The University of Nottingham

SCHOOL OF MATHEMATICAL SCIENCES

A LEVEL 2 MODULE, AUTUMN SEMESTER 2021-2022

ADVANCED MATHEMATICS AND STATISTICS FOR MECHANICAL ENGINEERS

Time allowed TWO Hours

Candidates may complete the front cover of their answer book and sign their desk card but must NOT write anything else until the start of the examination period is announced.

Answer ALL questions

Only a calculator from approved list A may be used in this examination.

List A

Basic Models	***Scientific Calculators***
Aurora HC133	Aurora AX-582
Casio HS-5D	Casio FX82 family
Deli – DL1654	Casio FX83 family
Sharp EL-233	Casio FX85 family
	Casio FX350 family
	Casio FX570 family
	Casio FX 991 family
	Sharp EL-531 family
	Texas Instruments TI-30 family
	Texas BA II+ family

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

DO NOT turn examination paper over until instructed to do so

ADDITIONAL MATERIAL: Formula Sheet, Table of Laplace Transforms, Table of Normal Distribution

1. (a) Solve the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 9\frac{\mathrm{d}y}{\mathrm{d}x} + 20y = 2\mathrm{e}^{3x},$$

subject to the initial conditions $y(0) = 3$ and $y'(0) = 12$. [9 marks]

(b) Find the general solution of the coupled equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sin t - 2x.$$

[11 marks]

2. The fully-rectified cosine wave is defined by

$$f(x) = |\cos x|.$$

(a) Sketch a graph of $f(x)$ for $-2\pi < x < 2\pi$.

[3 marks]

(b) What is the shortest period of $f(x)$?

[2 marks]

(c) Find the Fourier series for $f(x)$.
Hint:

$$\cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B).$$

[9 marks]

(d) State what values the Fourier series converges to when (i) $x = 0$ and (ii) $x = \pi/2$.

[2 marks]

(e) Hence evaluate the sum

$$S = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)}.$$

[4 marks]

3. The function $f(t)$ is defined by

$$f(t) = (t-1)H(t-1),$$

where H denotes a Heaviside step function.

(a) Sketch a graph of $f(t)$ for $t > 0$.

[3 marks]

(b) Find the Laplace transform $\bar{f}(s)$ of $f(t)$.

[3 marks]

(c) The function $y(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$

and the initial conditions

$$y(0) = 1 \quad \text{and} \quad y'(0) = 0.$$

Find its Laplace transform $\bar{y}(s)$. [6 marks]

(d) Hence find $y(t)$. [8 marks]

4. The function $\varphi(x,t)$ satisfies the partial differential equation

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial t^2} = \varphi, \qquad \text{for} \quad 0 < x < L \quad \text{and} \quad t > 0. \tag{1}$$

(a) Show that an appropriate separation of variables substitution leads to equations of the form

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ T''(t) + \lambda' T(t) &= 0, \end{aligned}$$

where λ and λ' are constants, and state the relationship between λ and λ'.

[5 marks]

(b) **Given** that the only solutions of interest are such that $\lambda > 0$, find the general forms of the corresponding solutions $X(x)$ and $T(t)$.

[4 marks]

(c) Find the most general solution of (1) consistent with the boundary conditions

$$\varphi(0,t) = 0 = \varphi(L,t) \qquad \text{for} \quad t > 0.$$

[6 marks]

(d) Find the solution of (1) when initial conditions are imposed in the form of the following Fourier series

$$\varphi(x,0) = 0 \quad \text{and} \quad \varphi_t(x,0) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \sin\left(\frac{n\pi x}{L}\right), \quad \text{for} \quad 0 < x < L,$$

where φ_t denotes a partial derivative of φ with respect to t.

[5 marks]

5. (a) For events A and B it is known that $P(A) = 0.32$, $P(B) = 0.67$ and $P(A \cup B) = 0.84$.
   i) Find the probability that both A and B occur.
   ii) Find the probability that neither A nor B occur.
   iii) Find the conditional probability of B, given that A occurs.

[6 marks]

(b) If the number of customers who arrive in a shop in 30-minute intervals follows a Poisson distribution with mean 3, independently over consecutive intervals, calculate
   i) the probability that no customers arrive in a given 30-minute period;
   ii) the probability that at least 2 customers arrive in a given 30-minute period;
   iii) the probability that no customers arrive over a 1-hour period.

[7 marks]

(c) A wall is made by stacking 3 bricks, separated by two layers of mortar. The bricks are taken from a batch in which heights can be modelled by a normal distribution with mean 20 cm and standard deviation 1 cm. A survey of bricklayers working on the project shows that they lay mortar with an thickness that can be modelled by a normal distribution with a mean of 1.5 cm and standard deviation 3 mm.
   i) What probability distribution describes the height of the wall?
   ii) What is the probability that the height of the wall exceeds 65 cm?

[4 marks]

(d) Determine a 95% confidence interval for the mean melting point of a new alloy based on 100 tests yielding a mean 621.54 and sample variance 1.41 (with measurements in $^\circ C$).

[3 marks]