1 Combined Loading

Learning Summary

1. Know how to use Mohr's circle to analyse a general state of plane stress (knowledge);
2. Recognise that the effect of combined loads on a component can be analysed by considering each load as initially having an independent effect (comprehension);
3. Employ the principle of superposition to determine the combined effect of these loads (application).

1.1 Introduction

Many engineering problems can be analysed as simple load situations e.g. uniaxial loading, beam bending, torsion etc. However, it is also very common in the real world for engineering components and structures to be subjected to several loads simultaneously. This is a **combined loading** situation and can be analysed by superposing the effects of the individual loads.

1.2 Mohr's Circle Recap

Mohr's circle for plane stress is a useful graphical technique for analysing plane stresses acting on an element in a material or structure. For combined loading situations, it is common to reduce the problem to such a plane stress problem and analyse using Mohr's circle. The analysis will give the principal stresses, the maximum shear stresses and the angles of the principal planes for the element. Figure 1.1 shows a shaft subjected to combined loading of a torque, T, and a compressive axial load, P.

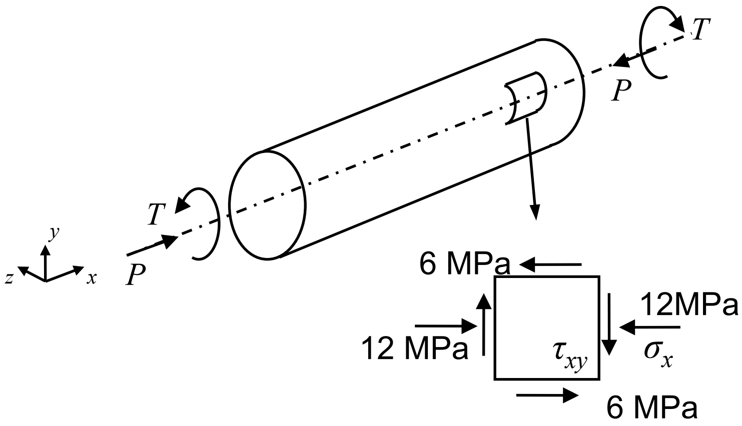

Figure 1.1

Let us assume that the loading gives rise to an axial stress of -12 MPa (i.e. compressive) and a shear stress of -6 MPa (i.e. causes element to rotate clockwise) acting on a surface element as shown in the figure. The Mohr's circle analysis is then as follows:

The known stresses on the element are:

$$\sigma_x = -12 \text{ MPa}$$
$$\sigma_y = 0 \text{ MPa}$$
$$\tau_{xy} = -6 \text{ MPa}$$
$$\tau_{yx} = 6 \text{ MPa}$$

Figure 1.2 shows the Mohr's circle for this stress system. To draw the circle, firstly draw the point **B**, which represents stresses on the x-plane (co-ordinates: -12, -6). Next draw point **E**, which represents stresses on the y-plane (co-ordinates: 0, +6). Join the two points with the line **BE**, which intersects the x-axis at the centre of the circle, **C**.

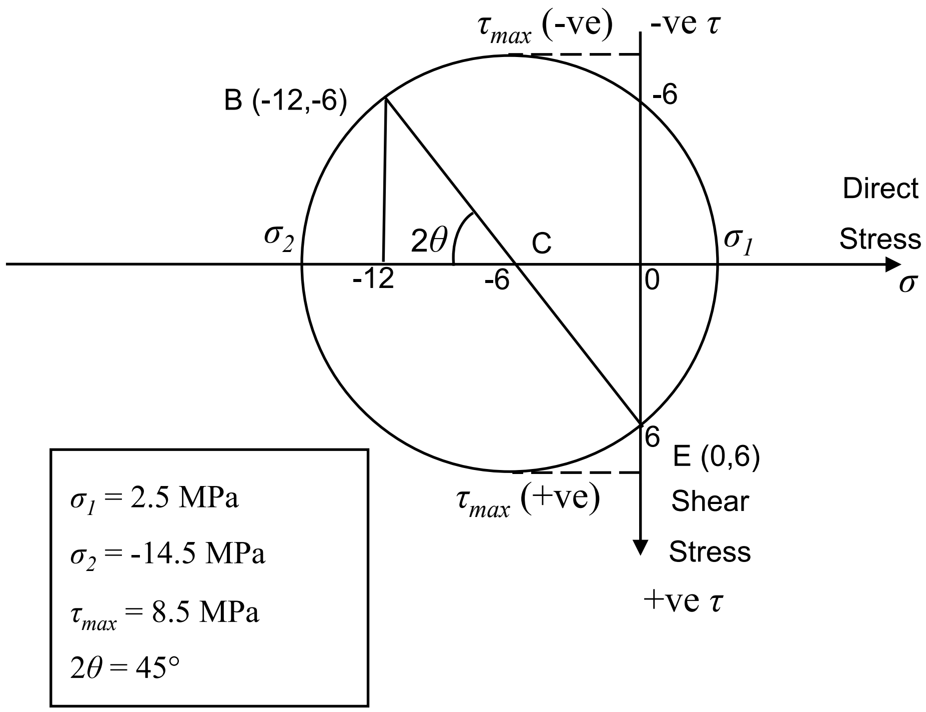

Figure 1.2

The circle can now be drawn and the following quantities measured:

$$\sigma_1 = 2.5 \text{ MPa}$$
$$\sigma_2 = -14.5 \text{ MPa}$$
$$\tau_{max} = 8.5 \text{ MPa}$$
$$2\theta = 45°$$

On the element, the angle of the principal plane (P1) from the y-plane is θ = 22.5° anticlockwise as shown in Figure 1.2.

Alternatively, the important parameters in the circle can be calculated analytically as follows:

The centre of the circle is given by:

$$C = (\sigma_x + \sigma_y)/2 = -6$$

The radius of the circle is given by:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = 8.5$$

The Principal stresses are:

$$\sigma_1 = C + R = 2.5 \text{ MPa}$$

$$\sigma_2 = C - R = -14.5 \text{ MPa}$$

$$\tau_{max} = R = 8.5 \text{ MPa}$$

The angle of the principal planes:

$$\tan 2\theta = \frac{\tau_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2}\right)} = 1$$

$$2\theta = 45°$$

$$\theta = 22.5°$$

If the analytical approach is taken (which does give the more accurate results), then it is always advisable to sketch the Mohr's circle in order to gain a clear understanding of the orientation of the principal planes and the maximum shear planes with respect to the x- or y-planes.

1.3 Superposition of Combined Loads

The Principal of Superposition states that:

$$\begin{bmatrix} \textit{The total effect of } \underline{\textit{combined}} \\ \textit{loads applied to a body} \end{bmatrix} = \sum \begin{bmatrix} \textit{The effects of the individual} \\ \textit{loads applied } \underline{\textit{separately}} \end{bmatrix}$$

[see volume 1 of 'Introduction to Mechanical Engineering']

Thus, when a body or structure is subjected to a combination of different types of loading simultaneously, we can consider the effect of each load on local stresses on an element separately. Stresses on the element can then be summed to determine the effect of the combined loading. A number of combined loading examples can be used to illustrate:

Combined bending and axial loads

Figure 1.3 shows a beam carrying a uniformly distributed load (UDL) along its span, while simultaneously being subjected to an axial force, F. Figure 1.3 shows how the effect of the combined loading, on the stress distribution through the thickness of the beam at the centre of its span, is determined. The effect of the UDL and the axial force are obtained separately and then summed to give the combined stress distribution in the beam. The symmetrical bending stress distribution about the neutral axis is essentially skewed to more by the effect of the axial stress

Combined bending and torsion

Figure 1.4 shows a similar beam to the above example, except now the beam carries a torque instead of the axial load. This loading situation is typical of a shaft with self-weight (UDL) transmitting a torque. In this case, the beam cross-section can be assumed to be solid circular with diameter d. The stresses at the centre of the span, at the bottom surface of the beam, are given by the usual bending and torsion equations as follows:

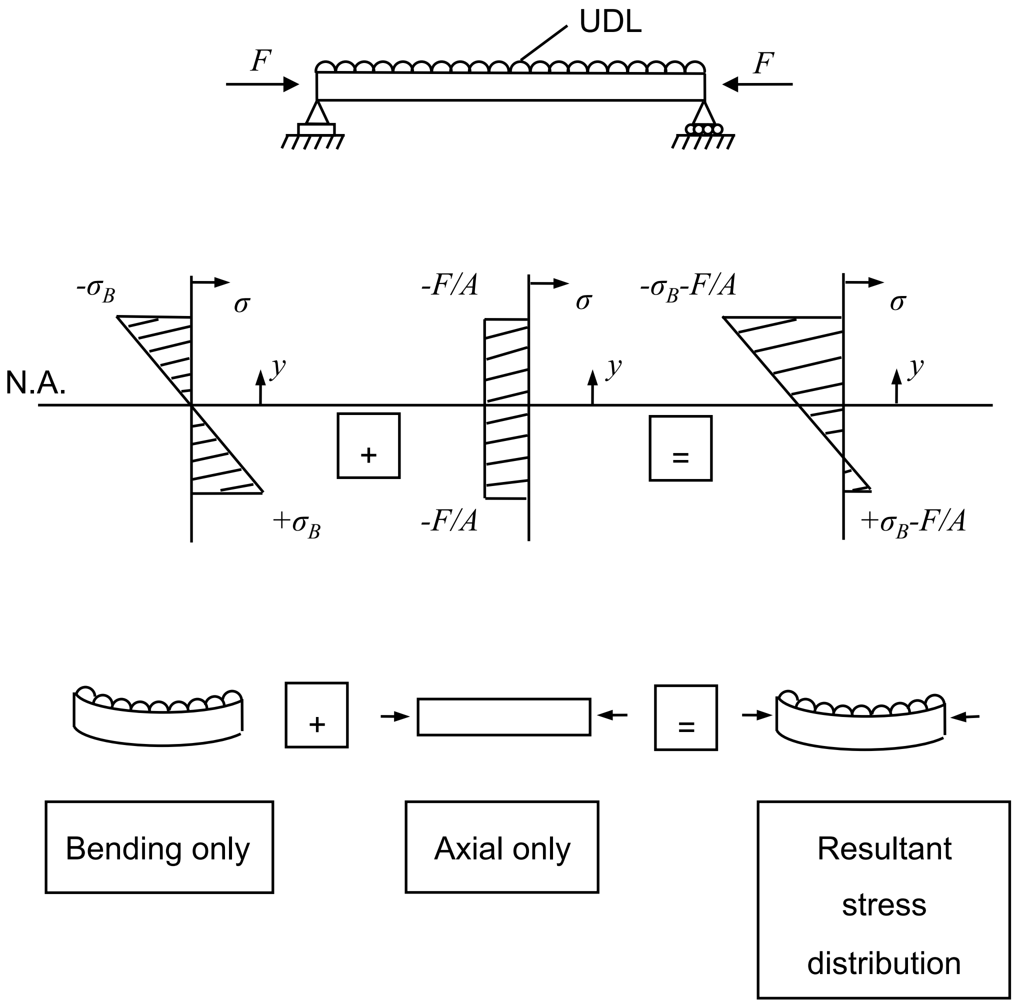
UDL
F
F
-σB
σ
y
N.A.
+σB
-F/A
σ
y
-F/A
-σB-F/A
σ
y
+σB-F/A
+
=
+
=
Bending only
Axial only
Resultant stress distribution

Figure 1.3

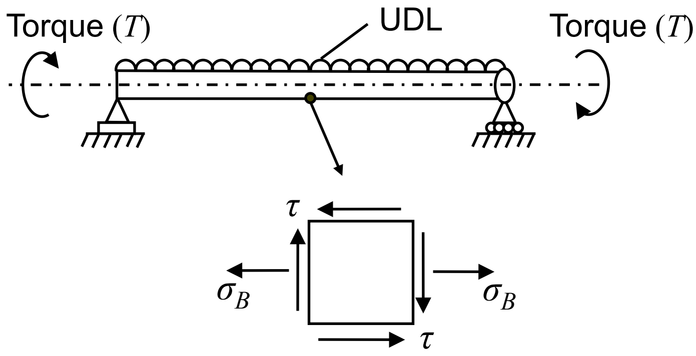

Figure 1.4

Arising from the UDL:

Bending stress (σ_B) $\sigma_B = \dfrac{My}{I}$ where y = d/2

Arising from the torque:

Torsional shear stress (τ) $\tau = \dfrac{Tr}{J}$ where r = d/2

These two stresses can be superposed and illustrated acting on an element at the surface of the beam, as shown in Figure 1.4**.**

Mohr's circle can now be used for this element to obtain the principal stresses and maximum shear stress at this position.

Combined pressure, axial and torsional loading

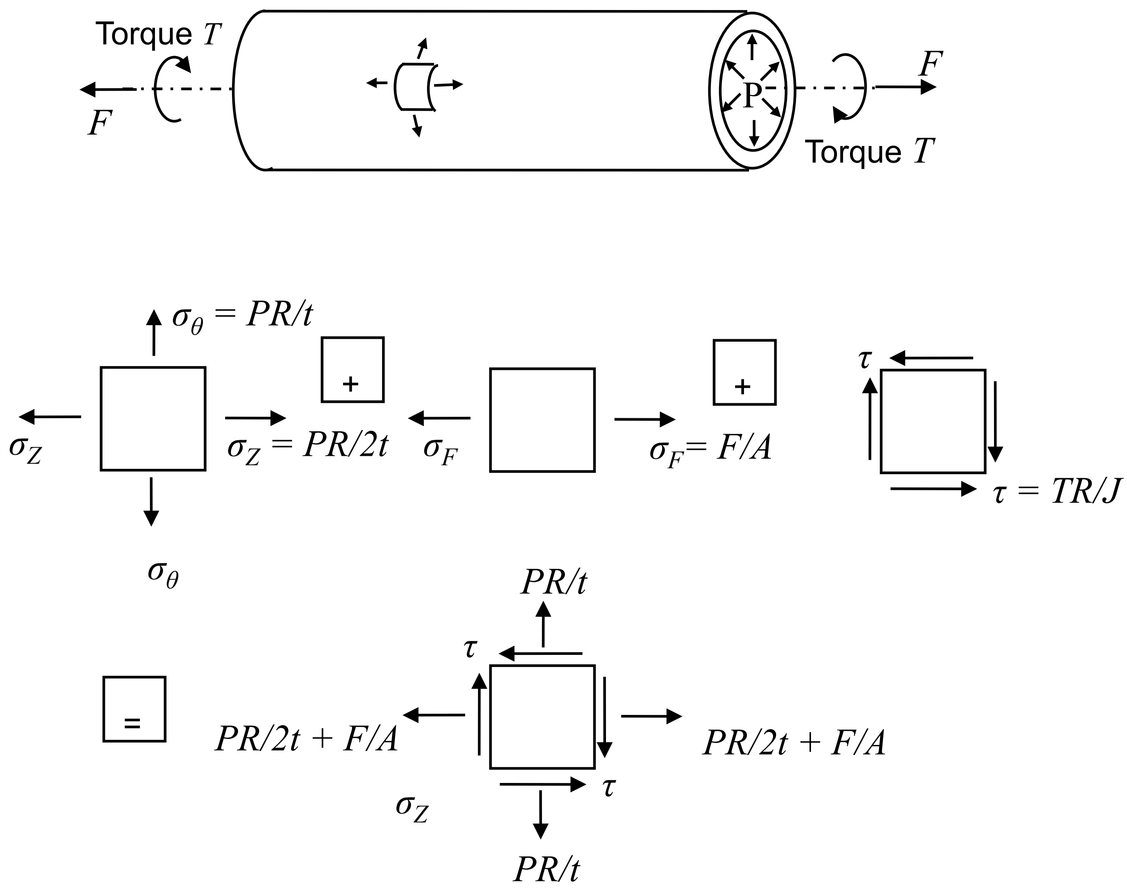

Figure 1.5

A combination of three loads can be illustrated by considering a thin-walled cylinder, as shown in Figure 1.5 subjected to an internal pressure, P, an axial tensile force, F, and a twisting torque, T. Figure 1.5 shows the stresses, arising from each load separately, acting on a surface element in the plane of the cylinder wall. Superposition of these three stresses are also shown on the element. Mohr's circle can again be used to obtain the principal stresses and maximum shear stress for the element.

1.4 Methodology for Combined Loading

The methodology for analysing components or structures under combined loading can now be summarised:

(i) Identify a 2D element at the location of interest in the component

(ii) Determine the stresses acting on the element arising from each individual load

(iii) Superpose the stresses from each individual load to obtain the combined stresses on the element

(iv) Use Mohr's circle to determine the principal stresses and the maximum shear stress on the element

1.5 Worked Example

Combined bending and torsion – offset loading on a cantilever

Figure 1.6 shows a solid circular cross-section cantilever beam, length, L, and diameter, d, fixed at one end. Attached at the free end of the beam is a crank arm which allows a vertical load, P, to be applied at an offset distance, a, from the axis of the cantilever.

Determine the maximum shear stress on the upper surface at the fixed support of the cantilever beam (position A).

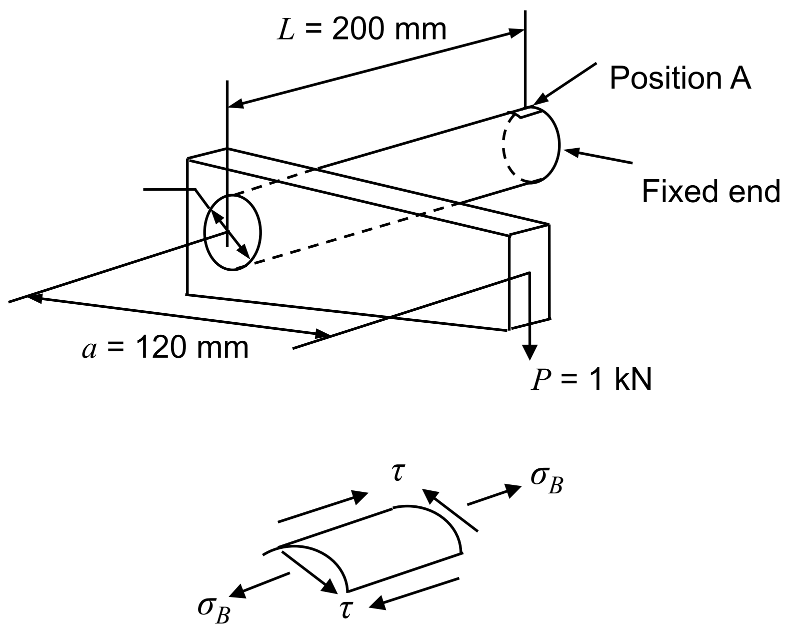

Figure 1.6

The following load and dimensions apply:

$$P = 1 \text{ kN}$$
$$L = 200 \text{ mm}$$
$$a = 120 \text{ mm}$$
$$d = 30 \text{ mm}$$

We consider the stresses acting on a small surface element at position A. The load gives rise to a bending moment and torsional moment at the cross-section at position A as follows:

Bending moment

$$M = PL$$

Torsional moment

$$T = Pa$$

These moments give rise to separate bending and shear stresses acting on the element at position A which can be superposed to give the total effect of the combined loading as shown in Figure 1.6 The stresses are:

Bending stress:

$$\sigma_B = \frac{My}{I} = \frac{PL\, {}^{d}\!/\!{}_{2}}{\pi d^4 /64} = \frac{32PL}{\pi d^3} = \underline{75.45\ MPa}$$

Torsional shear stress

$$\tau = \frac{Tr}{J} = \frac{Pa\, {}^{d}\!/\!{}_{2}}{\pi d^4 /32} = \frac{16Pa}{\pi d^3} = \underline{22.64\ MPa}$$

Mohr's circle can now be drawn for the element to determine the maximum shear stress, as shown in Figure 1.7. The co-ordinates of Point B on the circle (σ_B,τ) correspond to the stresses on the element in the longitudinal direction, i.e. along the cantilever. Point E corresponds to the stresses ($0,-\tau$) in the transverse direction to this. The line joining these two points defines the diameter of the circle and, where it crosses the σ-axis, the centre,

C. The circle can now be drawn and its radius measured to give the maximum shear stress as follows:

τ_{max} = Radius = <u>44 MPa</u>

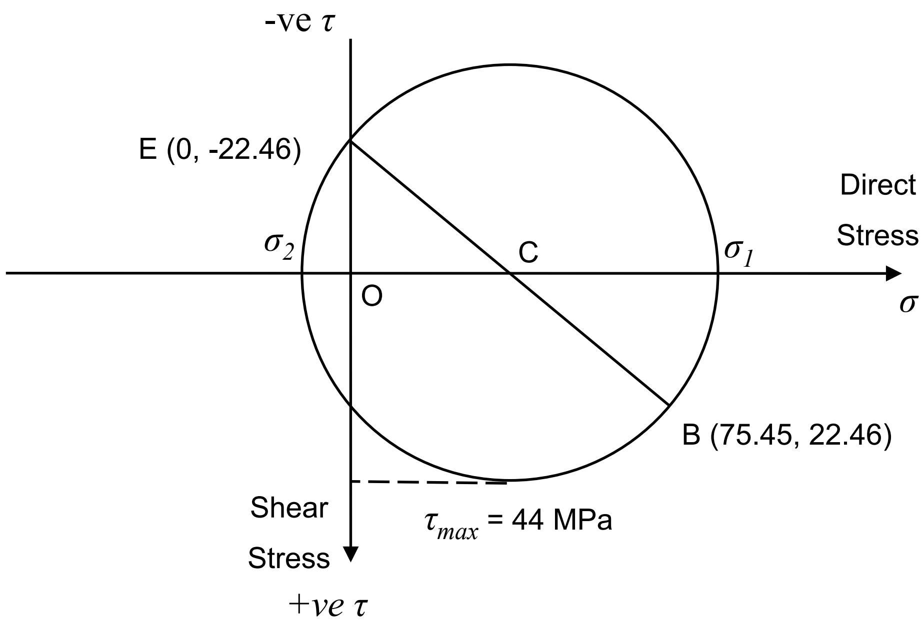

Figure 1.7

Alternatively, by calculation, given the element stresses:

σ_x = 75.45 MPa

σ_y = 0 MPa

τ_{xy} = 22.64 MPa

The maximum shear stress:

$$\tau_{max} = \text{Radius} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$= \sqrt{\left(\frac{75.45}{2}\right)^2 + 22.64^2}$$

= <u>44 MPa</u> as before