2 X \cos \omega t$

Hence, subbing in equation of motion:

$$m(-\omega^2 X \cos \omega t) + k X \cos \omega t = 0$$

$\Rightarrow \quad -m\omega^2 + k = 0$ } Vibrating at Natural frequency

$\Rightarrow \quad \omega^2 = \dfrac{k}{m} \quad \Longrightarrow \quad \boxed{\omega_n = \sqrt{\dfrac{k}{m}}}$ Resonance

$\Rightarrow \quad f_n\,[\text{Hz}] = \dfrac{\omega_R}{2\pi}\left[\dfrac{rad}{s}\right]$

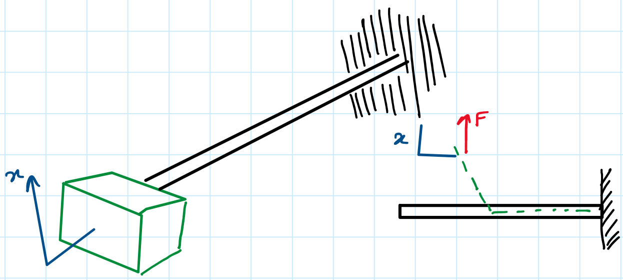
Obtaining Natural frequency of other systems:

$$M\ddot{z} + kz = 0$$

z - motion coordinate

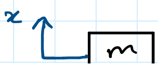
$$\omega_n = \sqrt{\frac{K}{M}}$$

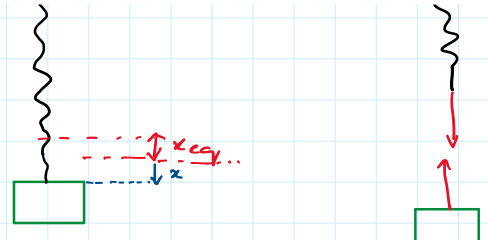
Example 2:



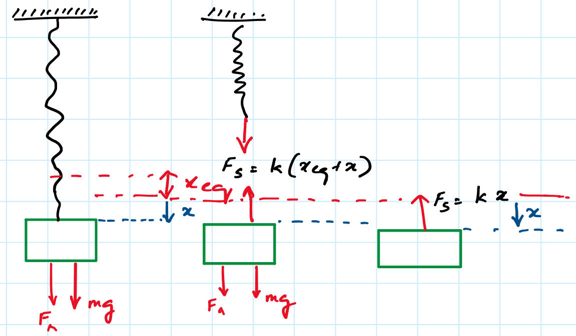
Bending Stiffness

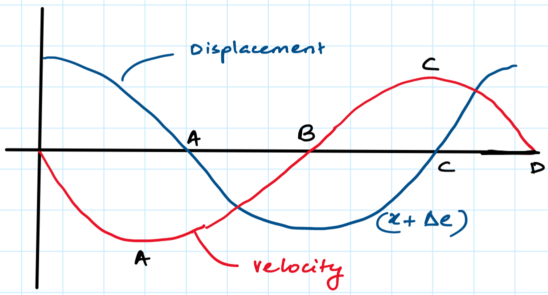
$= k_B = \dfrac{F}{x} = \dfrac{3EI}{L^3}$



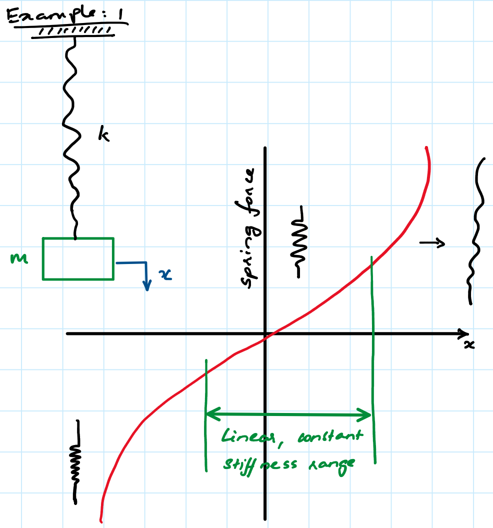


Spring forces will always resist the direction of travel & should therefore be shown to resist motion.

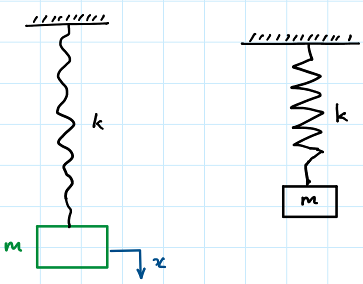




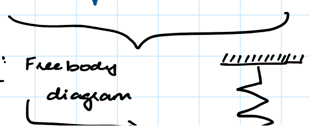
Example: 1



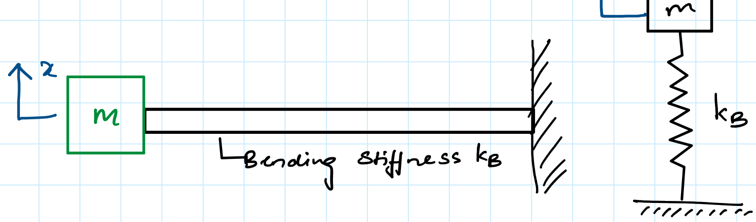
Step 1: Dynamic Spring mass model:

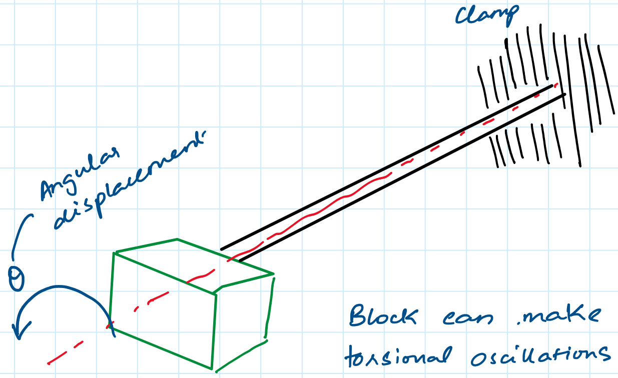


Step 2: Freebody diagram



$$= k_B = \frac{F}{x} = \frac{3EI}{L^3}$$





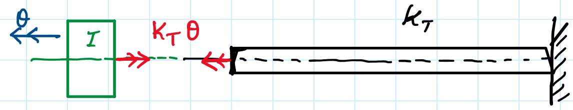
Block can make torsional oscillations

Torsional stiffness:

$$k_T = \frac{GJ}{L} \; [Nm/rad]$$

$$\omega_n = \sqrt{\frac{K_T}{I}}$$

Freebody diagram:



Resultant force in the direction of acceleration = Mass × Absolute acceleration of the centre of mass.

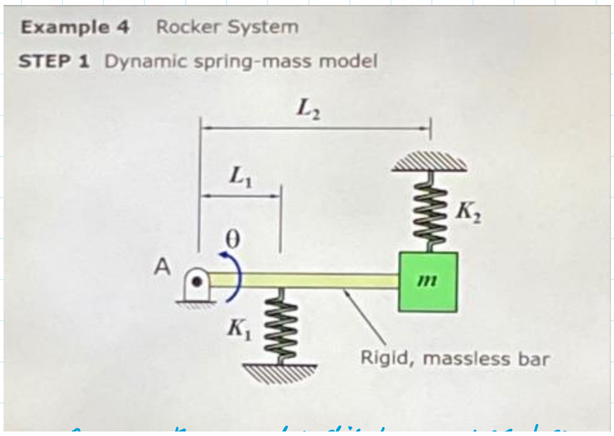
θ

$\dot{\theta}$

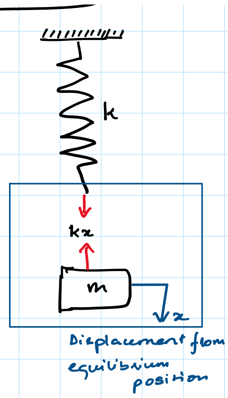
$\ddot{\theta}$

$$-k_T\theta = I\ddot{\theta}$$

or $$\boxed{I\ddot{\theta} + k_T\theta = 0}$$



Step 2: Free body diagram



Step 3: Equation of motion (Newton's 2nd)

Resultant force in the direction of acceleration = Mass × Absolute acceleration of the centre of mass.

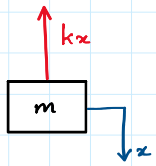
$kx = m\ddot{x}$

x

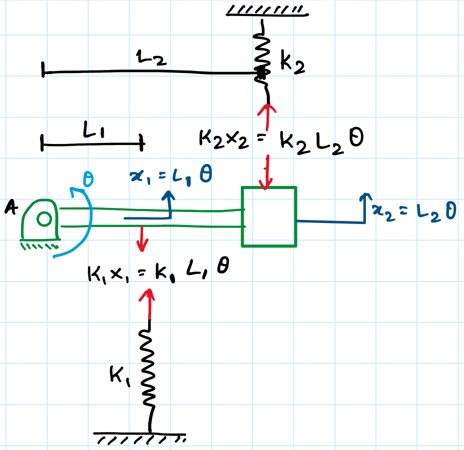
$\dot{x} — \frac{dx}{dt}$

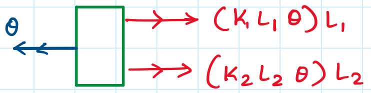
$\ddot{x} — \frac{d^2x}{dt^2}$

chosen positive direction of motion is downwards

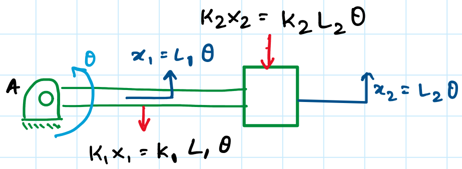


Assume the angular displacement of bar is small so $\cos\theta = 1$ & $\sin\theta = \tan\theta = 0$





$$-(K_1 L_1 \theta) L_1 - (K_2 L_2 \theta) L_2 = I_A \ddot{\theta}$$



$$I_A = m L_2^2$$

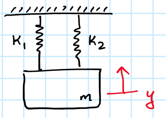
$$m L_2^2 \ddot{\theta} + (K_1 L_1^2 + K_2 L_2^2)\theta = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{K_1 L_1^2 + K_2 L_2^2}{m L_2^2}}$$

Questions:

Thursday, 10. November 2022 16:59



$F = K y$

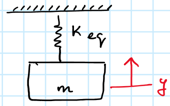
$F_{eq} = K_{eq} y$

$F_1 = K_1 y$

$F_2 = K_2 y$

$y = y_1 = y_2$

$F_{eq} \neq F_1 \neq F_2$

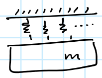


$\Rightarrow F_{eq} = F_1 + F_2$

$K_{eq} y = K_1 y + K_2 y$

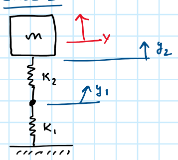
$K_{eq} = K_1 + K_2$

So,



$K_{eq} = K_1 + K_2 + K_3 + \dots + K_n$

Case 2:



$y = y_1 + y_2$

$F_{eq} = K_{eq} y$

$F_1 = K_1 y_1$

$F_2 = K_2 y_2$

$y_1 \neq y_2 \neq y$

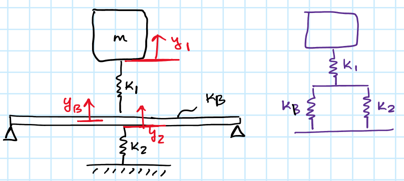
$F = F_1 = F_2$

$\Rightarrow y = \frac{F_{eq}}{K_{eq}}$, $y_1 = \frac{F_1}{K_1}$, $y_2 = \frac{F_2}{K_2}$

$\Rightarrow \frac{F_{eq}}{K_{eq}} = \frac{F_1}{K_1} + \frac{F_2}{K_2}$

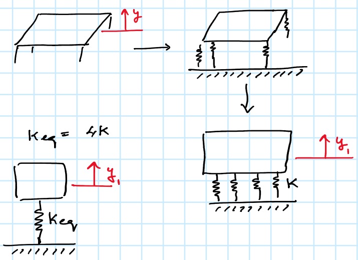
$\Rightarrow \frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2} + \dots + \frac{1}{K_n}$ (If n no. of springs)

Case 3:



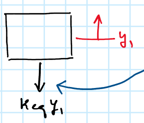
$$K_{eq} = \frac{K_1 (K_2 + K_3)}{K_1 + K_2 + K_3}$$

Example:



$K_{eq} = 4K$

$\Rightarrow$



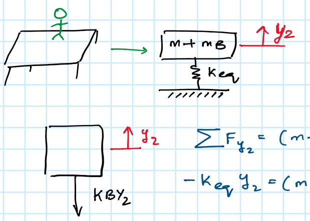
$= F_y = m\ddot{y}_1$

$-K_{eq} y_1 = m \ddot{y}_1$

$m\ddot{y}_1 + K_{eq} y_1 = 0$

$\omega_{n1} = \sqrt{\frac{K_{eq}}{m}} \longrightarrow (1)$

Example:



$\sum F_{y_2} = (m + m_B)\ddot{y}_B$

$-K_{eq} y_2 = (m + m_B)\ddot{y}_2$

$\Rightarrow (m + m_B)\ddot{y}_B + K_{eq} y_2 = 0$

1. Determine the equivalence spring stiffness, k_e, for the mass-spring system in Figure Q1a. Calculate the natural frequency of the system.

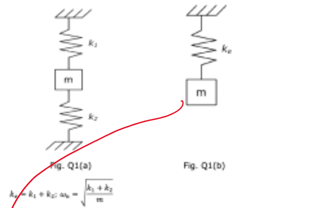
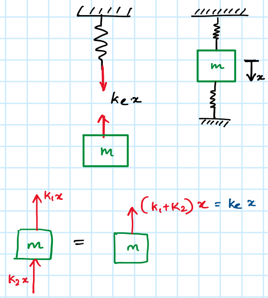


Fig. Q1(a) Fig. Q1(b)

$k_e = k_1 + k_2$; $\omega_n = \sqrt{\frac{k_1 + k_2}{m}}$



$\Rightarrow k_e \cdot x = k_1 x + k_2 x$

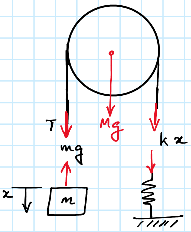
$\Rightarrow k_e = k_1 + k_2$

$\Rightarrow \omega_n = \sqrt{\frac{K_e}{M}} = \sqrt{\frac{K_1 + K_2}{m}}$

2. Draw the Free-Body-Diagram (FBD) of the mass-spring-pulley system in Figure Q2, and determine the natural frequency of the system. Note that the pulley has a uniform mass M with a radius of r.

Figure Q2

$\omega_n = \sqrt{\frac{k}{\frac{M}{2} + m}}$



Using Newton's second law:-

- for mass 'm':

$mg - T = m\ddot{x}$

$\Rightarrow T = mg - m\ddot{x} \longrightarrow (1)$

- for the pulley, the moment of inertia $J = \frac{1}{2} M r^2$

$J\ddot{\theta} = Tr - kr\theta \cdot r \longrightarrow (2)$

Subbing (1) into (2):

$\Rightarrow J\ddot{\theta} = (mg - m\ddot{x}) r - k r^2 \theta$

$\Rightarrow J\ddot{\theta} = mgr - m\ddot{x} r - k r^2 \theta$

$\Rightarrow J\ddot{\theta} + m\ddot{x} r + k r^2 \theta = mgr$

We know that:-

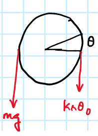
$\ddot{x} = \ddot{\theta} r$

$\therefore$ we get:-

$J\ddot{\theta} + m\ddot{\theta} r^2 + k r^2 \theta = mgr$

$(J + m r^2)\ddot{\theta} + k r^2 \theta = mgr \longrightarrow (3)$

$\Rightarrow$ If $\theta = \theta_0 + \theta_1$, where θ_0 is the static angular displacement caused by the weight, mg:



$mg = k r \theta_0$

$mgr = k r^2 \theta_0$

5. For the following systems, assume the beams are flexible and massless (i.e. they can be considered as a spring and therefore have a stiffness associated with them). Draw the correct FBD for the system about mass, M, then determine the resulting EOM and natural frequency for each system.

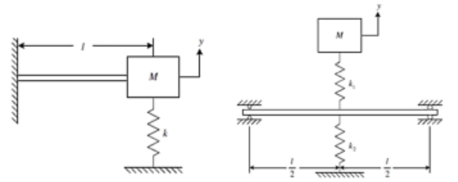
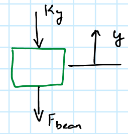


Fig. Q5(a) Fig. Q5(b)



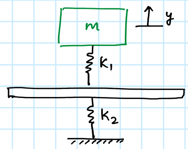
$F_{beam} = \frac{3EI}{l^3}$

EOM:

$m\ddot{y} + \left[\frac{3EI}{l^3} + K\right] y = 0$

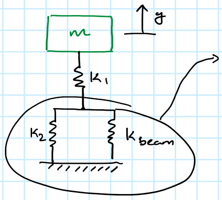
$\omega_n = \sqrt{\frac{\frac{3EI}{l^3} + K}{m}}$

(b)



Beam in parallel with k_2 because they see the same deflection.

$K_{beam} = \frac{48EI}{l^3}$



$K_{par} = K_2 + K_{beam}$

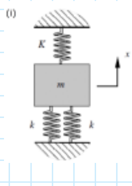
$K_{eq} = \frac{1}{\frac{1}{K_1} + \frac{1}{K_2 + K_{beam}}} = \frac{K_1 (K_2 + K_{beam})}{K_1 + K_2 + K_{beam}}$

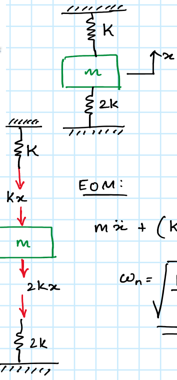
EOM

$m\ddot{y} + K_{eq} y = 0$

$\therefore \omega_n = \sqrt{\frac{K_1 (K_2 + K_{beam})}{m (K_1 + K_2 + K_{beam})}}$

6. Derive the equations of motion and hence find the natural frequencies for the vibrating systems shown in Figure Q6 (overleaf). Assume that all displacements are small. For system (iii), you should assume that the spring is pre-loaded in tension so that it never goes slack.

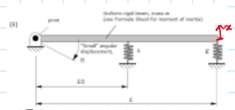




EOM:

$m\ddot{x} + (K + 2K)x = 0$

$\omega_n = \sqrt{\frac{K + 2K}{m}}$

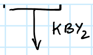


$\sum M_0 = J\ddot{\theta} = -\frac{k\theta_x L^2}{4} - K\theta_x L^2$

$\Rightarrow \frac{mL^2}{3}\ddot{\theta} = -\left(\frac{kL}{2} + KL\right)\theta_x$

$\Rightarrow \frac{mL^2}{3}\ddot{\theta} + \left(\frac{kL^2}{4} + KL^2\right)\theta_x = 0$

$\therefore \omega_n = \sqrt{\frac{\frac{kL^2}{4} + KL^2}{\frac{mL^2}{3}}} = \sqrt{\frac{3(4K + k)}{m}}$



$-K_{eq}\, y_2 = (m+m_B)\,\ddot y_2$

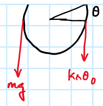
$\Rightarrow (m+m_B)\,\ddot y_B + K_{eq}\, y_2 = 0$

$\omega_{n_1} = \sqrt{\frac{K_{eq}}{m}} \rightarrow ①$ $\qquad \omega_{n_2} = \sqrt{\frac{K_{eq}}{m+m_B}} \longrightarrow ②$

$\Rightarrow K_{eq} = \omega_{n_1}^2\, m$ | $K_{eq} = \omega_{n_2}^2\,(m+m_b)$

$\Rightarrow \omega_{n_1}^2\, m = \omega_{n_2}^2\,(m+m_b)$

$\Rightarrow m = \frac{\omega_{n_2}^2\, m_B}{(\omega_{n_1}^2 + \omega_{n_2}^2)}$



$mg = kr\theta_0$

$mgr = kr^2\theta_0$

$\Rightarrow$ Eq (3) becomes:-

$\Rightarrow (J+mr^2)\,\ddot\theta_1 + kr^2(\theta_0+\theta_1) = kr^2\theta_0$

$\Rightarrow (J+mr^2)\,\ddot\theta_1 + kr^2\theta_1 = 0$

Note that $\ddot\theta = \ddot\theta_0 + \ddot\theta_1 \longleftrightarrow \ddot\theta = \ddot\theta_1$

Since $J = \frac{1}{2}Mr^2$

$$\omega_n = \sqrt{\frac{k}{\frac{M}{2}+m}}$$

3. A uniform stiff rod is restrained by linear and torsional springs as shown in Figure Q3. The stiffness of linear spring is k, while the stiffness of torsional spring is K. Calculate the natural frequency of the vertical oscillation of the rod.

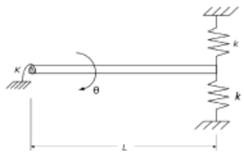
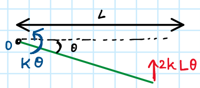


Figure Q3

$\omega_n = \sqrt{\frac{3K+6kL^2}{mL^2}}$



$\Rightarrow$ Taking moments at O:-

$\sum M_O = J\ddot\theta = -K\theta - 2kL\theta_x L$ [small θ is assumed $\sin\theta = \theta$]

$J = \frac{1}{3}mL^2$

$\Rightarrow \frac{1}{3}mL^2\ddot\theta = -K\theta_x - 2kL\theta_x L$
$= (-K - 2kL^2)\,\theta_x$

$\Rightarrow \frac{1}{3}mL^2\ddot\theta + (K+2kL^2)\,\theta = 0$

$\omega_n = \sqrt{\frac{k_T}{I}} = \sqrt{\frac{k+2kL^2}{\frac{1}{3}mL^2}}$

$\omega_n = \sqrt{\frac{3k+6kL^2}{mL^2}}$

4. For the system given below, draw the correct FBD for each mass, M, and the resulting EOM for the system. Then determine the resulting natural frequency for the systems. Assume the beam in Figure Q4(c) is rigid.

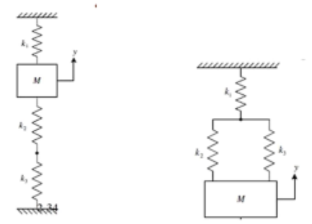
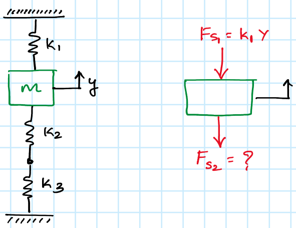
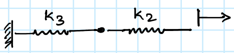


Fig. Q4(a) Fig. Q4(b)



For F_{s_2} find equivalent spring constant $F_{s_2} = K_{eq}\, y$



Spring in series

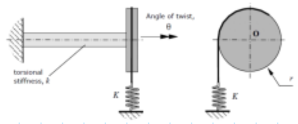
$\Delta_{tot} = \Delta_2 + \Delta_3$

$\Delta_2 = \frac{F}{K_2}$ $\qquad \Delta_3 = \frac{F}{K_3}$

$\Delta_{tot} = \frac{F}{K_2} + \frac{F}{K_3}$

$\therefore \omega_n = \sqrt{\frac{\frac{kL^2}{4} + KL^2}{\frac{mL^2}{3}}} = \sqrt{\frac{3(4K+k)}{m}}$

(iii)



moment of inertia = I

$F_{eq} = k + Kr^2$

EOM:

$I_0\ddot\theta + (k+kr^2)\,\theta = 0$

$\omega_n = \sqrt{\frac{k+Kr^2}{I_0}}$

7. A wheel (radius r, mass m, moment of inertia about its centre I) can roll without slipping on a horizontal plane. It is restrained by a horizontal spring (stiffness k) attached at one end to the centre of the wheel and at the other end to a rigid vertical wall, as in Figure Q7. Derive the equation of motion and hence find the natural frequency for the system.

What would the natural frequency be if there was no friction between the wheel and the plane?

$\omega_n = \sqrt{\frac{k}{m+\frac{I}{r^2}}}$; $\omega_n = \sqrt{\frac{k}{m}}$

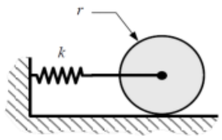
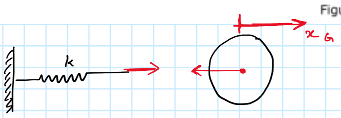
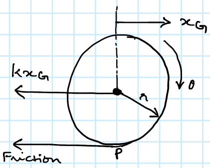


Figure Q7



FBD



$\sum$ Forces $= m\ddot x$

$-kx_G - F = m\ddot x_G$

$\Rightarrow Fr = I\ddot\theta$ | $x_G = r\theta \Rightarrow \ddot x_G = r\ddot\theta$

$F = \frac{I\ddot\theta}{r}$ | $\ddot\theta = \frac{\ddot x_G}{r}$

$\Rightarrow F = \frac{I\ddot x_G}{r^2}$ — 0

$\therefore$ E.O.M.:-

$\left(m + \frac{I}{r^2}\right)\ddot x_G + kx_G = 0$

$\therefore \omega_n = \sqrt{\frac{k}{m+\frac{I}{r^2}}}$ $\qquad$ Since F = 0 $\quad = \sqrt{\frac{K}{m}}$

8. In two of the examples from the lectures, a block is supported at its centre by a cantilever beam acting as a massless spring. The beam is 150 mm long, has a circular cross-section of diameter 16 mm and is made of steel (take Young's modulus, E = 200 GN/m2 and the shear modulus, G = 82.5 GN/m2). When tested, it is found that the natural frequencies for bending and torsional vibration of the block are 15 Hz and 23 Hz respectively.
Consider the two modes of vibration separately and calculate the mass of the block and its moment of inertia about the beam axis.

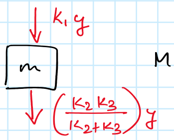
64.4 kg, 0.165 kg m²

$$\Delta_{tot} = \frac{F}{K_2} + \frac{F}{K_3}$$

$$K_{eq} = \frac{F}{\Delta_{tot}} = \frac{F}{\frac{F}{K_2} + \frac{F}{K_3}}$$

$$K_{eq} = \frac{1}{\frac{K_3 + K_2}{K_2 K_3}} = \frac{K_2 K_3}{K_3 + K_2}$$

(a)

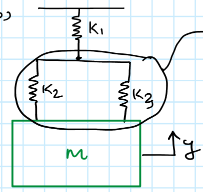


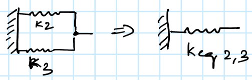
$$M\ddot{y} + (K)y = 0$$

$$M\ddot{y} + \left(K_1 + \frac{K_2 K_3}{K_2 + K_3}\right) y = 0$$

$$\omega_n = \sqrt{\frac{K_1 + \frac{K_2 K_3}{K_2 + K_3}}{m}}$$

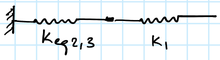
(b)





K_2 & K_3 are parallel

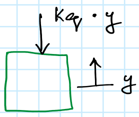
$\therefore \; K_{eq\,23} = K_2 + K_3$



K_1 & $K_{eq\,2,3}$ are in series

$$K_{eq} = \frac{1}{\frac{1}{K_1} + \frac{1}{K_{eq\,2,3}}}$$

$$= \frac{1}{\frac{1}{K_1} + \frac{1}{K_2 + K_3}}$$



$$K_{eq} = \frac{1}{\frac{K_2 + K_3 + K_1}{K_1 (K_2 + K_3)}}$$

Eq. of motion:

$$M\ddot{y} + \left[\frac{K_1 (K_2 + K_3)}{K_1 + K_2 + K_3}\right] y = 0$$

$$\therefore \; \omega_n = \sqrt{\frac{\left(\frac{K_1 (K_2 + K_3)}{K_1 + K_2 + K_3}\right)}{m}}$$

$$\omega_n = \sqrt{\frac{K_1 (K_2 + K_3)}{m (K_1 + K_2 + K_3)}}$$

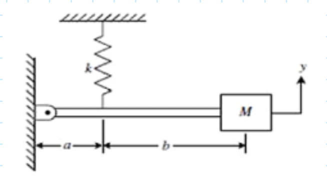


Fig. Q4(c)

Photo: Image 15.11.22 at 17.16

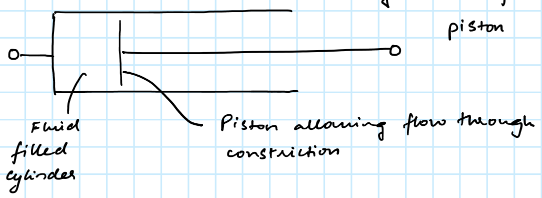
Tuesday, 15. November 2022 17:16

Damping:

Monday, 14. November 2022 09:02

Damping is a phenomenon of energy dissipation in a vibrating structure.

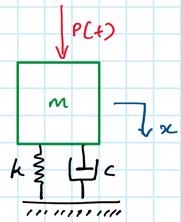
diagrammatic form of a piston

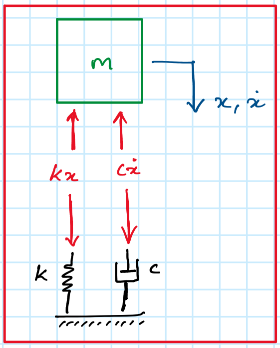


Assumption: Damper force is proportional to the relative velocity and acts in a direction to oppose the motion

$c(\dot{x} - \dot{y})$ $\quad c$ = damping coefficient

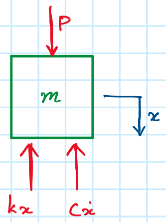
Example: Mass spring-Damper system





Equation of motion:

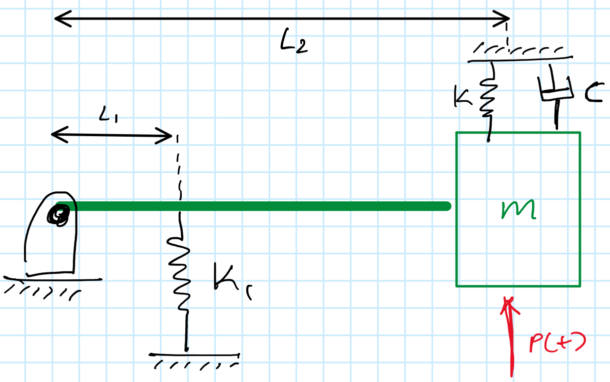
$\downarrow x, \dot{x}, \ddot{x}$ $\quad P - kx - c\dot{x} = m\ddot{x}$

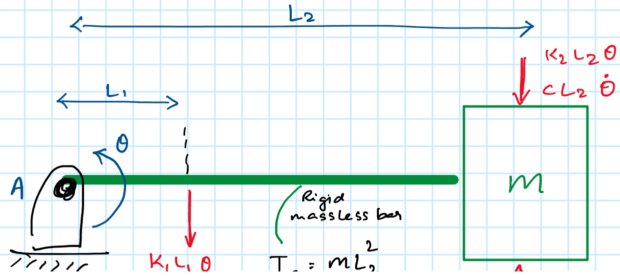


$$m\ddot{x} + c\dot{x} + kx = P(t)$$

(no (-) sign here)

System 2: Rocker system:





Four cases to consider:-

(i) Zero Damping

$$z(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4kM}}{2M}$$

For $c = 0$, $\quad \lambda = \pm \sqrt{\frac{-4kM}{2M}} =$

$$\therefore z(t) = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t}$$

$\Rightarrow e^{i\omega_n t} = \cos\omega_n t + i\sin\omega_n t$ & $e^{-i\omega_n t} = \cos\omega_n t - i\sin\omega_n t$

$$z(t) = A_1(\cos\omega_n t + i\sin\omega_n t) + A_2(\cos\omega_n t - i\sin\omega_n t)$$

$$z(t) = B\cos\omega_n t + C\sin\omega_n t$$

(ii) High Damping:

Damping ratio: ζ

$$\zeta = \frac{c}{\text{Critical Damping}} = \frac{c}{C_{cr}} = \frac{c}{2\sqrt{KM}}$$

Damping is high if $\boxed{\zeta > 1}$

A_1 & A_2 are found using initial conditions as usual.

$$A_1 = \frac{z_0 \lambda_2}{\lambda_2 - \lambda_1} \qquad A_2 = \frac{-z_0 \lambda_1}{\lambda_2 - \lambda_1}$$

when $\zeta = 1$

$$\Rightarrow C_{cr} = 2\sqrt{KM}$$

$$\therefore \quad \lambda_1 = \lambda_2 = -\frac{C_{crit}}{2M} = -\omega_n$$

$$z(t) = A_1 e^{-\omega_n t} + A_2 t e^{-\omega_n t}$$

Case 3: Light Damping

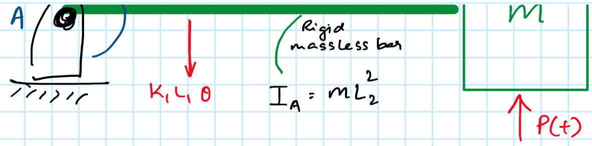
$\zeta < 1$ [Light Damping]

$$\lambda_{1,2} = -\frac{c}{2M} \pm i\frac{\sqrt{4KM - c^2}}{2M}$$

Damping ratio $\quad \zeta = \frac{c}{\text{critical damping}} = \frac{c}{2\sqrt{KM}}$

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$\therefore \quad z(t) = A_1 e^{(-\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2})} + A_2 e^{(-\zeta\omega_n - i\omega_n\sqrt{1-\zeta^2})}$$



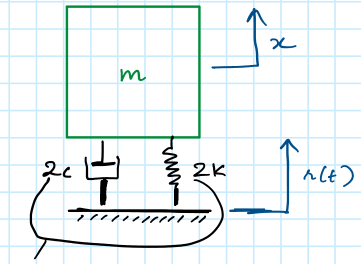
$$\Rightarrow -K_1 L_1 \theta \cdot L_1 - K_2 L_2 \theta \cdot L_2 - cL_2 \dot{\theta} \cdot L_2 + P \cdot L_2 = I_A \ddot{\theta}$$

$$\Rightarrow \boxed{mL_2^2 \ddot{\theta} + cL_2^2 \dot{\theta} + (K_1 L_1^2 + K_2 L_2^2)\theta = PL_2}$$

System 3: Single Axle Caravan:

Assumptions:

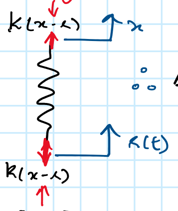
- tyres are very stiff compared to suspension springs
- tyres stay in contact with the road.



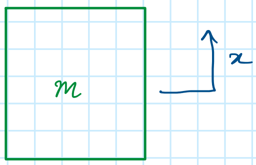
an example of Displacement excitement

Because assuming each wheel has 2 of each

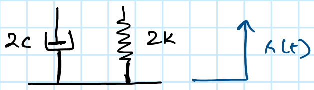
Spring force = Stiffness × change of length



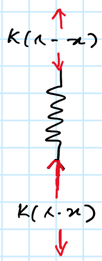
$\therefore \Delta \ell = x - r$

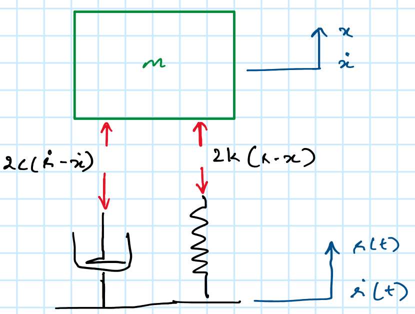


$\therefore F_{spring} = k(x - r)$



- If $(x-r)$ positive, spring is in Tension
- If $(r-x)$ positive, spring in compression

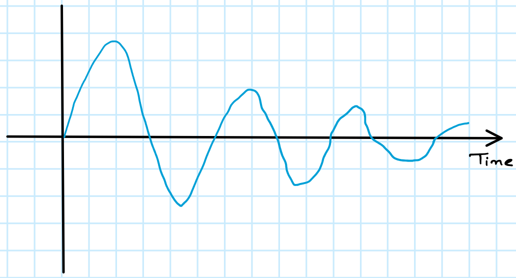




Eq of motion:

$$\therefore z(t) = A_1 e^{(-\xi\omega_n + i\omega_n\sqrt{1-\xi^2})} + A_2 e^{(-\xi\omega_n - i\omega_n\sqrt{1-\xi^2})}$$

$$z(t) = e^{-\xi\omega_n t}\left[B_1 \cos \omega_n\sqrt{1-\xi^2}\, t + B_2 \sin \omega_n\sqrt{1-\xi^2}\, t\right]$$



Alternative :-

$$z(t) = C_0 e^{-\xi\omega_n t} \cos\left(\omega_n\sqrt{1-\xi^2}\, t - \psi\right)$$

Frequency of vibrations:

$$\Omega_n = \omega_n \sqrt{1-\xi^2}$$

Damped natural frequency

To determine the free response of any system all you need to do is know what damping level it contains and choose the corresponding equation to solve.

Case (i) Zero Damping $C = 0$

$$z(t) = B \cos \omega_n t + C \sin \omega_n t \quad (4)$$

Case (ii) High Damping $\gamma > 1$ $(C^2 > 4KM)$

$$z(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \quad (3) \quad \text{where} \quad \lambda_{1,2} = \frac{-C \pm \sqrt{C^2 - 4KM}}{2M}$$

Case (iii) Critical damping $\gamma = 1$ $(C^2 = 4KM)$

$$z(t) = A_1 e^{-\omega_n t} + A_2 t e^{-\omega_n t} \quad (6)$$

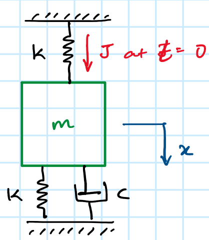
Case (iv) Light Damping $\gamma < 1$ $(C^2 < 4KM)$

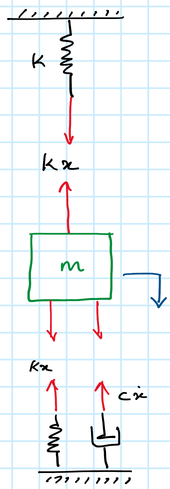
$$z(t) = e^{-\gamma\omega_n t}\left[B_1 \cos \omega_n\sqrt{1-\gamma^2}\, t + B_2 \sin \omega_n \sqrt{1-\gamma^2}\, t\right] \quad (10)$$

OR

$$z(t) = C_0 e^{-\gamma\omega_n t} \cos\left(\omega_n\sqrt{1-\gamma^2}\, t - \psi\right) \quad (11) \qquad \text{where} \quad \gamma = \frac{C}{2\sqrt{KM}}$$

Worked example:





$K = 500$ N/m

$c = 20$ Ns/m

$m = 10$ kg

$$\Rightarrow -2kx - c\dot{x} = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + 2kx = 0$$

$$\therefore M\ddot{z} + C\dot{z} + Kz = 0$$

$$\omega = \sqrt{\frac{K}{M}} = \sqrt{\frac{2k}{m}} = 10 \text{ rad s}^{-1}$$

Eq of motion:

$\uparrow x \quad + 2k(r - x) + 2c(\dot{r} - \dot{x}) = m\ddot{x}$

$\Rightarrow \quad m\ddot{x} + 2c\dot{x} + 2kx = 2c\dot{r}(t) + 2k\,r(t)$

Generic equation:

$$M\ddot{z} + C\dot{z} + Kz = F(t)$$

3 types of responses: depends of nature of excitation function.

A: "FREE Vibration" - no external forces

(i) Zero damping

(ii) High damping

(iii) Critical damping

(iv) Light damping

B: Forced Vibration - Response to sinusoidal excitation

C: Forced Vibration - Response to Periodic excitation

∴ A: "Free Vibration" - no external forces.

we say $F(t) = 0$, so :-

$$z(t) = A\cos\lambda t = Ae^{\lambda t}$$

$$\dot{z}(t) = -\lambda A\sin\lambda t = \lambda A e^{\lambda t}$$

$$\ddot{z}(t) = -\lambda^2 A\cos\lambda t = \lambda^2 A e^{\lambda t}$$

$\Rightarrow M\lambda^2 Ae^{\lambda t} + C\lambda Ae^{\lambda t} + KAe^{\lambda t} = 0$

$$M\lambda^2 + C\lambda + K = 0$$

$\Rightarrow \quad \lambda_{1,2} = \dfrac{-C \pm \sqrt{C^2 - 4KM}}{2M}$

$\Rightarrow$ Complete solution for position as a function of time:-

$$z(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

A_1 & A_2 are usually found from initial conditions given in numerical or plot form.

$$\omega = \sqrt{\frac{K}{M}} = \sqrt{\frac{2k}{m}} = 10\,\text{rad s}^{-1}$$

Damping ratio:-

$$\zeta = \frac{C}{2\sqrt{KM}} = \frac{c}{2\sqrt{2kM}} = \underline{\underline{0.1}}$$

Under damped system

$$\Rightarrow x(t) = e^{-\zeta\omega_n t}\left[B_1\cos\Omega_n t + B_2\sin\Omega_n t\right]$$

$$\Omega_n = \omega_n\sqrt{1-\zeta^2} = \underline{\underline{9.9\,\text{rad s}^{-1}}}$$

(i) initial condition:-

$t = 0, \quad x = 0 \quad \therefore B_1 = 0$

$$\therefore x(t) = B_2 e^{-\zeta\omega_n t}\sin\Omega_n t$$

(ii) Initial velocity: $J = 5\,\text{Ns}$

$$J = m(\dot{x}_0 - 0)$$

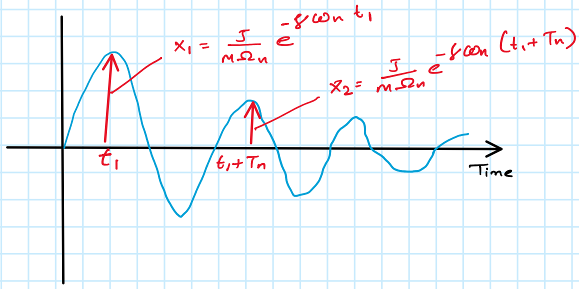
$$\dot{x}_0 = \frac{J}{m}$$

$$\dot{x} = B_2\left[\Omega_n e^{-\zeta\omega_n t}\cos\Omega_n t - \zeta\omega_n e^{-\zeta\omega_n t}\sin\Omega_n t\right]$$

$\Rightarrow \dot{x} = \dfrac{J}{m}$ at $t = 0$ $\quad \therefore \dfrac{J}{m} = B_2\left[\Omega_n - 0\right]$

$$\Rightarrow B_2 = \underline{\underline{\frac{J}{m\Omega_n}}}$$

$$\Rightarrow x(t) = \frac{J}{m\Omega_n}e^{-\zeta\omega_n t}\sin\Omega_n t$$



Ratio of amplitudes is $\dfrac{x_1}{x_2} = e^{\zeta\omega_n T_n}$

Period of damped Vibration: $T_n = \dfrac{2\pi}{\Omega_n}$

$$= \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} \approx \frac{2\pi}{\omega_n}$$

Assuming damping is very low:-

$$\frac{x_1}{x_2} = e^{\zeta(2\pi)}$$

$$\ln\left(\frac{x_1}{x_2}\right) = 2\pi\zeta$$

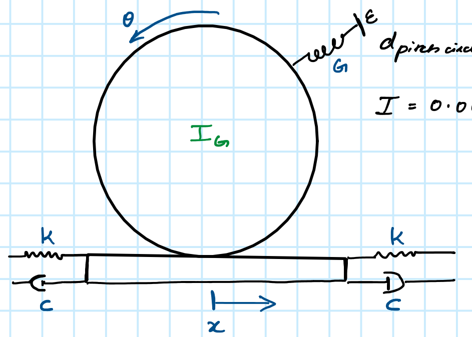
$x_1 = 0.0931$ m $\quad x_2 = 0.0229$ m

$\zeta = 0.101 \quad c = 20.1$ Ns/m

Questions

Thursday, 17. November 2022 17:03

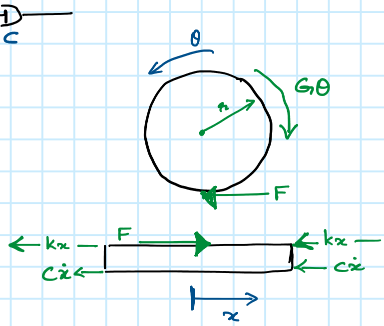
ES4 Q-6



$d_{\text{pitch circle}} = 200\,mm$

$I = 0.005\ kg\,m^2$

$$\left.\begin{aligned} x &= r\sin\theta \\ x &= r\theta \\ \dot{x} &= r\dot{\theta} \\ \ddot{x} &= r\ddot{\theta} \end{aligned}\right\} \quad (3)$$



⇒ Equations of motion:

$$\sum M = I_0\ddot{\theta}$$

$$-G\theta - F\cdot r = I_0\ddot{\theta}$$

$$I\ddot{\theta} + G\theta + F\cdot r = 0 \longrightarrow (1)$$

$$\sum F_x = m\ddot{x}$$

$$-2kx - 2c\dot{x} + F = m\ddot{x}$$

$$m\ddot{x} + 2c\dot{x} + 2kx = F \longrightarrow (2)$$

$$\Rightarrow (I_0 + mr^2)\ddot{\theta} + 2cr^2\dot{\theta} + (2kr^2 + G)\theta = 0$$

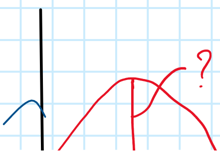
$$\omega_n = \sqrt{\frac{K}{M}} \qquad \gamma = \frac{c}{2\sqrt{KM}}$$

$$M\ddot{z} + C\dot{z} + kz = 0$$

$$\omega_n = \sqrt{\frac{2kr^2 + G}{I_0 + mr^2}} = 152.3\ rad\,s^{-1}$$

$$= 24.2\ Hz$$

$$\zeta = \frac{2cr^2}{2\sqrt{(kr^2 + G)(I_0 + mr^2)}} = 0.13$$



1. The mass-spring-damper system is shown in Figure Q1, with k = 500 N/m, c= 2 N.s/m and m=2 kg. An initial velocity of v is given to the mass at its equilibrium position ($x(0)$=0). Determine the expression for displacement of the mass, $x(t)$.

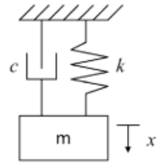


Figure Q1

⇒ checking damping ratio

$$\zeta = \frac{c}{2m\omega_n}$$

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{500}{2}} = 15.81\ rad\,s^{-1}$$

$$\zeta = \frac{2}{2\times 2\times 15.81} = 0.031 \ [\text{light damping}]$$

Damped free vibration: $x(t) = C\cdot e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$

Initial conditions: i) $x(0) = 0 = C\cdot\sin\phi \rightarrow \phi = 0$ since $c \neq 0$

ii) $\dot{x}(0) = v$

$$\dot{x}(t) = \zeta\omega_n C\cdot e^{-\zeta\omega_n t}\sin(\omega_d t) + \omega_d C\cdot e^{-\zeta\omega_n t}\cos(\omega_d t)$$

$$\dot{x}(0) = V = \omega_d\cdot C \longrightarrow C = \frac{V}{\omega_d}$$

Thus, $x(t) = \frac{V}{\omega_d}\cdot e^{-\zeta\omega_n t}\sin(\omega_d t)$

where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$

2. The system in Figure Q2 is initially at rest. The mass is displaced from its equilibrium position by 0.02 m. Find the expression for velocity of the mass, $\dot{x}(t)$, if c = 63.24 N.s/m, k=250 N/m and m= 2 kg.

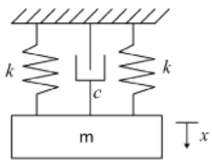


Figure Q2

$\dot{x}(t) = -4.995te^{-15.81t}$ [m/s]

⇒ $c = 63.24\ N\cdot s/m \qquad k = 250\ N/m \qquad m = 2\,kg$

⇒ Equal stiffness is :

$$K_e = k + k = 500\ N/m$$

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{500}{2}} = 15.81\ rad\,s^{-1}$$

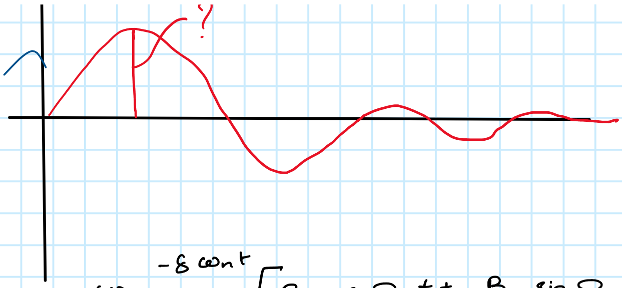
$$\Rightarrow \zeta\ (\text{damping ratio}) = \frac{c}{2\sqrt{KM}} = \frac{63.24}{2\sqrt{500\times 2}}$$

$$= 0.999 \approx 1 \ [\text{critical damping}]$$

$$x(t) = (C + Dt)\,e^{-\omega_n t}$$

Initial conditions:

(i) $x(0) = C = 0.02$



$$\theta(t) = e^{-\xi \omega_n t}\left[B_1 \cos \Omega_n t + B_2 \sin \Omega_n t\right]$$

$$\Omega_n = \omega_n \sqrt{1-\xi^2}$$

I.C. $t = 0$ $\theta = 0$

$\theta(0) = 0 = e$

Initial conditions:

(i) $x(0) = C = 0.02$

(ii) $\dot{x}(t) = D \cdot e^{-\omega_n t} + (C + Dt) e^{-\omega_n t}(-\omega_n)$

$\longleftrightarrow$ $\dot{x}(0) = D + C(-\omega_n) = 0$

$D = C \cdot \omega_n = 0.02 \times 15.81 = 0.316$

So, $x(t) = (0.02 + 0.316t)e^{-15.81t}$ [m]

$\dot{x}(t) = 0.316e^{-15.81t} - (0.316 + 4.995t)e^{-15.81t}$ [m]

$\therefore$ $\dot{x}(t) = -4.995t \cdot e^{-15.81t}$ [m/s]

3. If the same damper must be used for system in Q2, what parameters can be changed to ensure the system is lightly damped? What is the criteria for changing these parameters?

$\Rightarrow$ To ensure the system is lightly damped:

$$\xi < 1$$

$\rightarrow$ $\frac{c}{2\sqrt{KM}} < 1$ where $c = 63.24$ Ns/m

$\rightarrow$ $c < 2\sqrt{KM}$ $\longleftrightarrow$ $c^2 < 4KM$

$\rightarrow$ $KM > \left[\frac{c^2}{4} = \frac{63.24^2}{4}\right]$

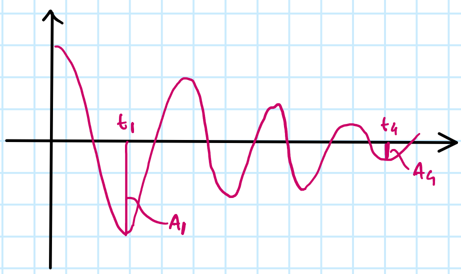
$\rightarrow$ $KM > 9.998 \times 10^2 \frac{Nkg}{m}$

Since the damper is not changed, either k or m can be increased to satisfy the above criterion

4. A mass of 2 kg is suspended from a spring of stiffness 1 kN/m in parallel with a viscous damper. When the mass is lifted and released, it is observed that in the ensuing oscillations the first downward displacement from the equilibrium position is four times as large as the fourth downward displacement. Find the damping ratio and the damping coefficient for the system by assuming small damping. Perform the same calculations without the small damping assumption and consider the difference.

$\Rightarrow$ $m = 2$kg $k = 1$ kN/m $c = ?$

$$A_1 = 4A_4$$



$A_1 = -C_0 e^{-\xi \omega_n t_1}$ $A_4 = -C_0 e^{-\xi \omega_n t_4}$

$$\frac{A_1}{A_4} = 4 = \frac{-C_0 e^{-\xi \omega_n t_1}}{-C_0 e^{-\xi \omega_n t_4}}$$

$$= e^{\xi \omega_n (t_4 - t_1)}$$

$\Rightarrow$ Based on the graph:

$$t_4 - t_1 = 3t_n$$

where t_n = period of free vibration

$\Rightarrow$ We can rewrite the eq. as:-

$-\xi \omega_n 3t_n$

⇒ We can rewrite the eq. as:-

$$e^{-\zeta \omega_n 3t_n} = 4$$

→ $$\log\left(e^{-\zeta\omega_n 3t_n}\right) = \log 4$$

$$\zeta \omega_n 3t_n = \log 4$$

⇒ Assuming small damping, i.e, $\zeta << 1$:- $C < 2\sqrt{kM}$

$$T_n \approx \frac{2\pi}{\omega_n} = 0.281$$

5. The rigid beam shown in Fig. Q5 has a moment of inertia of 10 kgm2 about the pivot at A. End C is displaced downwards by 10 mm from its equilibrium position and then released from rest. Find the maximum upward displacement of C from its equilibrium position and the elapsed time at which this occurs.

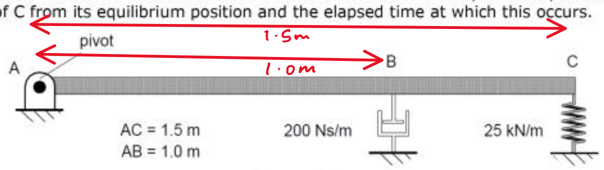
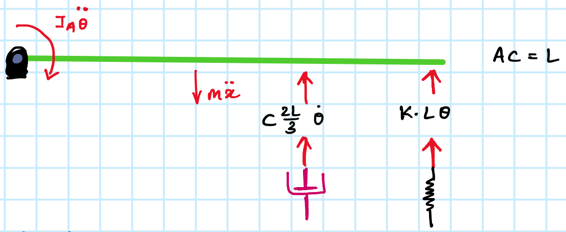


Figure Q5

⇒ $C = 200$ Ns/m $k = 25$ kN/m $I_A = 10$ kgm^2

$x = 0.010$m



At A:

↻ : $$-\left(c\frac{2L}{3}\dot\theta\right)\cdot\frac{2L}{3} - (KL\theta)\cdot L = I_A\ddot\theta$$

⇒ $$I_A\ddot\theta + \frac{4cL^2\dot\theta}{9} + KL^2\theta = 0$$

⇒ $$10\ddot\theta + 200\dot\theta + 56250\theta = 0$$

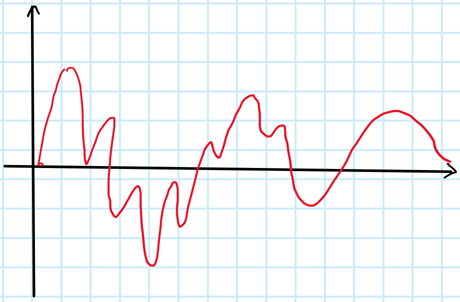
⇒ It's in the form:-

$$M\ddot z + C\dot z + kz = 0$$

$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{56250}{10}} = 75 \text{ rad s}^{-1}$$

$$\zeta = \frac{c}{2\sqrt{kM}} = 0.133$$

Monday, 21. November 2022 09:03



Forced vibration is when an alternating force or motion is applied to a mechanical system.

Harmonic Vibration is a type of forced vibration in which a force is repeatedly applied to a system

Eq. of motion:

$$M\ddot{z} + C\dot{z} + kz = f(t)$$

- complete solution of $z(t)$ consists of:-
  - Complementary function or Transient response & it's solution
  - Particular integral or steady state response.

$$z(t) = z(t)_{TR} + z(t)_{ss}$$

Complementary function is solution to the equivalent free vibration problem & provides the initial transient response.

P.I. provides the steady state part of the vibration.

Transient Response

Particular integral

$$M\ddot{z} + C\dot{z} + kz = 0$$

$$z(t)_{TR} = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

$$\lambda_{1,2} = \frac{-C \pm C\sqrt{C^2 - 4kM}}{2M}$$

$$z(t)_{ss} = Z\cos(\omega t + \alpha)$$

$$M\ddot{z} + C\dot{z} + kz = C\dot{r}(t) + kr(t)$$

$$= S_1 \cos\omega_1 t + S_2 \cos\omega_2 t$$

Method 1: Direct Substitution:

Z - amplitude of vibration

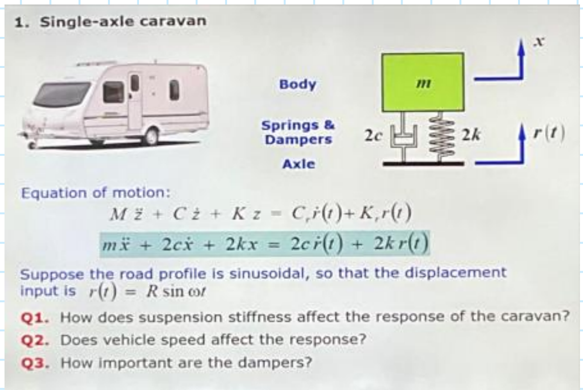
α - phase angle

Given:

$$z(t)_{ss} = Z\cos(\omega t + \alpha)$$

$$Z = \frac{F}{\sqrt{(k - M\omega^2)^2 + \omega^2 C^2}}$$

worked example:



E.O.M

$$m\ddot{x} + 2c\dot{x} + 2kx = 2c\dot{r}(t) + 2kr(t)$$

Subs:-

$r(t) = Re^{i\omega t}$

$\dot{r}(t) = i\omega Re^{i\omega t}$

$x(t)_{ss} = X^* e^{i\omega t}$

$\dot{x}(t)_{ss} = i\omega X^* e^{i\omega t}$

$\ddot{x}(t)_{ss} = -\omega^2 X^* e^{i\omega t}$

$$-m\omega^2 X^* e^{i\omega t} + 2ci\omega X^* e^{i\omega t} + 2kX^* e^{i\omega t} = 2c\omega iRe^{i\omega t} + 2kRe^{i\omega t}$$

$$-m\omega^2 X^* + 2ci\omega X^* + 2kX^* = 2c\omega iR + 2kR$$

$$(2k - m\omega^2)X^* + i2c\omega X^* = 2kR + 2c\omega iR$$

$$X^* = \frac{(K_r + iC_r\omega)R}{(K - M\omega^2) + iC\omega} = \frac{(2k + i2c\omega)R}{(2k - m\omega^2) + i2c\omega}$$

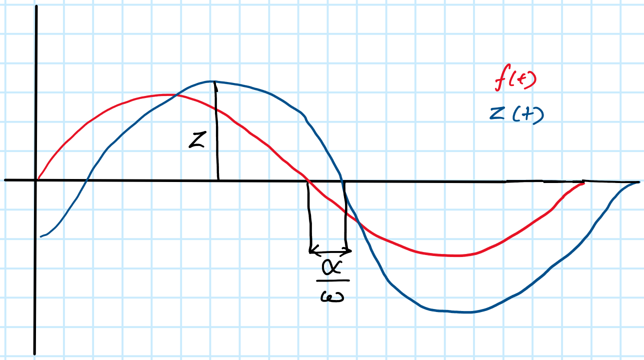
X^* in the form $= \frac{c + di}{e + fi}$

$$|X^*| = \frac{\sqrt{c^2 + d^2}}{\sqrt{e^2 + f^2}} = \frac{\sqrt{K_r^2 + C_r^2\omega^2}\ R}{\sqrt{(K - M\omega^2)^2 + C^2\omega^2}}$$

$$= \frac{\sqrt{4k^2 + 4c^2\omega^2}\ R}{\sqrt{(2k - m\omega^2)^2 + 4c^2\omega^2}}$$

$\sqrt{(K - M\omega^2)^2 + \omega^2 c^2}$

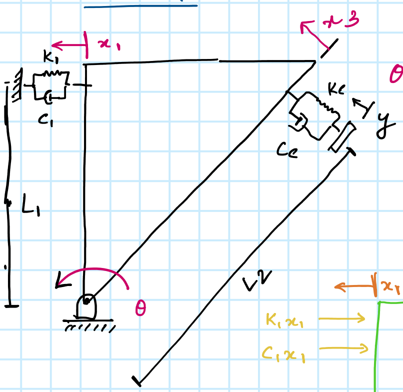
$\alpha = \tan^{-1}\left(\dfrac{\omega c}{K - M\omega^2}\right)$



Questions:

Thursday, 24. November 2022 17:01

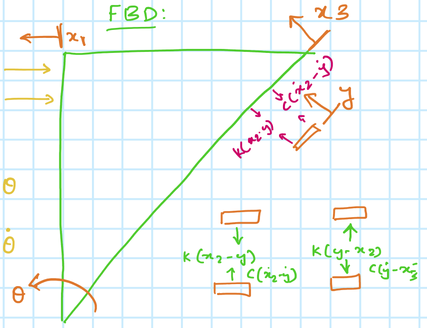
ES5 - Q3



$\theta(t) = |\Theta^*| \sin(\omega t + \alpha)$

↑ Θ^*

FBD:



$x_1 = l_1\theta$ $\quad x_2 = l_2\theta$

$\dot{x}_1 = l_1\dot{\theta}$ $\quad \dot{x}_2 = l_2\dot{\theta}$

$\ddot{x}_1 = l_1\ddot{\theta}$ $\quad \ddot{x}_2 = l_2\ddot{\theta}$

$\Rightarrow \sum T_0 = I_0\ddot{\theta}$

$\Rightarrow -(l_1 k_1 x_1) - (l_1 c_1 \dot{x}_1) - (l_2 k_2(x_2 - y)) - (l_2 c_2(\dot{x}_2 - \dot{y})) = I_0\ddot{\theta}$

$\Rightarrow -l_1 k_1 l_1\theta - l_1 c_1 l_1\dot{\theta}_1 - l_2 k_2(l_2\theta - y) - l_2 c_2(l_2\dot{\theta} - \dot{y}) = I_0\ddot{\theta}$

$\Rightarrow I_0\ddot{\theta} + (c_1 l_1^2 + c_2 l_2^2)\dot{\theta} + (k_1 l_1^2 + k_2 l_2^2)\theta = (k_2 l_2)y + (c_2 l_2)\dot{y}$

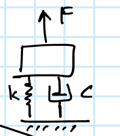
M, C, K, K_A, C_A

Equation of motion

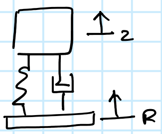
$\Rightarrow M\ddot{z} + C\dot{z} + kz = 0$

$\Rightarrow M\ddot{z} + C\dot{z} + kz = F(t)$

↳ $F \sin \omega t$, cos, $e^{\omega t}$



$\Rightarrow$



$M\ddot{z} + C\dot{z} + kz = k_A\, r(t) + C_A\, \dot{r}(t)$

$$H(\omega) = \frac{(H)^*}{y} = \frac{k_A - C_A\omega i}{(k - M\omega^2) + C\omega i}$$

$$= \frac{k_2 l_2 + c_2 l_2 \omega i}{((k_1 l_1^2 + k_2 l_2^2) - I_0\omega^2) + i(c_1 l_1^2 + c_2 l_2^2)}$$

1. A circular steel shaft (length 1 m, diameter 40 mm) is clamped at one end and carries a flywheel with a moment of inertia of 2 kgm² at the other end. The torsion formula is $\frac{2\tau_{MAX}}{d} = \frac{32T}{\pi d^4} = \frac{G\theta}{L}$, and assume that the shear modulus, G, is 80 GPa.

   i) Use the torsion formula to find the maximum shear stress in the shaft due to a ***static*** torque of 800 Nm applied to the flywheel.

   ii) Calculate the undamped natural frequency for torsional vibration.

   iii) If, instead of the static torque, a ***sinusoidally alternating*** torque with amplitude 800 Nm and frequency 12 Hz is applied to the flywheel, solve the equation of motion to find the *steady-state* amplitude of the twist in the shaft, neglecting damping. Hence find the corresponding maximum shear stress in the shaft.

   iv) A torsional damper with damping coefficient 100 Nms/rad is now connected between the flywheel and ground. Re-solve the equation of motion for the sinusoidal excitation case in part (*iii*) to obtain the new steady-state maximum shear stress.

   v) Calculate the phase angle between the angular displacement of the flywheel and the applied torque for the problem in part (*iv*).

(*i*) 63.66 MPa; (*ii*) 15.96 Hz; (*iii*) 0.09153 rad; 146.4 MPa; (*iv*) 0.06931 rad; 110.9 MPa; (*v*) -40.78°

$\Rightarrow$ we have:-

$$\frac{2\tau_{max}}{d} = \frac{32T}{\pi d^4} = \frac{G\theta}{L}$$

(i)

$$\Rightarrow \frac{2\tau_{max}}{(0.04)} = \frac{32(800)}{\pi(0.04)^4} \Rightarrow \tau_{max} = 63.66$$

(ii) Calculating torsional stiffness:-

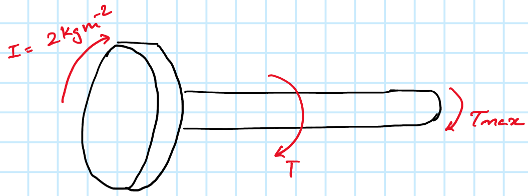
$$q = \frac{G\pi D^4}{32L} = \frac{80\times10^9 \times \pi \times (0.04)^4}{32\times 1}$$

$q = 20106.2$

$I_{flywheel} = 2\ kg\ m^{-2}$

$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}} = \frac{1}{2\pi}\sqrt{\frac{20106.2}{2}}$$

$f_n = 15.96\ Hz$



(iii) Equations of motions as derived from lectures:-

$I\ddot{\theta} + k\theta = T$

Put $T = Te^{i\omega t}$ and $\theta = \theta^* e^{i\omega t}$

$\therefore \dot{\theta} = i\omega\theta^* e^{i\omega t}$

$\ddot{\theta} = -\omega^2\theta^* e^{i\omega t}$

$\Rightarrow$ Subbing into the formula:-

$\Rightarrow -I\omega^2\theta^* e^{i\omega t} + k\theta^* e^{i\omega t} = Te^{i\omega t}$

$\Rightarrow -I\omega^2\theta^* + k\theta^* = T$

$$\Rightarrow \theta^* = \frac{T}{k - I\omega^2} = \frac{800}{}$$

$$+\, i\,(c_1 l_1^2 + c_2 l_2^2)$$

$$V_m E_f + V_f E_m = 0.34$$

$$(72.5\times10^9)V_m + (3\times10^9)V_f = 30\times10^9$$

$$V_m + V_f = 0.8$$

$$k - I\omega^2$$

$$\omega = 2\pi f = 2\pi\times 12 = 75.4\ \text{rad s}^{-1}$$

$$k = \frac{G\pi d^4}{32L} = \frac{G\pi(0.04)^4}{32(1)} =$$

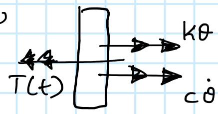
$$\theta^* = \frac{800}{(\quad - (2\times 75.4^2))}$$

$$=$$

QED

(iv) $c = 100$ Nms/rad

(v)



$$\ddot\theta:\ T - k\theta - c\dot\theta = I\ddot\theta$$

$$I\ddot\theta + c\dot\theta + k\theta = T$$

$$\therefore\ \theta = \theta^* e^{i\omega t}$$

$$\dot\theta = i\omega\,\theta^* e^{i\omega t}$$

$$\ddot\theta = -\omega^2\theta^* e^{i\omega t}$$

$$\Rightarrow -I\omega^2\theta^* e^{i\omega t} + c i\omega\,\theta^* e^{i\omega t} + k\theta^* e^{i\omega t} = T$$

$$\Rightarrow \theta^*(-I\omega^2 + c i\omega + k) = T$$

$$\Rightarrow \theta^* = \frac{T}{(k - I\omega^2) + i\omega c}$$

2. Derive the frequency response function $\left(=\frac{\Theta^*}{P}\right)$ for the system in Figure Q2.

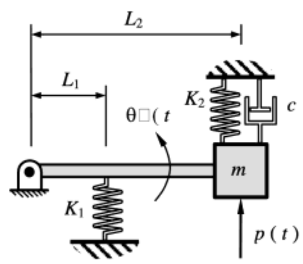


Figure Q2

$$H(\omega) = \frac{L_2}{\left(K_1 L_1^2 + K_2 L_2^2 - m L_2^2 \omega^2\right) + i\omega c L_2^2}$$

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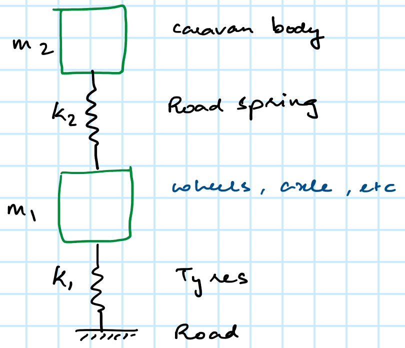
Sunday, 27. November 2022 13:51

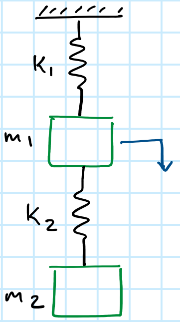
Multiple Degrees of Freedom:

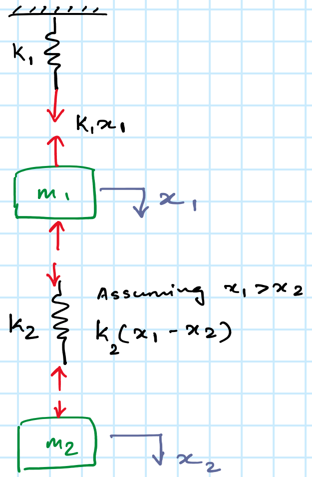
Monday, 28. November 2022 09:03



Definition of mode shape: characteristic deflection pattern for a structure when it vibrates at one of its natural frequencies.

Example 1:





If $(x_1 - x_2)$ is +ve, the spring is in compression.

Equation of motion:

$\downarrow x_1 \qquad -k_1 x_1 - k_2(x_1 - x_2) = m_1 \ddot{x}_1$

$\downarrow x_2 \qquad +k_2(x_1 - x_2) = m_2 \ddot{x}_2$

$\rightarrow \quad m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$

$\qquad m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$

In matrix form :-

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or (in gen. form)

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

To obtain natural frequencies & mode shapes:

using $x_1(t) = X_1 \cos \omega t$ & $x_2(t) = X_2 \cos \omega t$

$\ddot{x}_1(t) = -\omega^2 X_1 \cos \omega t$ & $\ddot{x}_2(t) = -\omega^2 X_2 \cos \omega t$

$\therefore \longrightarrow$ $-\omega^2 m_1 X_1 + (k_1 + k_2) X_1 - k_2 X_2 = 0$

$-\omega^2 m_2 X_2 - k_2 X_1 + k_2 X_2 = 0$

$$\begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$([K] - \omega^2 [M]) \{X\} = \{0\}$$

$$[Z] \{X\} = \{0\}$$

$$[Z] = [K] - \omega^2 [M]$$

Eigen values : natural frequencies

Eigen vectors : mode shape

$$([A] - \lambda [B]) \{X\} = \{0\}$$

For a non-trivial soltn. of eq. $\det [Z] = 0$

Multiplying out the determinant gives:-

$$m_1 m_2 \omega^4 - (m_1 k_2 + m_2 (k_1 + k_2)) \omega^2 + k_1 k_2 = 0$$

Frequency equation

$$A x^2 + B x + C = 0$$

$$x = \omega^2$$

$\therefore$ we will have $x = \omega_{n1}^2$ & $x = \omega_{n2}^2$

ω_{n1} & ω_{n2} are the two natural frequencies

$$\begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \begin{matrix} (2a) \\ (2b) \end{matrix}$$

$\Rightarrow$ Let $X_2 = 1$. Eq 2b gives:-

$$-k_2 X_1 + (k_2 - \omega^2 m_2) \times 1 = 0$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{k_2 - \omega^2 m_2}{k_2} \\ 1.0 \end{Bmatrix}$$

$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ is called the required Mode shape expression.

Example 1: Demonstration System:

$m_1 = m_2 = 2kg$ $k_1 = k_2 = 200 \, N/m$

$$\left[m_1 m_2 \omega^4 - (m_1 k_2 + m_2(k_1 + k_2))\omega^2 + k_1 k = 0 \right]$$

$\therefore \omega_{n_1}^2 = 38.1 \, s^{-2} \longrightarrow \omega_{n_1} = 6.18 \, rad \, s^{-1}$

$\omega_{n_2}^2 = 261.8 \, s^{-2} \longrightarrow \omega_2 = 16.18 \, rad \, s^{-2}$

$\therefore \omega_{n_1} = 0.98 \, Hz$

$\omega_{n_2} = 2.58 \, Hz$

Mode shapes:

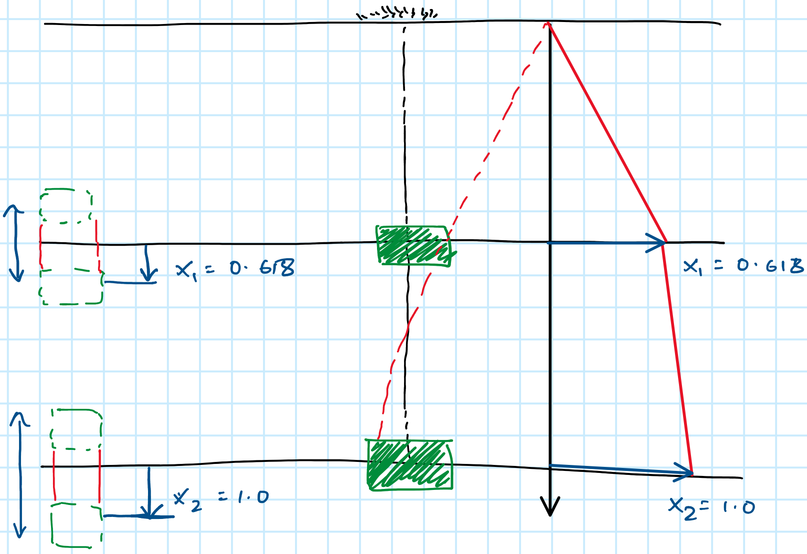
$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \frac{k_2 - \omega^2 m_2}{k_2} \\ 1.0 \end{Bmatrix}$$

when $\omega_{n_1}^2 = 38.1 \, s^{-2}$

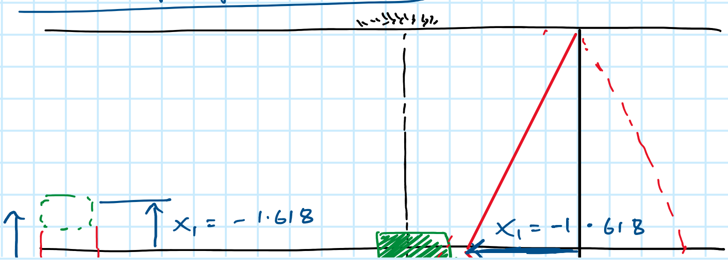
$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0.618 \\ 1.0 \end{Bmatrix}$$

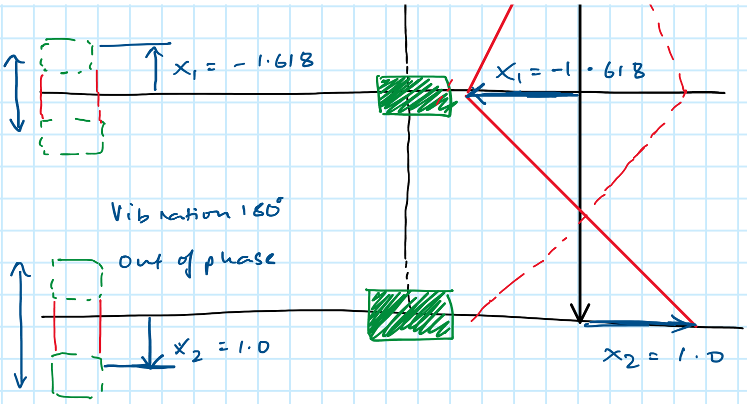
when $\omega_{n_2}^2 = 261.8 \, s^{-2}$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} -1.618 \\ 1.0 \end{Bmatrix}$$



Mode shape for mode 2:

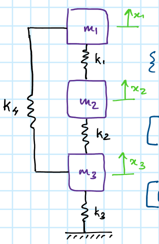




$x_1 = -1.618$
$x_1 = -1 \cdot 618$
Vibration 180°
out of phase
$x_2 = 1.0$
$x_2 = 1.0$

Questions:

Thursday, 1. December 2022 17:04



$[Z]\{X\} = \{0\}$

$\{X\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$ $\{0\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$[Z] = [k] - \omega^2 [M]$

$[M]\{\ddot{x}_i\} + [k]\{x_i\} = \{0\}$

$\Rightarrow x_1 >> x_2 >> x_3$

$\Sigma F_{x_1} = m_1 \ddot{x}_1$

$-k_1(x_1 - x_2) - k_4(x_1 - x_3) = m_1\ddot{x}_1$

① $m_1\ddot{x}_1 + (k_1 + k_4)x_1 - k_4 x_3 - k_1 x_2 = 0$

$k_4(x_1 - x_3)$, $k_1(x_1 - x_2)$

FBD 2

$\Sigma F_{x_2} = m_2 \ddot{x}_2$

$+k_1(x_1 - x_2) - k_2(x_2 - x_3) = m_2 \ddot{x}_2$

② $m_2\ddot{x}_2 + (k_1 + k_2)x_2 - k_1 x_1 - k_2 x_3 = 0$

$k_4(x_1 - x_3)$, $k_2(x_2 - x_3)$

$\Sigma F_{x_3} = m_3 \ddot{x}_3$

$+k_4(x_1 - x_3) + k_2(x_2 - x_3) - k_3 x_3 = m_3 \ddot{x}_3$

③ $m_3\ddot{x}_3 + (k_2 + k_3 + k_4)x_3 - k_4 x_1 - k_2 x_2 = 0$

$k_3 x_3$

$\Rightarrow [M]\{\ddot{x}\} + [k]\{x\} = \{0\}$

$\rightarrow \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1+k_4 & -k_1 & -k_4 \\ -k_1 & k_1+k_2 & -k_2 \\ -k_4 & -k_2 & k_2+k_3+k_4 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

always +ve   always +ve

$\rightarrow [Z] = [k] - \omega^2 [M]$

$\Rightarrow \begin{bmatrix} k_1+k_4 & -k_1 & -k_4 \\ -k_1 & k_1+k_2 & -k_2 \\ -k_4 & -k_2 & k_2+k_3+k_4 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = [Z]$

$\Rightarrow \begin{bmatrix} k_1+k_4-m_1\omega^2 & -k_1 & -k_4 \\ -k_1 & k_1+k_2-m_2\omega^2 & -k_2 \\ -k_4 & -k_2 & k_2+k_3+k_4-m_3\omega^2 \end{bmatrix} = [Z]$

$\Rightarrow x_1(t) = x_1 \sin\omega t$ $\quad x_2(t) = x_2 \sin\omega t$ $\quad x_3 = x_3 \sin\omega t$

$x_1 \cos\omega t$

$x_1 e^{i\omega t}$

$\ddot{x}_1(t) = -\omega^2 x_1 \sin\omega t$ $\quad \ddot{x}_2(t) = -\omega^2 x_2 \sin\omega t$ $\quad \ddot{x}_3 = -\omega^2 x_3 \sin\omega t$

$\Rightarrow$ Rewriting equations of motion:

$m_1 - \omega^2 x_1 \sin\omega t + (k_1+k_4)x_1 \sin\omega t - k_1 x_2 \sin\omega t - k_4 x_3 \sin\omega t = 0$

$\rightarrow (k_1 + k_4 - m_1\omega^2)x_1 - k_1 x_2 - k_4 x_3 = 0$ (1b)

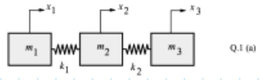
$\rightarrow (k_1 + k_2 - m_2\omega^2)x_2 - k_1 x_1 - k_2 x_3 = 0$ (2b)

$\rightarrow (k_2 + k_3 + k_4 - m_3\omega^2)x_3 - k_4 x_1 - k_2 x_2 = 0$ (3b)

$[Z]\{x_i\} = \{0\}$

$\rightarrow \begin{bmatrix} k_1+k_4-m\omega^2 & -k_1 & -k_4 \\ -k_1 & (k_1+k_2)-m_2\omega^2 & k_2 \\ -k_4 & -k_4 & k_2+k_3+k_4-m_3\omega^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

1 Derive the equations of motion for each of the following systems. Assume that all displacements and angles are small.



Q.1 (a)

$x_3 >> x_2 >> x_1$

$\Rightarrow [Z]\{X\} = \{0\}$

$\rightarrow \{X\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$ $\quad \{0\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$\Rightarrow [Z] = [k] - \omega^2 [M]$

$\Rightarrow [M]\{\ddot{x}_i\} + [k]\{x_i\} = \{0\}$

$k_1(x_1 - x_2)$

$\Sigma F_{x_1} = m_1\ddot{x}_1$

$k_1(x_2 - x_1) = m\ddot{x}_1$

$\Rightarrow m\ddot{x}_1 + kx_1 - kx_2 = 0 \longrightarrow$ ①

$-k_1(x_1 - x_2)$ $\quad k_2(x_2 - x_3)$

$\Sigma F_{x_2} = m_2\ddot{x}_2$

$\Rightarrow -k(x_2 - x_1) + k_2(x_3 - x_2) = m_2\ddot{x}_2$

$\Rightarrow m_2\ddot{x}_2 - k_1x_1 + k_1x_2 - k_2x_3 + k_2x_2 = 0$

$\Rightarrow m_2\ddot{x}_2 - k_1x_1 + (k_1 + k_2)x_2 - k_2x_3 = 0 \longrightarrow$ ②

$-k_2(x_2 - x_3)$

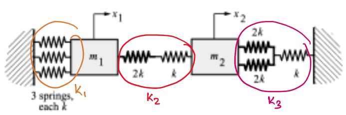
$\Sigma F_{x_3} = m_3\ddot{x}_3$

$\Rightarrow -k_2(x_3 - x_2) = m_3\ddot{x}_3$

$\Rightarrow m_3\ddot{x}_3 - k_2x_2 + k_2x_3 = 0 \longrightarrow$ ③

$[M]\{\ddot{x}\} + [k]\{x\} = \{0\}$

$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_2 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$



Q.1 (b)

$\Rightarrow$ For each case where the springs are in series:-

$\frac{1}{k_2} = \frac{1}{2k} + \frac{1}{k}$

$= \frac{1+2}{2k} = \frac{3}{2k}$

$\therefore k_2 = \frac{2k}{3}$

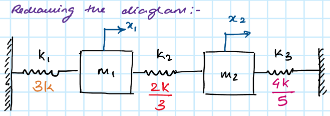
$\frac{1}{k_3} = \frac{1}{4k} + \frac{1}{k}$

$= \frac{1+4}{4k} = \frac{5}{4k}$

$\therefore k_3 = \frac{4k}{5}$

$k_1 = 3k$

Redrawing the diagram:-

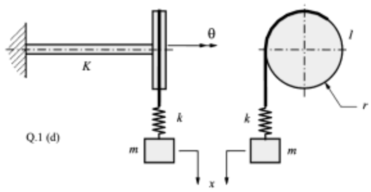


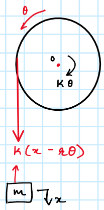
Assuming $x_2 >> x_1$:-

FBD 1:-

$-k_1x_1$ $\quad k_2(x_2 - x_1)$

$\Sigma F_{x_1} = m_1\ddot{x}_1$





Equations of motion

Disc: moment around O $\ddot{\theta}$

$\Sigma T = \Sigma I\ddot{\theta}$

$\Rightarrow k(x - r\theta)r - K\theta = I\ddot{\theta}$

$\Rightarrow I\ddot{\theta} + K\theta - kxr + kr^2\theta = 0$

$\Rightarrow I\ddot{\theta} + (K - kr^2)\theta - kxr = 0 \longrightarrow$ ①

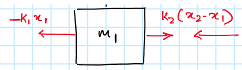
Mass:

$\Sigma F_x = m\ddot{x}$

$-k(x - r\theta) = m\ddot{x}$

$\Rightarrow m\ddot{x} - kr\theta + kx = 0 \longrightarrow$ ②

$\Rightarrow \begin{bmatrix} I & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x} \end{Bmatrix} + \begin{bmatrix} K \cdot kr & -kr \\ -kr & k \end{bmatrix} \begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$



$$\Sigma F_{x_1} = m_1 \ddot{x}_1$$

we want

$$[M]\{\ddot{x}\} + [K]\{x\} = 0$$

⇒ EOM:-

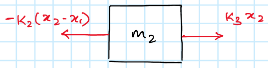
$$-k_1 x_1 + k_2(x_2 - x_1) = m_1 \ddot{x}_1$$

$$\Rightarrow -3k x_1 + \frac{2k}{3}(x_2 - x_1) = m_1 \ddot{x}_1$$

$$\Rightarrow -3k x_1 + \frac{2k x_2}{3} - \frac{2k x_1}{3} = m_1 \ddot{x}_1$$

$$\Rightarrow m_1 \ddot{x}_1 + \frac{11 k x_1}{3} - \frac{2k x_2}{3} = 0 \longrightarrow (1)$$

FBD2:



EOM:

$$\Sigma F_{x_2} = m_2 \ddot{x}_2$$

$$\Rightarrow -k_2(x_2 - x_1) + k_3 x_2 = m_2 \ddot{x}_2$$

$$\Rightarrow -\frac{2k}{3}(x_2 - x_1) + \frac{4k}{5} x_2 = m_2 \ddot{x}_2$$

$$\Rightarrow -\frac{2k x_2}{3} + \frac{2k x_1}{3} + \frac{4k}{5} x_2 = m_2 \ddot{x}_2$$

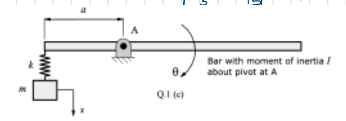
$$\Rightarrow \frac{2k x_2}{15} + \frac{2k x_1}{3} = m_2 \ddot{x}_2$$

$$\Rightarrow m_2 \ddot{x}_2 - \frac{2k x_1}{3} - \frac{2k x_2}{15} = 0 \longrightarrow (2)$$

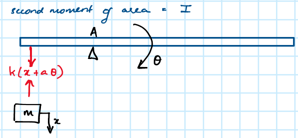
∴ We can write:-

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k \begin{bmatrix} \frac{11}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{15} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



⇒ second moment of area = I



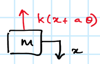
Bar around A:

$$\Sigma T_\theta = I \ddot{\theta}$$

$$\Rightarrow -k(x + a\theta)a = I \ddot{\theta}$$

$$\Rightarrow I\ddot{\theta} + kxa + ka^2\theta = 0 \longrightarrow (1)$$

Mass:



$$\Sigma F_x = m\ddot{x}$$

$$\Rightarrow -k(x + a\theta) = m\ddot{x}$$

$$\Rightarrow m\ddot{x} + kx + ka\theta = 0 \longrightarrow (2)$$

$$\Rightarrow [M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

$$[I]\{\ddot{\theta}\} + [K]\{\theta\} = \{0\}$$

$$\Rightarrow \begin{bmatrix} I & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x} \end{Bmatrix} + k \begin{bmatrix} a^2 & a \\ a & 1 \end{bmatrix} \begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

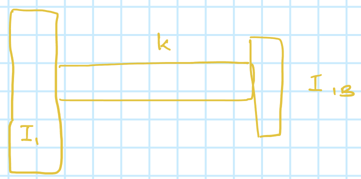
Photo: Image 07.12.22 at 17.14

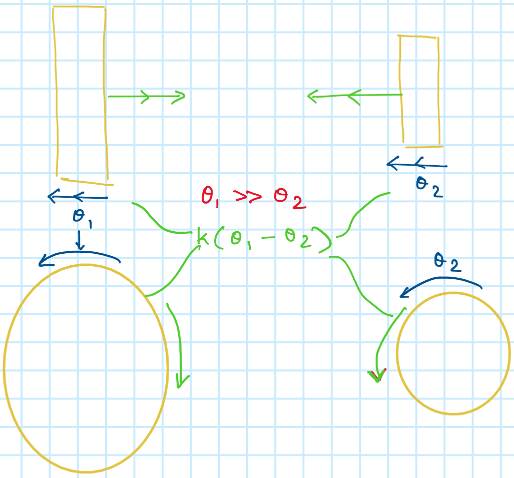
Wednesday, 7. December 2022 17:14

Multiple Degrees of Freedom:

Monday, 5. December 2022 09:02

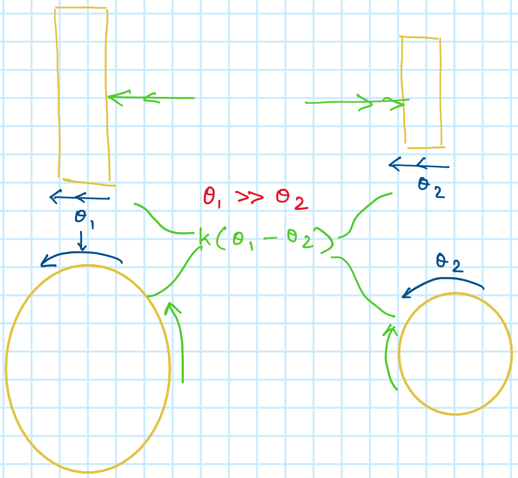
Dynamic mass-spring model:





If $\theta_2 >> \theta_1$

$k(\theta_2 - \theta_1)$



⇒ E.O.M:-

Fan:

θ_1 $\quad -k(\theta_1 - \theta_2) = I_1 \ddot{\theta}_1$

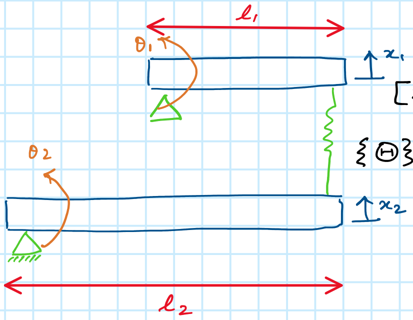
Turbine:

θ_2 $\quad +k(\theta_1 - \theta_2) = I_2 \ddot{\theta}_2$

⇒ $I_1 \ddot{\theta}_1 + k\theta_1 - k\theta_2 = 0$

$I_2 \ddot{\theta}_2 - k\theta_1 + k\theta_2 = 0$

⇒ $$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



$[Z]\{\Theta\} = \{0\}$

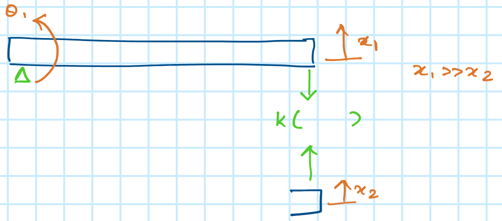
$\{\Theta\} = \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}$ $\quad \{0\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

⇒ $x_1 = l_1 \theta_1$ $\qquad x_2 = l_2 \theta_2$

$\dot{x}_1 = l_1 \dot{\theta}_1$ $\qquad \dot{x}_2 = l_2 \dot{\theta}_2$

$\ddot{x}_1 = l_1 \ddot{\theta}_1$ $\qquad \ddot{x}_2 = l_2 \ddot{\theta}_2$

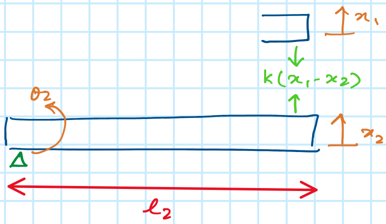
⇒ FBD



EOM: $\quad \sum T = I_1 \ddot{\theta}_1$

$-l_1 k(x_1 - x_2) = I_1 \ddot{\theta}_1$

$$I_1 \ddot{\theta}_1 + l_1^2 k \theta_1 - l_1 l_2 k \theta_2 = 0 \quad (1)$$



E.O.M:

$\sum T = I_2 \ddot{\theta}_2$

$l_2 k(x_1 - x_2) = I_2 \ddot{\theta}_2$

$$I_2 \ddot{\theta}_2 + l_2^2 k \theta_2 - l_1 l_2 k \theta_1 = 0 \quad (2)$$

$[M]\{\ddot{\theta}(t)\} + [k]\{\theta(t)\} = \{0\}$

$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \end{Bmatrix} + \begin{bmatrix} l_1^2 k & -l_1 l_2 k \\ -l_1 l_2 k & l_2^2 k \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$[Z] = [k] - \omega^2 [M]$

$$[Z] = \begin{bmatrix} l_1^2 k - \omega^2 I_1 & -l_1 l_2 k \\ -l_1 l_2 k & l_2^2 k - \omega^2 I_2 \end{bmatrix}$$

⇒ $I_1 \ddot{\theta}_1 + l_1^2 k \theta_1 - l_1 l_2 k \theta_2 = 0$

$$\Rightarrow \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

⇒ Subbing

$$\theta_1(t) = \Theta_1 \cos\omega t \quad \& \quad \theta_2(t) = \Theta_2 \cos\omega t$$

$$\rightarrow \begin{bmatrix} k - I_1\omega^2 & -k \\ -k & k - I_2\omega^2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$Z\{\Theta\} = \{0\}$$

For natural frequencies: $\det[Z] = 0$

Frequency eq:-

$$I_1 I_2 \omega^4 - k(I_1 + I_2)\omega^2 = 0$$

∴ The roots are:

$$\omega_{n_1}^2 = 0 \quad \& \quad \omega_{n_2}^2 = \frac{k(I_1 + I_2)}{I_1 I_2}$$

⇒ To find mode shapes:-

$$\begin{bmatrix} k - I_1\omega^2 & -k \\ -k & k - I_2\omega^2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

⇒ If $\Theta_2 = 1$:-

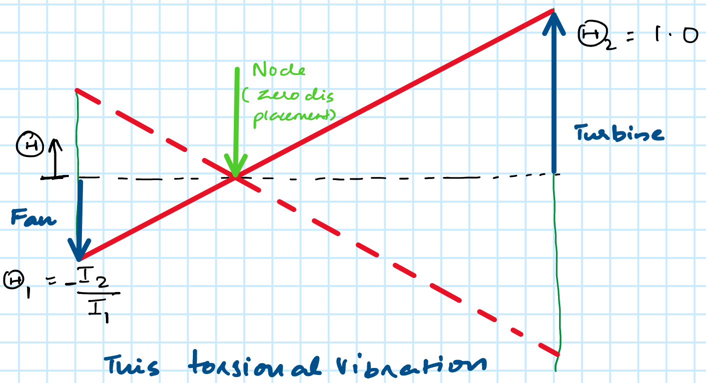
$$\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} (k - I_2\omega^2)/k \\ 1.0 \end{Bmatrix} \qquad (2)$$

Mode 1: has $\omega_{n_1} = 0$

$$\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \text{or} \quad \theta_1 = \theta_2$$

Mode 2:

$$\omega_{n_2} = \sqrt{\frac{k(I_1 + I_2)}{I_1 I_2}} \quad \text{so} \quad \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} -I_2/I_1 \\ 1.0 \end{Bmatrix}$$



This torsional vibration is superimposed on the continuous rotation

$$\Rightarrow I_1\ddot{\theta}_1 + l_1^2 k\theta_1 - l_1 l_2 k\theta_2 = 0$$

$$I_2\ddot{\theta}_2 + l_2^2 k\theta_2 - l_1 l_2 k\theta_1 = 0$$

$$\Rightarrow \theta_1 = \Theta_1 \sin\omega t \qquad \theta_2 = \Theta_2 \sin\omega t$$

$$\ddot{\theta}_1 = -\omega^2 \Theta_1 \sin\omega t \qquad \ddot{\theta}_2 = -\omega^2 \Theta_2 \sin\omega t$$

(1b) ⇒ $-\omega^2 \Theta_1 \sin\omega t + l_1^2 k \Theta_1 \sin\omega t - l_1 l_2 k \Theta_2 \sin\omega t = 0$

Similarly

(2b) $-\omega^2 \Theta_2 + l_2^2 k \Theta_2 - l_1 l_2 k \Theta_1 = 0$

$$\begin{bmatrix} k l_1^2 - \omega^2 I_1 & -l_1 l_2 k \\ -l_1 l_2 k & l_2^2 k - \omega^2 I_2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$(k l_1^2 - \omega^2 I_1)(l_2^2 k - \omega^2 I_2) - (l_1 l_2 k)(-l_1 l_2 k) = 0$$

⇒ $I_1 I_2 \omega^4 - (k l_1^2 I_2 + k l_2^2 I_1)\omega^2 = 0$
↳ Frequency equation.

$$\omega_{n_1}^2 = 0 \qquad \omega_{n_2}^2 = \frac{k l_1^2 I_2 - k l_2^2 I_1}{I_1 I_2}$$

⇒ $\Theta_1 = 1$

$$-l_1 l_2 k(1) + (l_2^2 k - \omega^2 I_2)\Theta_2 = 0$$

$$\Theta_2 = \frac{l_1 l_2 k}{l_2^2 k - \omega^2 I_2}$$

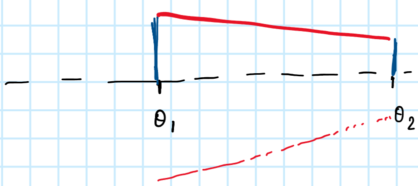
$$(k l_1^2 - \omega^2 I_1)(1) - l_1 l_2 k \Theta_2 = 0$$

$$\Theta_2 = \frac{-\omega^2 I_1 + k l_1^2}{l_1 l_2 k}$$

$$\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ \dfrac{l_1 l_2 k}{l_2^2 k - \omega^2 I_2} \end{Bmatrix}$$

$$\omega_{n_1}^2 = 0$$

$$\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ \dfrac{l_1 l_2 k}{l_2^2 k} \end{Bmatrix} = \begin{Bmatrix} 1 \\ \dfrac{l_1}{l_2} \end{Bmatrix}$$

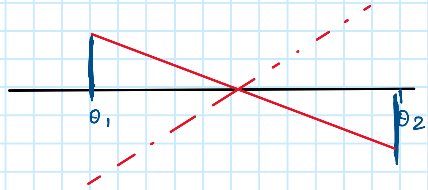


$$\omega_n^2 = \frac{k l_1^2 I_2 + k l_2^2 I_1}{I_1 I_2}$$

$$\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ \dfrac{l_1 l_2 k}{l_2^2 k - (k l_2^2 I \ldots + k l_1^2 I \ldots)} \end{Bmatrix}$$

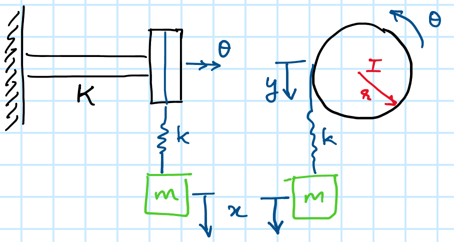
This torsional vibration is superimposed on the continuous rotation of the shaft.

$$\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ \dfrac{l_1 l_2 K}{l_2^2 K - \left(\dfrac{K l_1^2 I_2 + K l_2^2 I_1}{I_1 I_2}\right) I_2} \end{Bmatrix}$$



Questions:

Thursday, 8. December 2022 17:00

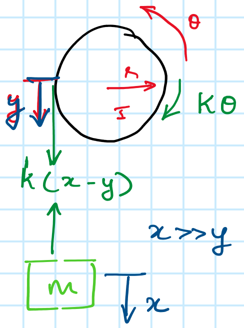


$y = r\theta$

$\ddot{y} = r\ddot{\theta}$

$y = r\theta$

FBD θ



EOM θ: $\sum M_\theta = I\ddot{\theta}$

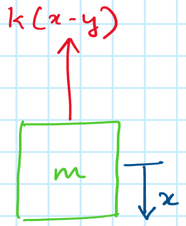
$$-K\theta + k(x-y)r = I\ddot{\theta}$$

$$-K\theta + kr(x - r\theta) = I\ddot{\theta}$$

$$I\ddot{\theta} + K\theta - krx + kr^2\theta = 0$$

$$I\ddot{\theta} + (K + kr^2)\theta - krx = 0 \rightarrow (1)$$

FBD x:



EOM x: $\sum F = m\ddot{x}$

$$-k(x-y) = m\ddot{x}$$

$$= -k(x - r\theta) = m\ddot{x}$$

$$= m\ddot{x} + kx - kr\theta = 0 \longrightarrow (2)$$

$$\longrightarrow [M]\{\ddot{z}\} + [k]\{z\} = \{0\}$$

$$\rightarrow \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x} \end{Bmatrix} + \begin{bmatrix} K + kr^2 & -kr \\ -kr & k \end{bmatrix} \begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\rightarrow [Z] = [k] - \omega^2 [M]$$

$\rightarrow x = X \cos\omega t$ $\qquad$ $\theta = \Theta \cos\omega t$

$\ddot{x} = -\omega^2 X \cos\omega t$ $\qquad$ $\ddot{\theta} = -\omega^2 \Theta \cos\omega t$

Subbing:–

$$\Rightarrow -I\omega^2 \Theta \cos\omega t + (K + kr^2)\Theta \cos\omega t - kr X \cos\omega t = 0$$

$$\Rightarrow [kr^2 + K - I\omega^2]\Theta - kr X = 0 \longrightarrow (1b)$$

$$\Rightarrow -m\omega^2 X \cos\omega t + kX \cos\omega t - kr\Theta \cos\omega t = 0$$

$$\Rightarrow (k - m\omega^2) X - kr\Theta = 0 \longrightarrow (2b)$$

$$\begin{bmatrix} kr^2 + K - I\omega^2 & -kr \\ -kr & k - m\omega^2 \end{bmatrix} \begin{Bmatrix} \Theta \\ X \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$[Z]$

$\Rightarrow$ Solve for ω_n :-

$\det|Z| = 0$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 0$$

$\Rightarrow$ $a = kr^2 + K - I\omega^2$

$b = -kr$

$c = -kr$

$d = k - m\omega^2$

$$\Rightarrow I m \omega^4 - (Ik + kmr^2 + Km)\omega^2 + Kk = 0$$

$\rightarrow Au^2 + Bu + C = 0$

$u = \omega^2$

$A = Im$ $\quad$ $B = -Ik + kmr^2 + Km$

$C = Kk$

$$\Rightarrow \begin{bmatrix} kr^2 + K - I\omega^2 & -kr \\ -kr & k - m\omega^2 \end{bmatrix} \begin{Bmatrix} \Theta \\ X \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\Rightarrow$ Giving $X = 1$

$-kr\Theta + (k - m\omega^2)(1) = 0$

$$\Rightarrow \Theta = -\frac{m\omega^2 + k}{kr}$$

$$\therefore \begin{Bmatrix} \Theta \\ X \end{Bmatrix} = \begin{Bmatrix} -\frac{m\omega^2 + k}{kr} \\ 1 \end{Bmatrix}$$

---

using eq (2b)

$$\begin{Bmatrix} \Theta \\ X \end{Bmatrix} = \begin{Bmatrix} \frac{kr}{kr^2 + K - I\omega^2} \\ 1 \end{Bmatrix}$$

---

if $\Theta = 1$

2b) $$\begin{Bmatrix} \Theta \\ X \end{Bmatrix} = \begin{Bmatrix} 1 \\ \frac{kr}{k - m\omega^2} \end{Bmatrix}$$

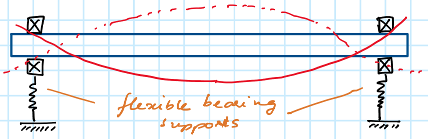
1b) $\begin{Bmatrix} \Theta \\ X \end{Bmatrix} = \begin{Bmatrix} 1 \\ \dfrac{kr^2 + K - I\omega^2}{kr} \end{Bmatrix}$

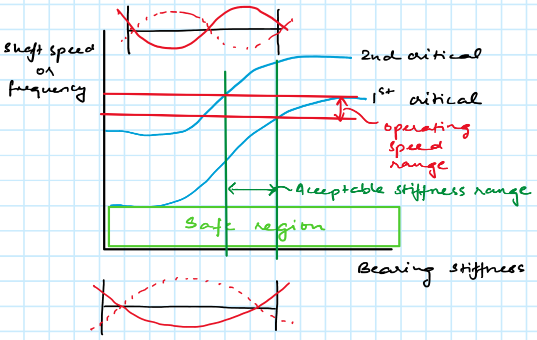
Beam Vibrations:

Monday, 12. December 2022 09:06

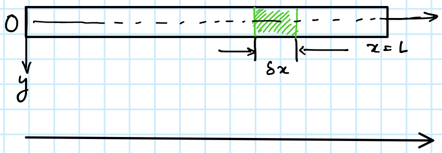
- Shaft whirl is potentially destructive, self sustaining flexural vibration observed in rotating shafts.
- It occurs if the rotational frequency of the shaft coincides with a resonant frequency for flexural vibrations.
- These speeds are called critical speeds.

Case study:





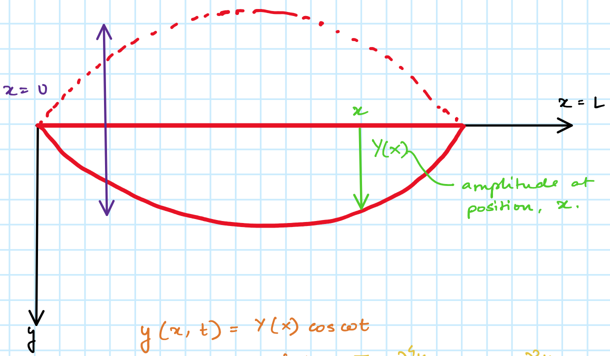
Theory for the flexural vibration of uniform Beams



analysis would lead to:-

$$EI\frac{\partial^4 y}{\partial x^4} = -\rho A\frac{\partial^2 y}{\partial t^2}$$

motion of each point on the beam is gonna be sinusoidal but amplitude of vibration will vary along the length.



$y(x,t) = Y(x)\cos\omega t$ into $EI\frac{\partial^4 y}{\partial x^4} = -\rho A\frac{\partial^2 y}{\partial t^2}$

$$\Rightarrow \frac{d^4 Y}{dx^4} = \frac{\rho A\omega^2}{EI}Y(x)$$

For a uniform cross-section, A & I are constant & its convenient to introduce the so-called wavenumber, 'λ'.

∴ Assembling the four conditions into matrix:-

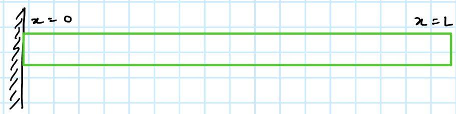
$$[Z]\{C\} = \{0\}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -\lambda^2 & 0 & \lambda^2 \\ \sin\lambda L & \cos\lambda L & \sinh\lambda L & \cosh\lambda L \\ -\lambda^2\sin\lambda L & -\lambda^2\cos\lambda L & \lambda^2\sinh\lambda L & \lambda^2\cosh\lambda L \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

For frequency: $\det[Z] = 0$ which has roots

$\lambda_n L = n\pi$ where $n = 1, 2, 3 \ldots\ldots$

Example 2: Cantilever



Boundary conditions:

Clamped end at $x = 0$, $y = 0$ & $\frac{\partial y}{\partial x} = 0$

Free end at $x = L$, $M = 0$ $\therefore \frac{\partial^2 y}{\partial x^2} = 0$

$S = 0$ $\therefore \frac{\partial^3 y}{\partial x^3} = 0$

Since $y(x,t) = Y(x)\cos\omega t$, Boundary conditions become

At $x = 0$, $Y = 0$ & $\frac{dY}{dx} = 0$

At $x = L$, $\frac{d^2 Y}{dx^2} = 0$ & $\frac{d^3 Y}{dx^3} = 0$

Assemble into Matrix form

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \lambda & 0 & \lambda & 0 \\ -\lambda^2\sin\lambda L & -\lambda^2\cos\lambda L & \lambda^2\sinh\lambda L & \lambda^2\cosh\lambda L \\ -\lambda^3\cos\lambda L & \lambda^3\sin\lambda L & \lambda^3\cosh\lambda L & \lambda^3\sinh\lambda L \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Setting up frequency eq.:-

$$\det[Z] = 0$$

After manipulation:-

$$1 + \cos\lambda L\cosh\lambda L = 0$$

Mode shapes:-

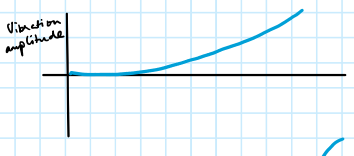
Subbing $\lambda = \lambda_n$

From the matrix above:-

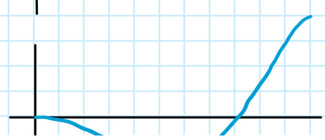
$$C_3 = -C_1 \quad \& \quad C_4 = -C_2$$

$$C_2 = -\frac{\sin\lambda_n L + \sinh\lambda_n L}{\cos\lambda_n L + \cosh\lambda_n L}C_1 = \sigma_n C_1$$

$$Y(x) = \sin\lambda_n x - \sinh\lambda_n x + \sigma_n(\cos\lambda_n x - \cosh\lambda_n x)$$



Mode 1



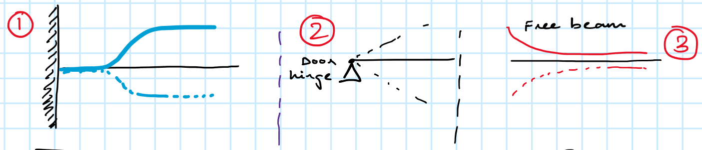
Mode 2

$\frac{d^4 Y}{dx^4} = \frac{\rho A \omega^2}{EI} Y(x)$

For a uniform cross-section, A & I are constant & its convenient to introduce the so-called wavenumber, 'λ'.

$$\lambda^4 = \frac{\rho A \omega^2}{EI}$$

$$\therefore \boxed{Y(x) = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 \sinh \lambda x + C_4 \cosh \lambda x}$$



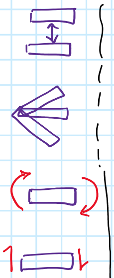
①	②	③
$\boxed{Y = 0}$ no displacement at wall	$\boxed{Y = 0}$	$Y = ?$
$\boxed{\theta = 0}$ no angular	$\theta = ?$	$\theta = ?$
$M = ?$	$\boxed{M = 0}$	$\boxed{M = 0}$ no resistance
$S = ?$	$S = ?$	$\boxed{S = 0}$ no resisting forces

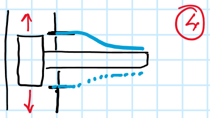
$Y(x) = d\,B\,p$

$\frac{dY}{dx} = $ slope

$\frac{d^2Y}{dx^2} = $ moment

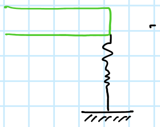
$\frac{d^3Y}{dx^3} = $ shear





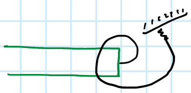
$Y = ?$
$\boxed{\theta = 0}$
$M = ?$
$\boxed{S = 0}$

Other situations:



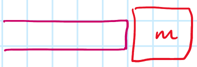
$y = ?$
$\theta = ?$
$\boxed{m = 0}$

$\boxed{S \neq 0}$ Shear force equals spring force



$Y = ?$
$\theta = ?$

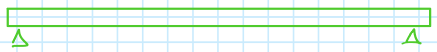
→ related to spring stiffness
$\boxed{m \neq 0}$
$\boxed{S = 0}$



$Y = ?$
$\theta = ?$

$\boxed{M \neq 0}$, $\boxed{S \neq 0}$ } due to $m \times a$ [force] acting

Example 1:



1. Boundary conditions: at $x = 0$ & $x = L$

$y = 0$ & $M = 0 \Longrightarrow \frac{\partial^2 y}{\partial x^2} = 0$

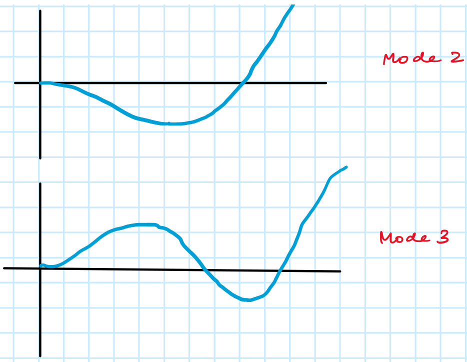
$y(x,t) = Y(x) \cos \omega t$

$Y = 0$ & $\frac{d^2Y}{dx^2} = 0$

$Y(x) = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 \sinh \lambda x + C_4 \cosh \lambda x$

$\frac{d^2y}{dx^2} = -\lambda^2 C_1 \sin \lambda x - \lambda^2 C_2 \cos \lambda x + \lambda^2 C_3 \sinh \lambda x + \lambda^2 C_4 \cosh \lambda x$

At $Y = 0$, $x = 0$ & $\frac{d^2Y}{dx^2} = 0$



dx^2

$$Y(0) = c_1 \times 0 + c_2 \times 1 + c_3 \times 0 + c_4 \times 1 = 0$$

$$\left(\frac{d^2Y}{dx^2}\right)_{x=0} = -\lambda^2 c_1 \times 0 - \lambda^2 c_2 \times 1 + \lambda^2 c_3 \times 0 + \lambda^2 c_4 \times 1 = 0$$

At $x = L$, $Y = 0$ & $\frac{d^2Y}{dx^2} = 0$

$$Y(L) = c_1 \sin\lambda L + c_2 \cos\lambda L + c_3 \sinh\lambda L + c_4 \cosh\lambda L = 0$$

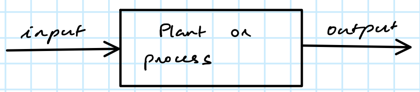
$$\frac{d^2Y}{dx^2} = -\lambda^2 c_1 \sin\lambda L - \lambda^2 c_2 \cos\lambda L + \lambda^2 c_3 \sinh\lambda L + \lambda^2 c_4 \cosh\lambda L = 0$$

Systems Modelling and Control:

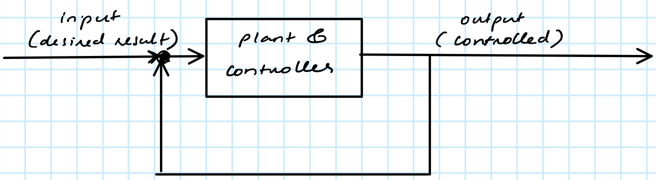
Monday, 30. January 2023 15:57

Systems and block diagrams:

- Open-loop system:

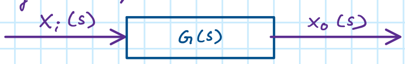


- Closed loop (feedback) system:



Representation of control systems:

- Transfer function of a linear system is formally defined as the ratio of Laplace transform of the output to the Laplace transform of the input, where initial conditions are zero.



$$X_o = G(s)X(s)$$

$$G(s) = \frac{X_o(s)}{X_i(s)} = \frac{P(s)}{Q(s)} \rightarrow \text{characteristic function when } s=0.$$

- Laplace:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

where $s = \alpha + j\omega$

$$e^{-st} = e^{-\alpha t}e^{-j\omega t} = e^{-\alpha t}(\sin\omega t + j\cos\omega t)$$

Useful Results relating Laplace:

- final value theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

theorem only valid if the final value is finite & constant.

If $\mathcal{L}[f(t)] = F(s)$ then $\mathcal{L}\{f(t-\tau)\} = e^{-s\tau}F(s)$

Example 1:

$\mathcal{L}\{f(t)\}$ where $f(t) = \frac{d^2x}{dt^2}$, $x = 2$, $\frac{dx}{dt} = 1$ & $t=0$

⇒ $F(s) = s^2X(s) - sx(0) - \dot{x}(0)$

⇒ Subbing ICs:

$$F(s) = s^2X(s) - 2s - 1$$

If ICs are each zero:

$$F(s) = s^2X(s)$$

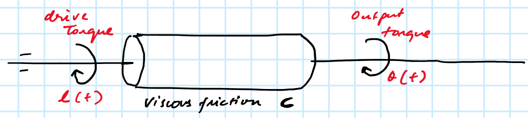
Example 2:

$$\frac{d^2x}{dt^2} + \omega_n^2 x = \cos p(t)$$

no ICs.

⇒ Taking Laplace:

(c) Rotor with Viscous Drag:



Equation of motion of this system:

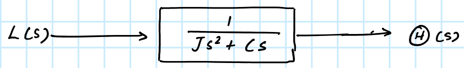
$$\ell(t) - c\dot{\theta}(t) = J\ddot{\theta}(t)$$

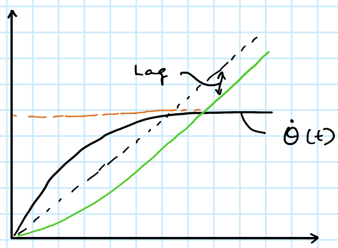
Assuming zero ICs & taking Laplace:

$$Js^2\Theta(s) + Cs\,\Theta(s) = L(s)$$

Rearranging:-

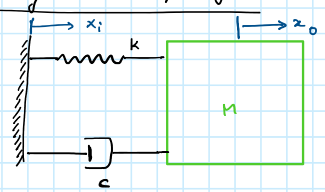
$$G(s) = \frac{\Theta(s)}{L(s)} = \frac{1}{Js^2 + Cs}$$





$$L(s) = \frac{1}{s} \qquad \Theta(s) = \frac{1}{s}\left(\frac{1}{Js^2 + Cs}\right)$$

(d) Spring mass-Damper System:



⇒ Forces on M: → $k(x_i - x_o)$

$c\frac{d}{dt}(x_i - x_o)$ or $c(\dot{x}_i - \dot{x}_o)$

$$\therefore M\frac{d^2x_o}{dt^2} - c\frac{d}{dt}(x_i - x_o) + k(x_i - x_o) = 0$$

$$M\frac{d^2x_o}{dt} - \frac{C\,dx_o}{dt} + kx_o = \frac{C\,dx_i}{dt} + kx_i$$

↓ taking Laplace:

$$(Ms^2 + Cs + k)X_o(s) = (Cs + k)X_{in}(s)$$

$$\therefore G(s) = \frac{X_o}{X_i} = \frac{Cs + k}{Ms^2 + Cs + k}$$

$$= \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Given that

$$\omega_n^2 = \frac{k}{M} \quad \& \quad \zeta = \frac{c}{2\sqrt{KM}}$$

no ICs.

⇒ Taking Laplace:

$$s^2X(s) + \omega_n^2 X(s) = \frac{s}{s^2+p^2}$$

⇒ Rearranging gives:

$$X(s) = \frac{s}{(s^2+p^2)(s^2+\omega_n^2)}$$

⇒ Reverse Laplace gives:-

$$x(t) = \frac{1}{\omega_n^2 - p^2}\left[\cos(pt) - \cos(\omega_n t)\right]$$

Example 3:

Transfer function of the system:

$$\dot{x}_0 = \frac{dx}{dt}$$

$$\dot{x}_0 + a x_0 = a x_i$$

where x_0 is output & x_i is input.

⇒ taking Laplace:

$$sX_0(s) + aX_0(s) = aX_i(s)$$

$$x_0(s)(s+a) = a x_i(s)$$

⇒ Rearranging for the transfer function:

$$G(s) = \frac{X_0(s)}{X_i(s)} = \frac{a}{s+a}$$

⇒ Output can be deduced from:

$$X_0 = G(s)\, X_i(s)$$

⇒ If the input X_i is a unit step function, then:

$$X_i(s) = \frac{1}{s}$$

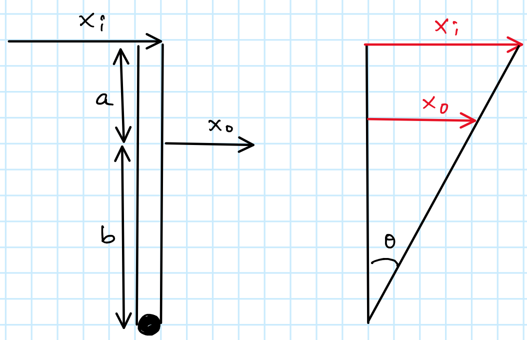
⇒ from transfer function the Laplace:

$$X_0(s) = \frac{a}{s(s+a)}$$

⇒ inverse Laplace:

$$x(t) = 1 - e^{-at}$$

Simple Lever system:



Assume displacements are 0:-

$$\tan\theta = \frac{x_i}{(a+b)} = \frac{x_0}{b}$$

$$\frac{x_0}{x_i} = \left(\frac{b}{a+b}\right)$$

Taking Laplace:

$X_0(s)$ b

$$\omega_n^2 = \frac{K}{M} \quad \& \quad \zeta = \frac{C}{2\sqrt{KM}}$$

(e) Hydraulic Ram:

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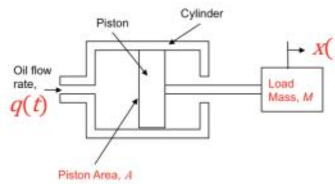


Determine the **transfer function** between the input $q(t)$ and the output $x(t)$.

Assumptions:

i) Neglect any leakage past the piston

ii) Neglect the compressibility of the oil

$$\frac{d(vol)}{dt} = \text{Area} \times \frac{dx}{dt} = q(t)$$

To obtain the transfer function the continuity eq for the oil flow is formed, such that:-

$$q(t) = q_{piston} = A\frac{dx}{dt}$$

Taking Laplace transforms with zero initial conditions and rearranging:-

$$G(s) = \frac{X(s)}{Q(s)} = \frac{1}{As}$$

In simplified case the load mass M does not appear in the transfer function and ram acts as integrator

$$q(t) = A\frac{dx}{dt} \quad \text{or} \quad x(t) = \frac{1}{A}\int q(t)\,dt$$

Taking Laplace:

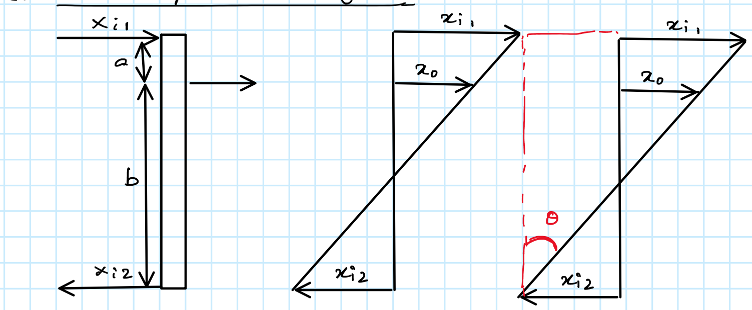
$$\frac{X_o(s)}{X_i(s)} = \frac{b}{a+b}$$

Transfer function given by:

$$G(s) = \frac{b}{a+b}$$

$$X_i(s) \longrightarrow \frac{b}{a+b} \longrightarrow X_o(s)$$

(b) <u>More Complex Lever System:</u>



$$\tan\theta = \frac{x_{i_1} + x_{i_2}}{a+b} = \frac{x_o + x_{i_2}}{b}$$

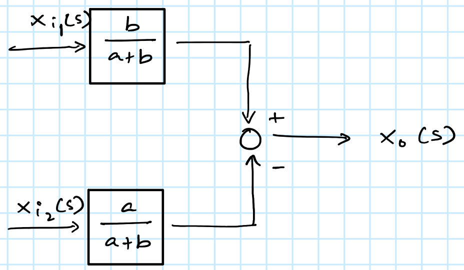
Rearranging:

$$x_o = \frac{b}{a+b} x_{i_1} - \frac{a}{a+b} x_{i_2}$$

Taking Laplace:

$$X_o(s) = \frac{b}{a+b} X_{i_1}(s) - \frac{a}{a+b} X_{i_2}(s)$$

Transfer function of the system.



Questions

Tuesday, 31. January 2023 15:11

1. (a) $f(t) = 0.5\frac{dx}{dt} + 4x$, and $x = 4$ & $t = 0$.

$\Rightarrow F(s) = 0.5sX(s) - X(0) + 4X(s)$

$= (0.5s + 4)X(s) - 2$

(b) $f(t) = \frac{d^2x}{dt^2} + 0.1\frac{dx}{dt} + 3x$ & $x = 10$ & $\frac{dx}{dt} = 2$ when $t = 0$