MMME2046 Vibrations Review

This is intended to be a general review of the topics covered in MMME2046. It does not contain all information required to do the exam successfully, and some topics may be missing.

All slides here are derived from lecture slides in case you wish to see more detail.

Information specific to the Exam (length, time to complete, question types, etc.) can be found on the module moodle page.

What did we cover?

- Single Degree of Freedom Systems
  - Free body diagrams
  - Equations of motion
  - Determining natural frequency

- Free Vibration
  - Response of a single DOF system to External vibration
  - Effect of damping on displacement response
    - Zero damping
    - High damping
    - Critical damping
    - Light damping
  - Undamped and damped natural frequencies
  - Damping ratio
  - Estimation of damping from experimental results

What did we cover?

- Forced Vibration (in our case largely Harmonic Vibration)
  - Determination of steady state response for various systems
  - Frequency Response Function (FRF)

- Multiple Degrees of Freedom
  - Free body diagrams
  - Equations of motion and resulting generalized equation in matrix form $[Z]\{C\} = \{0\}$
  - Determination of natural frequencies and mode shapes

- Approximate methods
  - Rayleigh's
  - Simplification to single DOF systems

What did we cover?

- Beam Vibrations
  - Assemble general matrix
  - Solving for C_1- C_4 based on boundary conditions
  - Natural frequencies
  - Mode shapes

- Vibration isolation
  - Transmissibility of forces
    - To surrounding structure
    - To the ground
  - Isolator values
    - Stiffness
    - Static deflection
    - Design of the isolator given manufacturers specifications

Single Degrees of Freedom

Starting steps for any vibration system

1) Convert the **physical structure** into a **dynamic mass-spring model**

2) Draw a **free body diagram**

   This is the key step in the analysis and should be tackled systematically

   There are three stages

   (i) Create the “free” body by removing any restraining springs

   (ii) Select a motion coordinate and mark it on the diagram

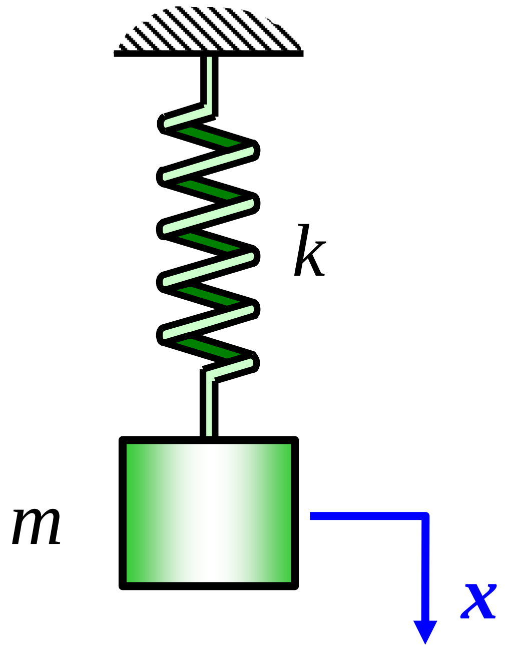
   (iii) Apply a positive deflection in the chosen motion coordinate, identify the forces (and/or moments) that result and draw them on the diagram

   !!!ASSUME WE ARE LOOKING AT SMALL DEFLECTIONS FROM STATIC EQUILIBRIUM!!!

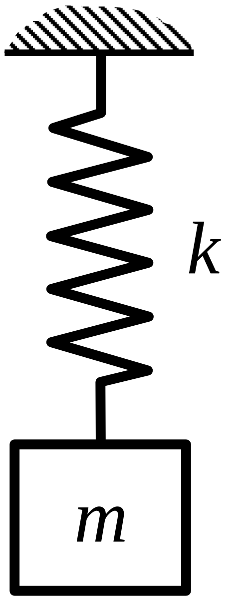
3) Apply the appropriate form of Newton's 2nd Law of motion to give the **equation of motion** for the system

Example 1 Mass suspended on a massless spring

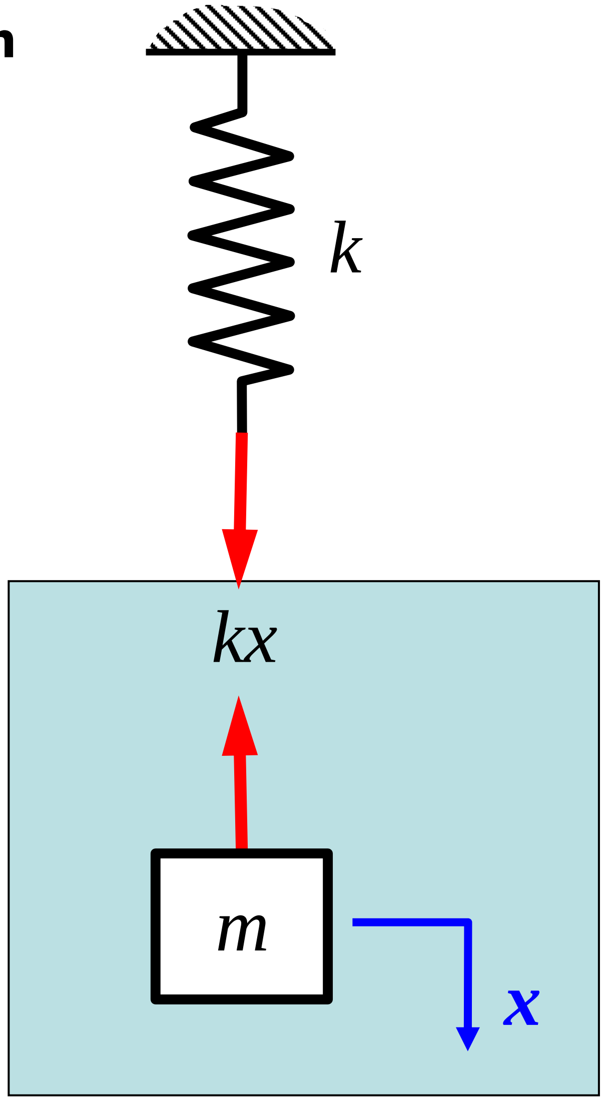
Physical System
Equilibrium position



STEP 1 Dynamic Spring-Mass Model



STEP 2: Free Body Diagram



Completed free body diagram

Displacement from equilibrium position

STEP 3: Equation of motion

Resultant force in the direction of the acceleration = Mass x Absolute acceleration of the centre of mass

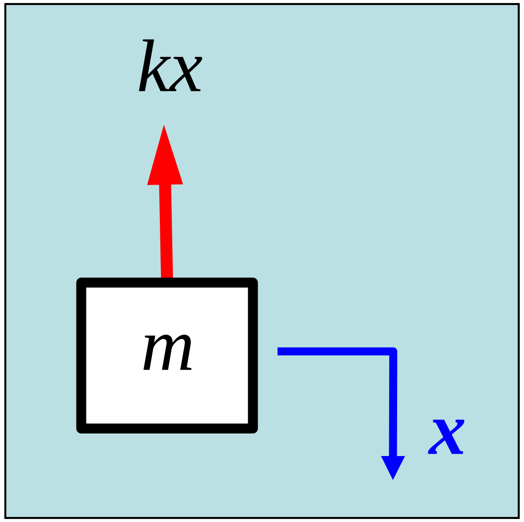
x

$\dot{x}$

$\ddot{x}$

The chosen positive direction of motion is downwards

$$-kx = m\ddot{x}$$



Re-arrange as

$$m\ddot{x} + kx = 0$$

UNITS **Don't be caught out**

When used in the equation of motion, $\boldsymbol{\omega}$ must have the units of **rad/s** to make the equation consistent

However, the value would normally be quoted (in a report, for example) using the units of **Hz** (Hertz or cycles/s)

The two are linked by the equation:

$$f_n \left[\mathrm{Hz} \right] = \frac{\omega_n}{2\pi} \left[\mathrm{rad/s} \right]$$

The rule is:

When **substituting frequency values into formulae**, use units of **rad/s**

When **quoting frequency values** in answers or reports, use units of **Hz**

Obtaining the natural frequencies of other systems

We will find that the equation of motion of all single-degree-of-freedom systems has the form

$$M\ddot{z} + Kz = 0$$

where z is the chosen motion coordinate

The natural frequency is given by

$$\omega_n = \sqrt{\frac{K}{M}} \quad [\text{rad/s}]$$

Hence, as soon as you've derived the equation of motion and obtained **the coefficients of the displacement and acceleration**, you can immediately write down the expression for the natural frequency of a system

!!!THERE IS NO DIFFERENCE BETWEEN SOLVING OF LINEAR AND ROTATIONAL SYSTEMS!!!

All that changes are the symbols used.

Free Vibration

All linear, single-degree-of-freedom systems have this form, which can be written generically as:

$$\mathrm{M}\ddot{z} + C\dot{z} + Kz = F(t)$$

z is the response coordinate

M is the Mass coefficient of acceleration $\ddot{z}$

C is the Damping coefficient of velocity $\dot{z}$

K is the Stiffness coefficient of displacement z

$F(t)$ is the excitation function (independent of z)

Some systems covered in class

Mass-spring-damper system	$\mathrm{M}\ddot{z} + C\dot{z} + Kz = F(t)$
Rocker system	$\mathrm{m}l_2^2\ddot{\theta} + cl_2^2\dot{\theta} + (k_1 l_1^2\theta + k_2 l_2^2)\theta = l_2 P(t)$
Single-axle caravan	$m\ddot{x} + 2c\dot{x} + 2kx = 2c\dot{r}(t) + 2kr(t)$

A: "FREE" VIBRATION

"Free" vibration means that there is no external applied force or moment acting on the structure

I.e. $F(t)=0$, general solution is $z(t)=A\,\mathrm{e}^{\lambda t}$

$$M\lambda^2 A\mathrm{e}^{\lambda t} + C\lambda A\mathrm{e}^{\lambda t} + KA\mathrm{e}^{\lambda t} = 0$$

The complete solution for the transient response is then

$$z(t) = A_1\,\mathrm{e}^{\lambda_1 t} + A_2\,\mathrm{e}^{\lambda_2 t}$$

Where $$\lambda_{1,2} = \frac{-C \pm \sqrt{C^2 - 4KM}}{2M}$$

The integration constants, $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$, are found from the "initial conditions" specified in the problem and derivations for z(t) with respect to time on the formula sheet.

There are **FOUR CASES** to consider

Case (i) **Zero damping**

$$C = 0$$

Case (ii) **High damping**

$\gamma > 1$ $(C^2 > 4KM)$

Case (iii) **Critical damping**

$\gamma = 1$ $(C^2 = 4KM)$

Case (iv) **Light damping**

$\gamma < 1$ $(C^2 < 4KM)$

You should be able to identify and use the correct equations from the formula sheet given the damping level of your system

The most common in this class being **light damping**

Light Damping

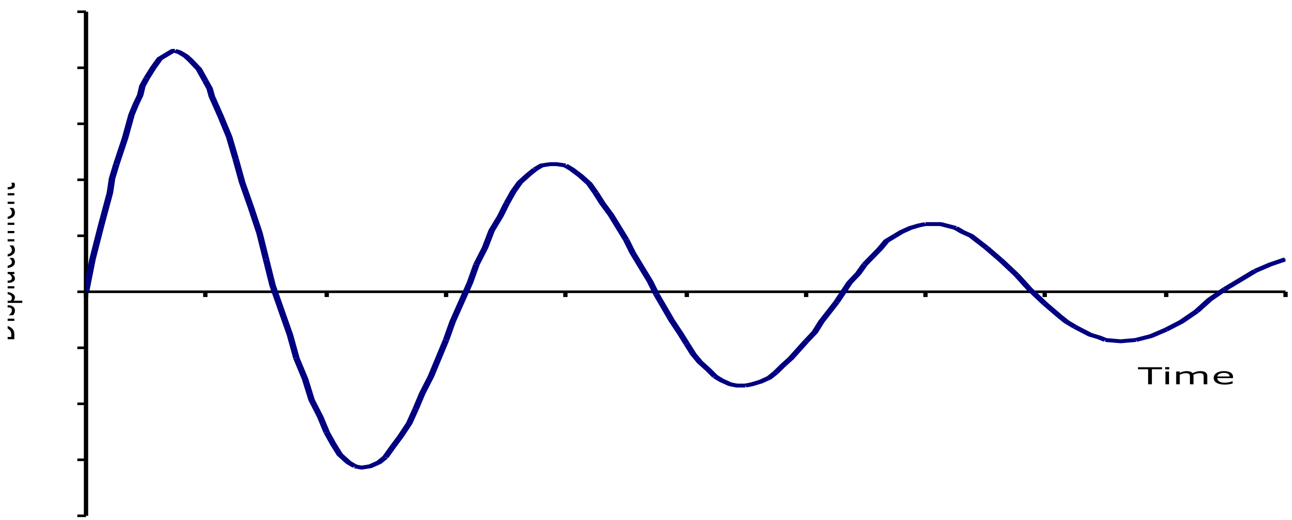
Gives

$$z(t) = A_1\, \mathrm{e}^{\left(-\gamma\omega_n + \mathbf{i}\omega_n\sqrt{1-\gamma^2}\right)t} + A_2\, \mathrm{e}^{\left(-\gamma\omega_n - \mathbf{i}\omega_n\sqrt{1-\gamma^2}\right)t}$$

Using of the complex exponential identities and the fact that $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are a complex conjugate pair, this becomes

$$z(t) = \mathrm{e}^{-\gamma\omega_n t}\left[B_1 \cos\omega_n\sqrt{1-\gamma^2}\,t + B_2 \sin\omega_n\sqrt{1-\gamma^2}\,t\right]$$

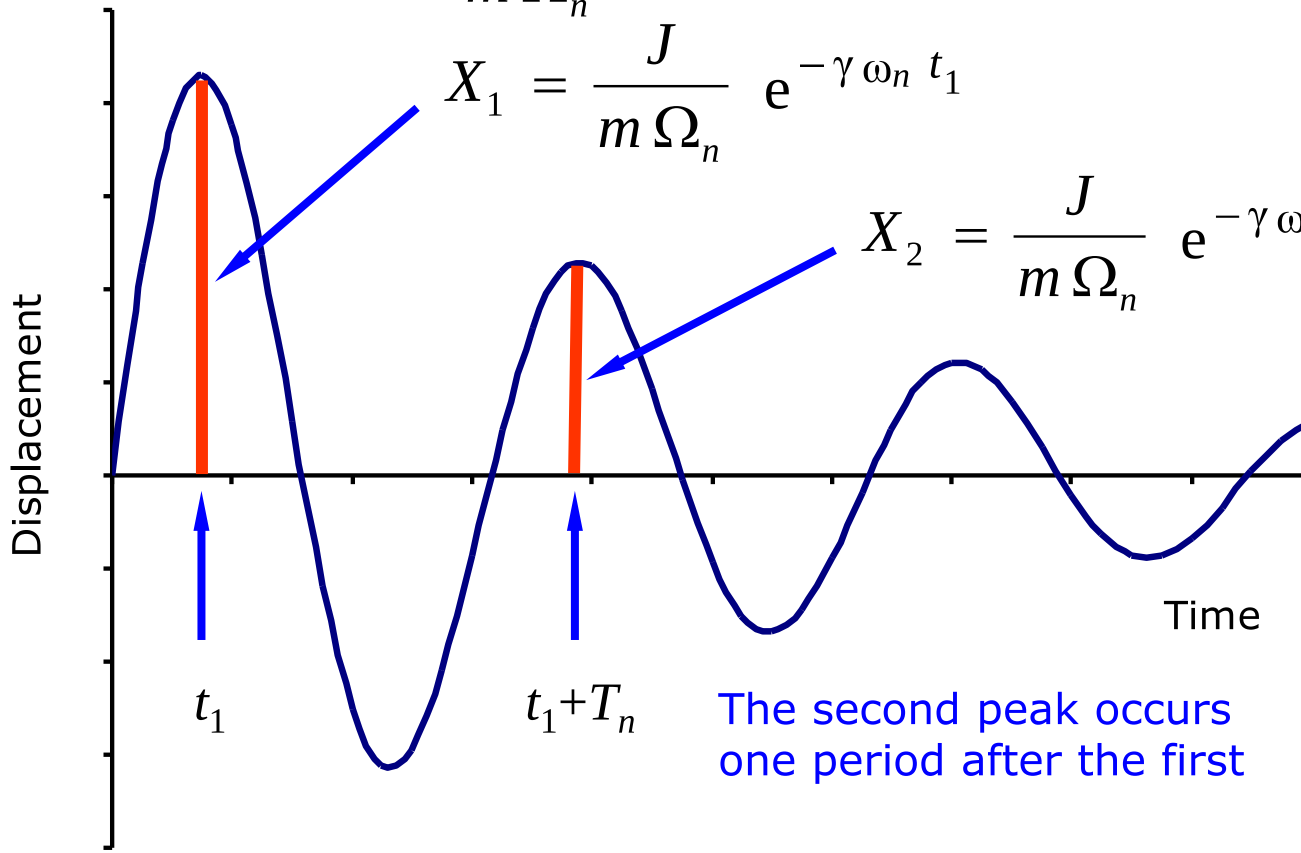
These equations describe a sinusoidal waveform (indicated by the terms in the square brackets) with an exponentially decaying term that will cause the amplitude of the sinusoid to decrease



Estimating Damping – Logrithmic Decrement Method

In the worked example from class,

$$x(t) = \frac{J}{m\Omega_n} e^{-\gamma \omega_n t} \sin \Omega_n t$$



$$X_1 = \frac{J}{m\Omega_n} e^{-\gamma \omega_n t_1}$$

$$X_2 = \frac{J}{m\Omega_n} e^{-\gamma \omega_n (t_1 + T_n)}$$

The second peak occurs one period after the first

The ratio of the amplitudes is $\dfrac{X_1}{X_2} = e^{\gamma \omega_n T_n}$

Where, $T_n = \dfrac{2\pi}{\Omega_n} = \dfrac{2\pi}{\omega_n \sqrt{1-\gamma^2}}$

Forced Vibration/Harmonic Response

Forced Versus Harmonic Vibration

Forced Vibration is when an alternating force or motion is applied to a mechanical system, for example when a washing machine shakes due to an imbalance.

Harmonic Vibration is a type of Forced Vibration in which a force is repeatedly applied to a system.

Harmonic Vibration is what we typically considered.

$$M\ddot{z} + C\dot{z} + Kz = f(t)$$

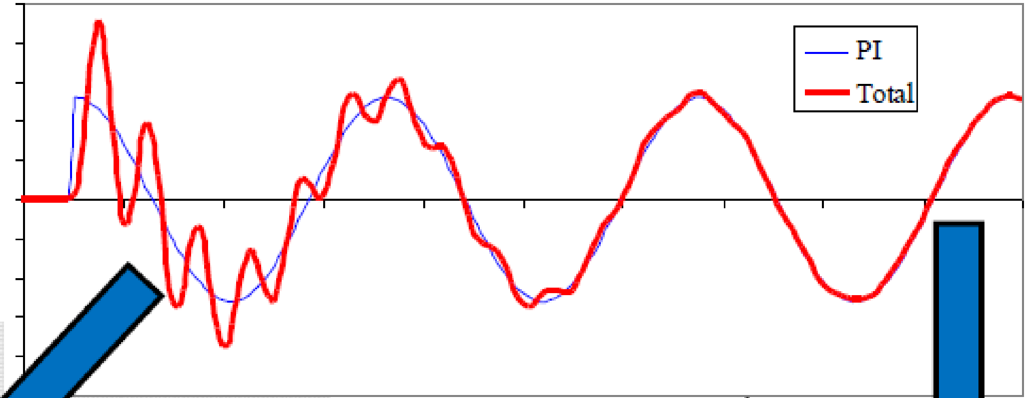
$$M\ddot{z} + C\dot{z} + Kz = C_r\dot{r}(t) + K_r r(t)$$

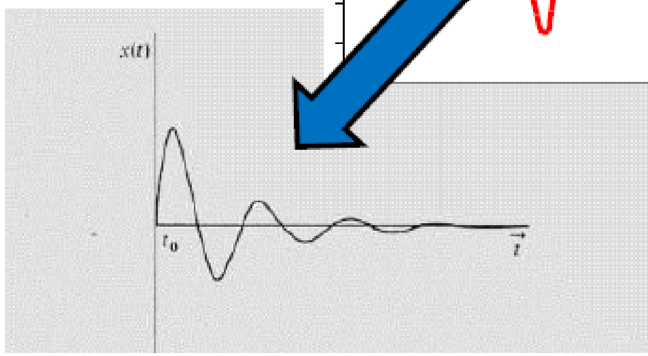
$$M\ddot{z} + C\dot{z} + Kz = S_1\cos\omega_1 t + S_2\cos\omega_2 t$$

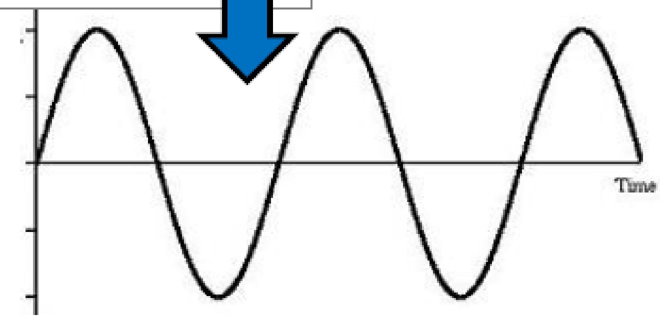
The complete solution for $\boldsymbol{z(t)}$ consists of

- The solution to the ***Complementary Function*** or ***Transient Response***
- The ***Particular Integral*** or ***Steady-State Response***

$$\boxed{z(t) = z(t)_{TR} + z(t)_{SS}}$$







Complementary Function or Transient Response

$$M\,\ddot{z} + C\,\dot{z} + K\,z = 0$$

$$z(t)_{TR} = A_1\,e^{\lambda_1 t} + A_2\,e^{\lambda_2 t}$$

$$\lambda_{1,2} = \frac{-C \pm \sqrt{C^2 - 4KM}}{2M}$$

Particular Integral or Steady State Response

$$z(t)_{SS} = Z\cos(\omega t + \alpha)$$

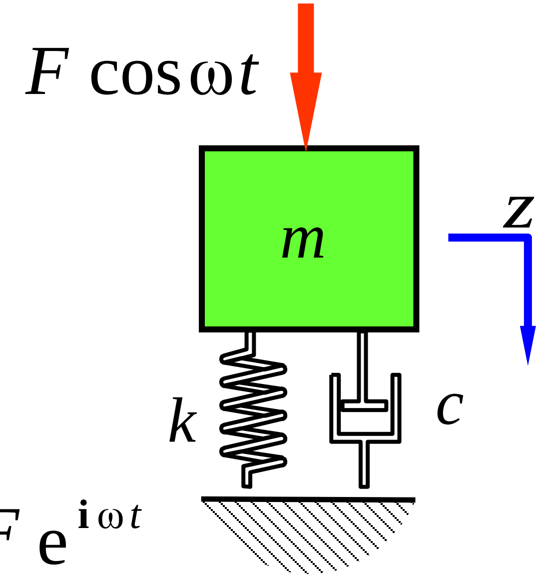
$$M\,\ddot{z} + C\,\dot{z} + K\,z = f(t)$$

$$M\,\ddot{z} + C\,\dot{z} + K\,z = C_r\dot{r}(t) + K_r r(t)$$

$$M\,\ddot{z} + C\,\dot{z} + K\,z = S_1\cos\omega_1 t + S_2\cos\omega_2 t$$

Total Displacement is then $z(t) = z(t)_{TR} + z(t)_{SS}$

For a simple single degree-of-freedom mass-spring-damper system



$$M\ddot{z} + C\dot{z} + Kz = f(t)$$

Substitute the following into the EOM

$$f(t) = F\,e^{i\omega t}$$

$$z_{ss}(t) = Z\cos(\omega t + \alpha)$$

$$\dot{z}_{ss}(t) = -\omega Z\,sin(\omega t + \alpha)$$

$$\ddot{z}_{ss}(t) = -\omega^2 Z\cos(\omega t + \alpha)$$

OR

$$f(t) = F\,e^{i\omega t}$$

$$z_{ss}(t) = Z^* e^{i\omega t}$$

$$\dot{z}_{ss}(t) = i\omega Z^* e^{i\omega t}$$

$$\ddot{z}_{ss}(t) = -\omega^2 Z^* e^{i\omega t}$$

Leading to: $$H(\omega) = \frac{Z^*}{F} = \frac{1}{\left(K - M\omega^2\right) + i\,\omega C} = \frac{1}{K}\,\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + i\,2\gamma\frac{\omega}{\omega_n}}$$

This is also called the Frequency Response Function (FRF):
Response / Unit Applied Force

From complex algebra the amplitude of response is then:

$$\left|Z^*\right| = \frac{F}{\sqrt{\left(K - M\,\omega^2\right)^2 + \omega^2\,C^2}} = \frac{F}{K\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\,\gamma^2\,\frac{\omega^2}{\omega_n^2}}}$$

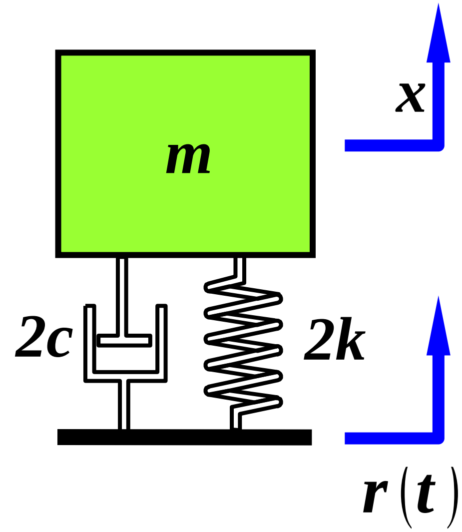
The phase lag is:

$$\alpha = \tan^{-1}\left(\frac{-\omega\,C}{K - M\,\omega^2}\right) = \tan^{-1}\left(\frac{-2\,\gamma\,\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}\right)$$

$$z(t)_{SS} = \left|Z^*\right|\,\cos\left(\omega t + \alpha\right)$$

Equation of motion for a simple mass-spring-damper with displacement input:

$$M\ddot{x} + C\dot{x} + Kx = C_r\,\dot{r}(t) + K_r\,r(t)$$



Substitutions: $r(t) = Re^{i\omega t}$

$\dot{r}(t) = i\omega Re^{i\omega t}$

$$x_{ss}(t) = X^* e^{i\omega t}$$
$$\dot{x}_{ss}(t) = -\omega X^* e^{i\omega t}$$
$$\ddot{x}_{ss}(t) = -\omega^2 X^* e^{i\omega t}$$

The FRF is

$$\frac{Z^*}{R} = \frac{K_r + \mathbf{i}\,C_r\,\omega}{\left(K - M\,\omega^2\right) + \mathbf{i}\,C\omega} = \frac{1}{K}\,\frac{K_r + \mathbf{i}\,C_r\,\omega}{\left(1 - \dfrac{\omega^2}{\omega_n^2}\right) + \mathbf{i}\,2\gamma\,\dfrac{\omega}{\omega_n}}$$

Where

$$C_r = 2c, \quad K_r = 2k, \quad C = 2k, \quad and\ K = 2k$$

The amplitude of response is then:

$$\left|Z^*\right| = \frac{\sqrt{K_r^2 + C_r^2\omega^2}\ \ R}{\sqrt{\left(K - M\omega^2\right)^2 + C^2\omega^2}} = \frac{\sqrt{K_r^2 + C_r^2\omega^2}\ \ R}{K\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\gamma^2\frac{\omega^2}{\omega_n^2}}}$$

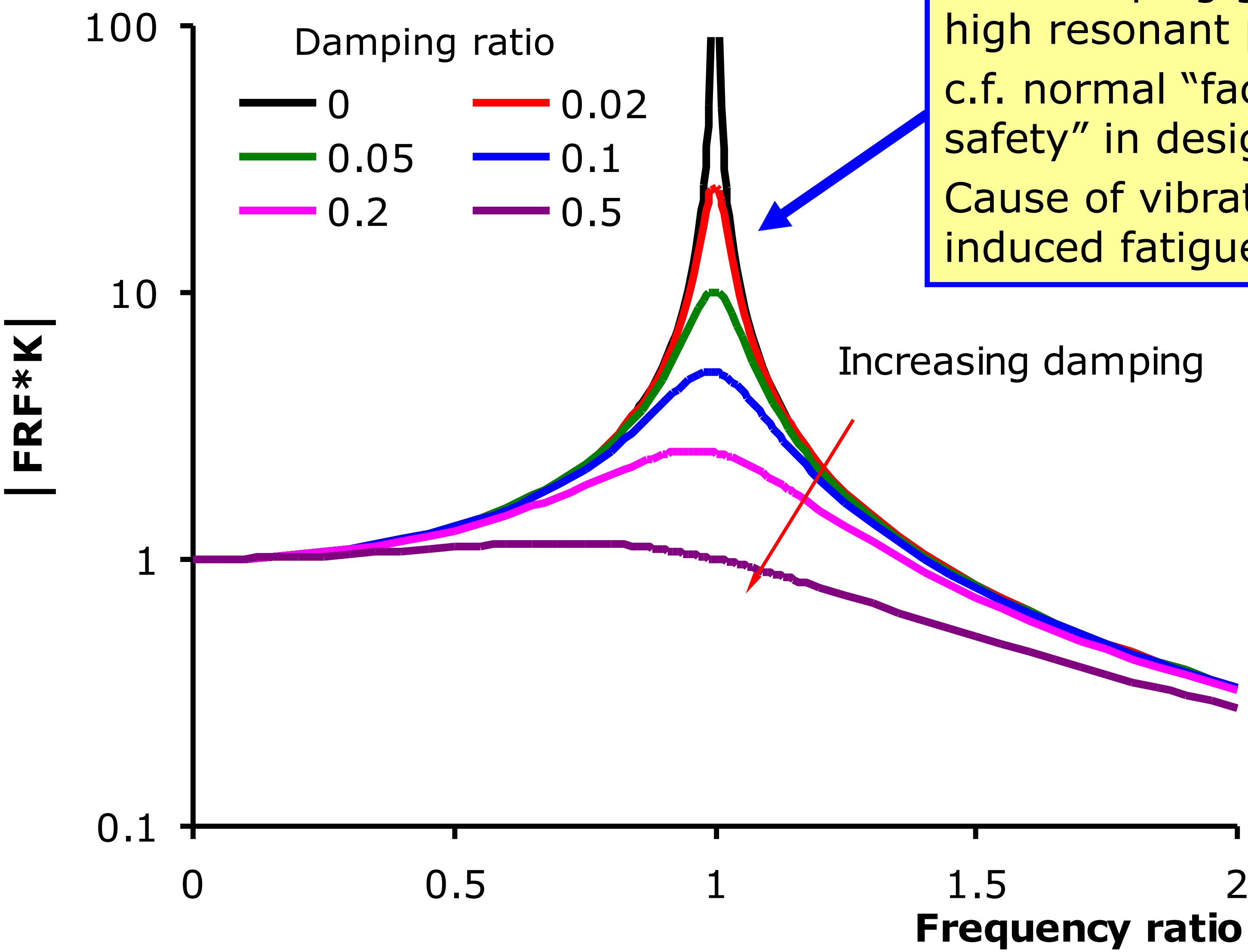
The phase angle is:

$$\alpha = \tan^{-1}\left(\frac{C_r\omega\left(K - M\omega^2\right) - K_r C\omega}{K_r\left(K - M\omega^2\right) + C_r C\omega^2}\right)$$

$$x(t)_{SS} = \left|X^*\right| \cos\left(\omega t + \alpha\right)$$

There are further forms of these equations, but these are the ones on your equation sheet.

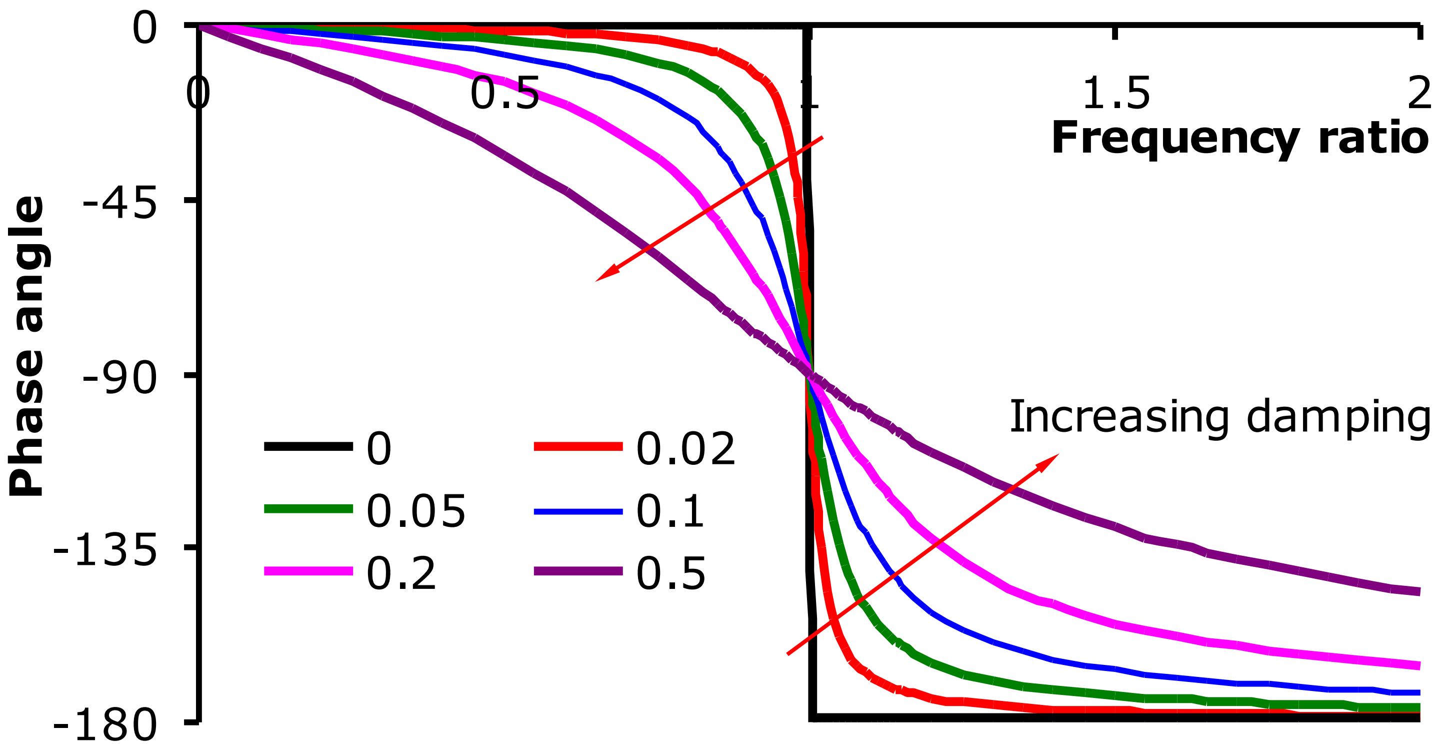
The response depends on ω / ω_n and on damping ratio γ



Low damping gives a high resonant peak

c.f. normal “factors of safety” in design

Cause of vibration-induced fatigue



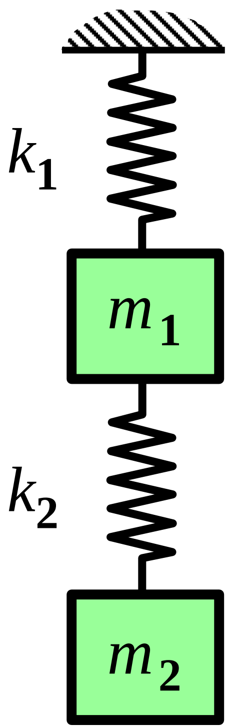
At low frequencies, the phase angle is approximately zero

It drops by 180º as the system passes through resonance

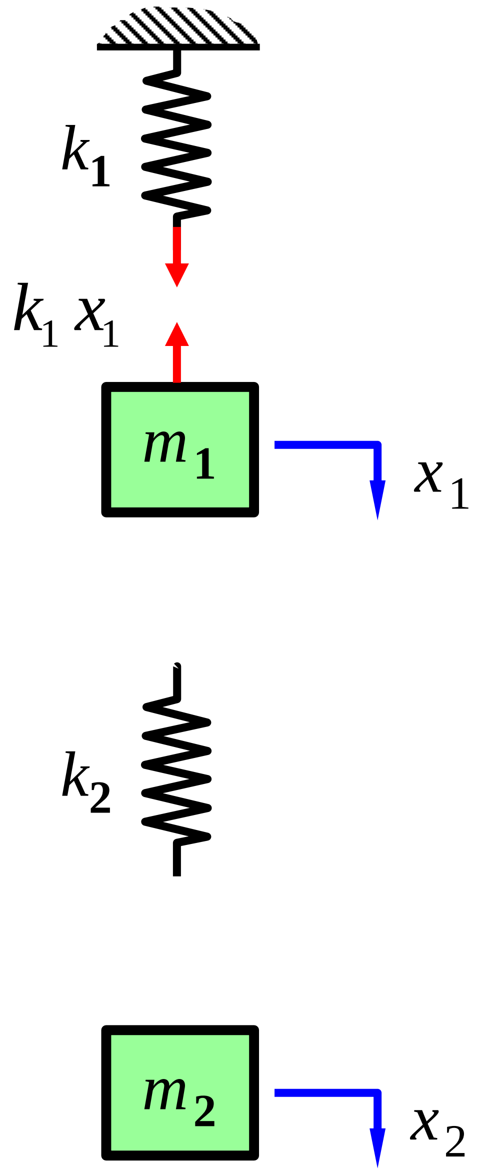
Multiple Degrees of Freedom

2 Degrees of Freedom

STEP 1:
Dynamic Mass-Spring Model



STEP 2: Free-body Diagrams



Force in spring k_2 ?

Force in spring k_2?

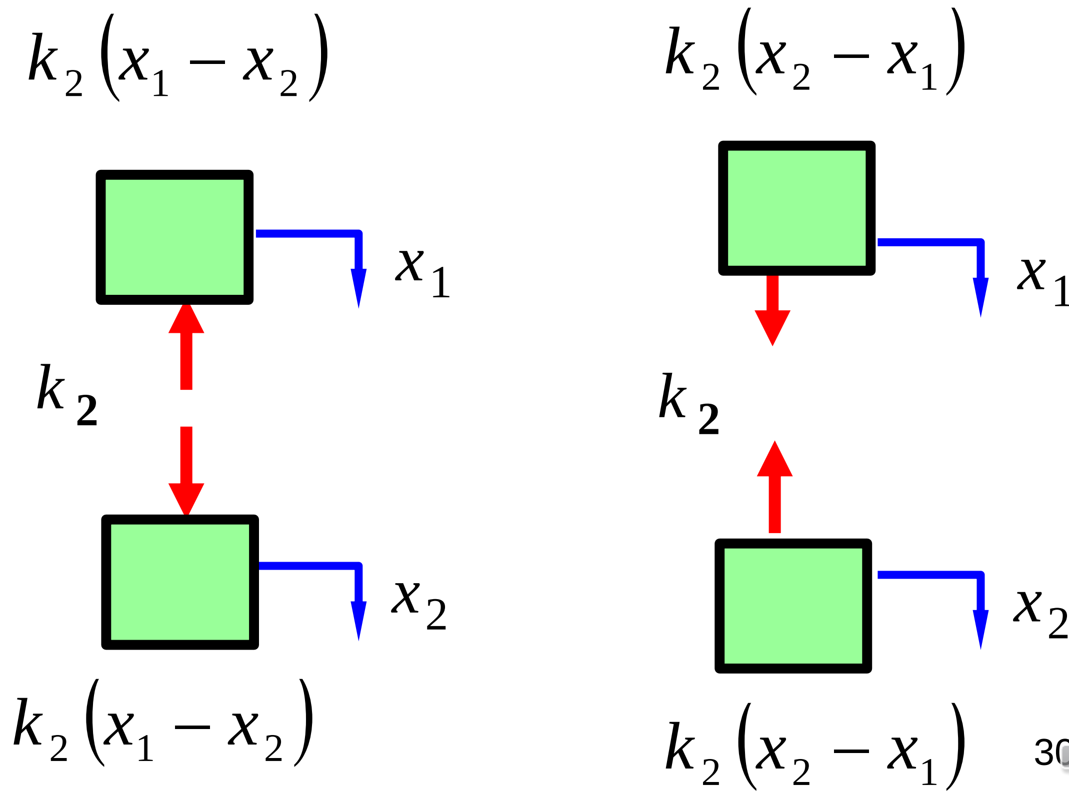
The expression for the force in a spring is

Spring force = Stiffness x Change of length

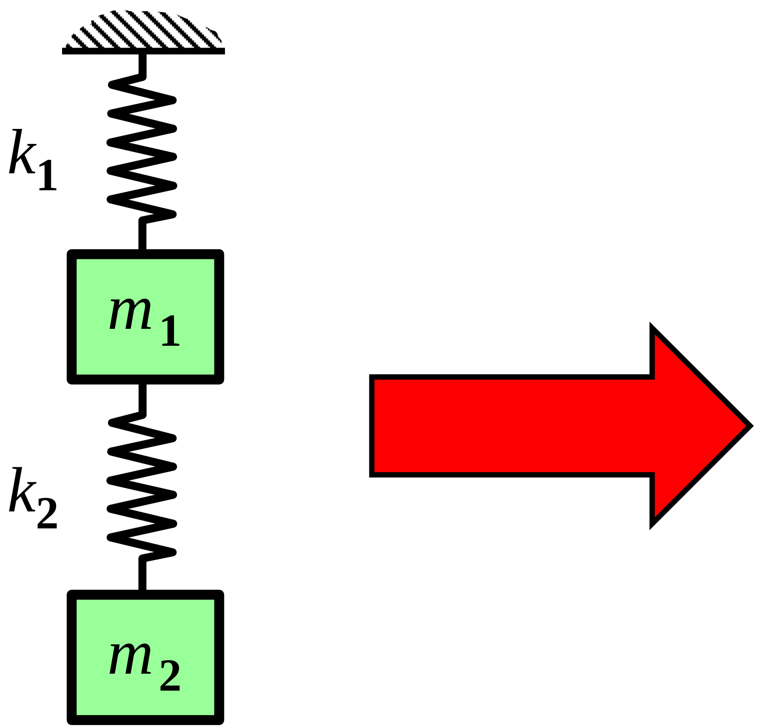
What is the **change of length** of the spring?

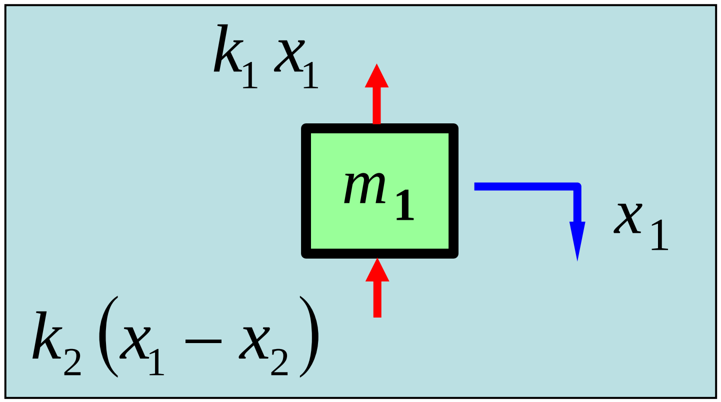
Is the spring in **tension** or **compression**?

The two are exactly equivalent

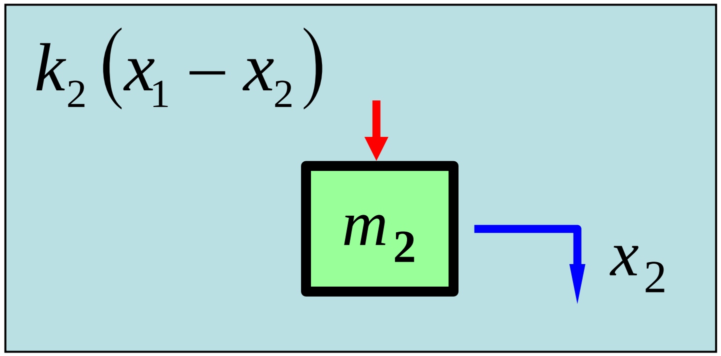


STEP 2: Free-body Diagrams





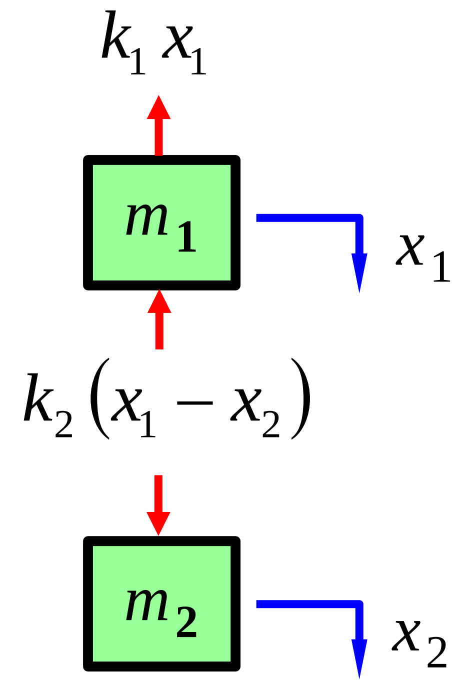
Completed free body diagrams



STEP 3: Equations of motion

$$x_1 \quad m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \qquad \text{(1a)}$$

$$x_2 \quad m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0 \qquad \text{(1b)}$$



In matrix form (& remembering the general formulation $M\ddot{z} + Kz = 0$)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

In matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or $$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

As with single-degree-of-freedom systems, we can check for errors in the equations at this stage.

In particular,

(1) the terms in the leading diagonals of $\mathbf{[M]}$ and $\mathbf{[K]}$ are **always positive**

(2) the off-diagonal terms may be positive **or** negative

(3) $\mathbf{[M]}$ and $\mathbf{[K]}$ are **often symmetric** about the leading diagonal

Solution to obtain **natural frequencies** and **mode shapes**

For free vibration of the system at one of its natural frequencies, the motion of each mass will be sinusoidal.

Use as substitutions, $x_1(t) = X_1 \cos\omega t$ and $x_2(t) = X_2 \cos\omega t$

$\ddot{x}_1(t) = -\omega^2 X_1 \cos\omega t$ $\ddot{x}_2(t) = -\omega^2 X_2 \cos\omega t$

Substituting into EOM and cancelling the common factor $\cos\omega t$

In matrix form

$$\begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{matrix} (2a) \\ (2b) \end{matrix}$$

Resulting matrix is an **eigenvalue problem**; normally presented in maths books as $\left(\left[K\right]-\omega^2\left[M\right]\right)\left\{X\right\} = \left\{0\right\}$

$$[Z]\{X\}=\{0\}$$

Where the **eigenvalues** give the **natural frequencies** and the **eigenvectors** give the corresponding **mode shapes**

Obtaining the Natural Frequencies

For a non-trivial solution find the $\det\left[Z\right] = 0$ and solve for ω

Obtaining the Mode Shapes

To find the corresponding mode shapes, we substitute each root back into the Matrix to get the relationship between X_1 and X_2

Give one value (X_1 or X_2) an amplitude of UNITY (i.e. X_1 or $X_2 = 1$) and then find the amplitude of the other (X_2 or X_1) relative to this

Be aware of in-phase and out-of-phase relationships at various mode shapes, and be able to sketch them.

In the exam you are expected to be able to form the generalized matrices ($[Z]\{X\}=\{0\}$) for any DOF system, and solve for values of ω and $\{X\}$ up to 2 DOF. If you are given a 3 DOF system you will be provided with enough information that your 3 equations will only have 2 unknowns to solve for.

You are also expected to be able to sketch basic mode shapes given boundary conditions, without solving for values.

Approximate Methods

RAYLEIGH'S METHOD

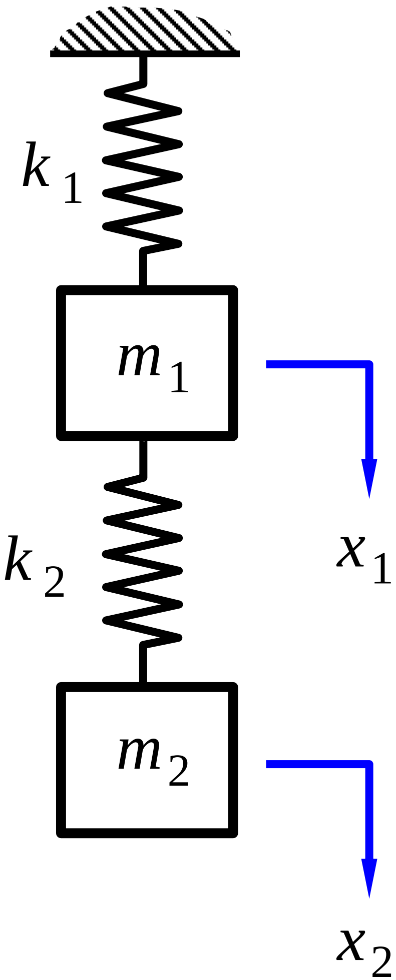
For an undamped system in free vibration, energy is conserved, so that

Max. Kinetic Energy = Max. Strain Energy

- These can be found if we know the deflected shape (i.e., the mode shape) of the system
- Normally, we do NOT know the exact mode shape, so we need to make an estimate
- Accuracy depends on making a good guess

$$\omega_{\text{Exact}} \leq \omega_{\text{Rayleigh}}$$

Example 1 **Two-degree-of-freedom System**



The **instantaneous** kinetic energy of mass 1 is

$$\frac{1}{2} m_1 \ddot{x}_1$$

If $x_1(t) = X_1 \sin \omega t$ then $\dot{x}_1(t) = \omega X_1 \cos \omega t$

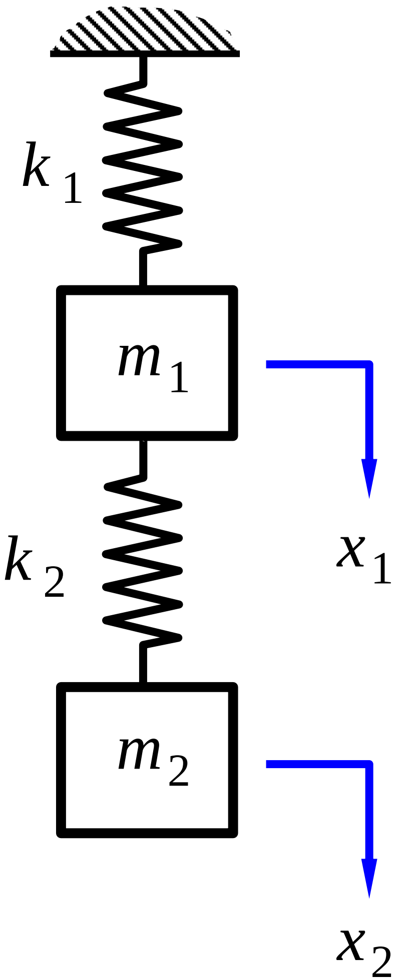
That is, the maximum velocity is (ωX_1)

Therefore, the **maximum** kinetic energy of mass 1 is

$$T_{\max} = \tfrac{1}{2} m_1 (\omega X_1)^2 = \tfrac{1}{2} m_1 \omega^2 X_1^2$$

For both masses

$$\boxed{T_{\max} = \tfrac{1}{2} \omega^2 \left(m_1 X_1^2 + m_2 X_2^2 \right)}$$



The **instantaneous** strain energy of a spring is

$$\tfrac{1}{2} \times \text{Stiffness} \times (\text{Change of length})^2$$

For the top spring, this is $U = \tfrac{1}{2} k_1 x_1^2$

If $x_1(t) = X_1 \sin \omega t$ then $U_{\max} = \tfrac{1}{2} k_1 X_1^2$

For the bottom spring

$$U_{\max} = \tfrac{1}{2} k_2 (X_1 - X_2)^2$$

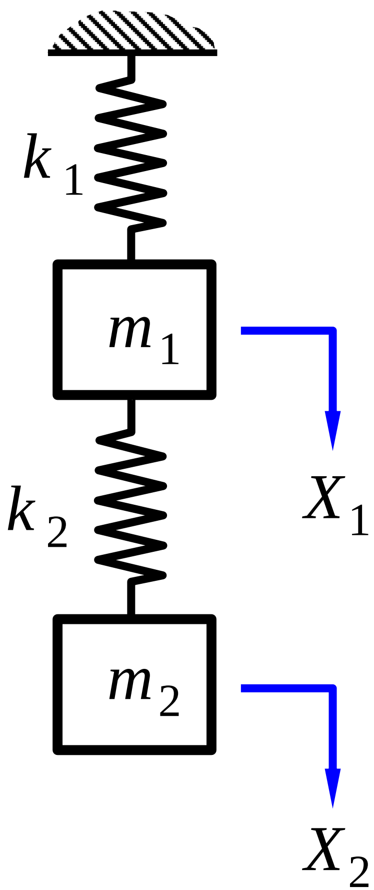
For both springs

$$\boxed{U_{\max} = \tfrac{1}{2} k_1 X_1^2 + \tfrac{1}{2} k_2 (X_1 - X_2)^2}$$

Equating $T_{\max} = U_{\max}$, we get

$$\omega^2 = \frac{k_1 X_1^2 + k_2 (X_1 - X_2)^2}{m_1 X_1^2 + m_2 X_2^2}$$

If we can estimate the mode shape, we will have values of $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ that can be substituted into the equation



We need an educated guess based on experience

For $\boldsymbol{m_1} = \boldsymbol{m_2} = 2$ kg and $\boldsymbol{k_1} = \boldsymbol{k_2} = 200$N/m,

we know that the two masses vibrate in phase and

$$X_2 > X_1$$

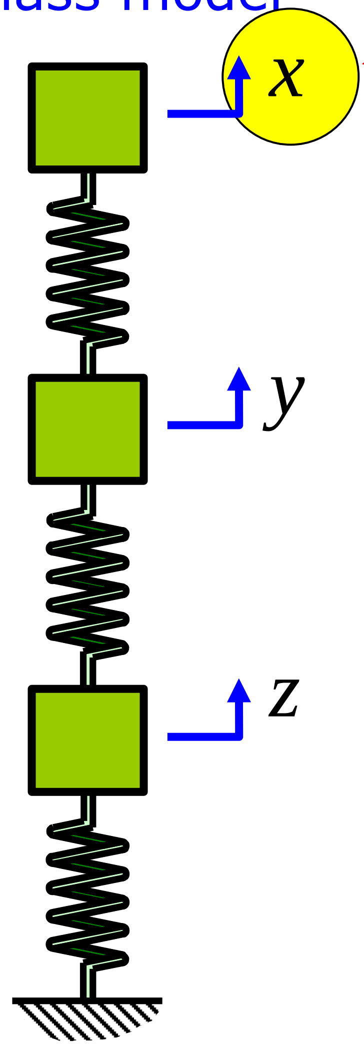
Let's guess that $\boldsymbol{X_1} = 1$ and $\boldsymbol{X_2} = 2$

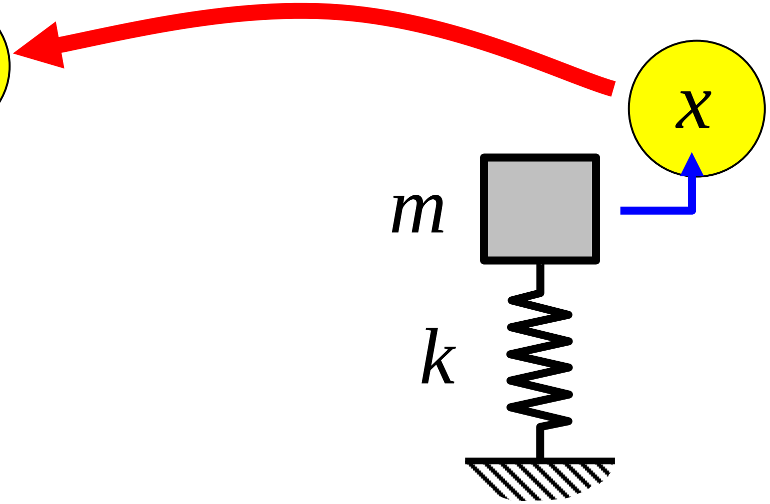
This gives a value for ω_n of 1.007 Hz

This is an error of 2.3% (exact value is 0.984 Hz)

Single DOF approximations

- Lumped mass model

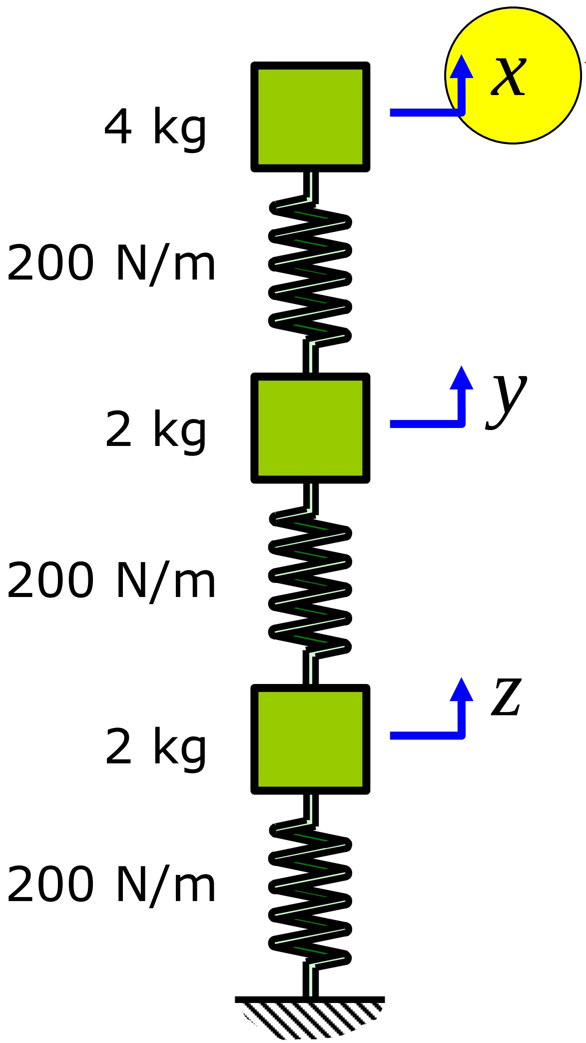


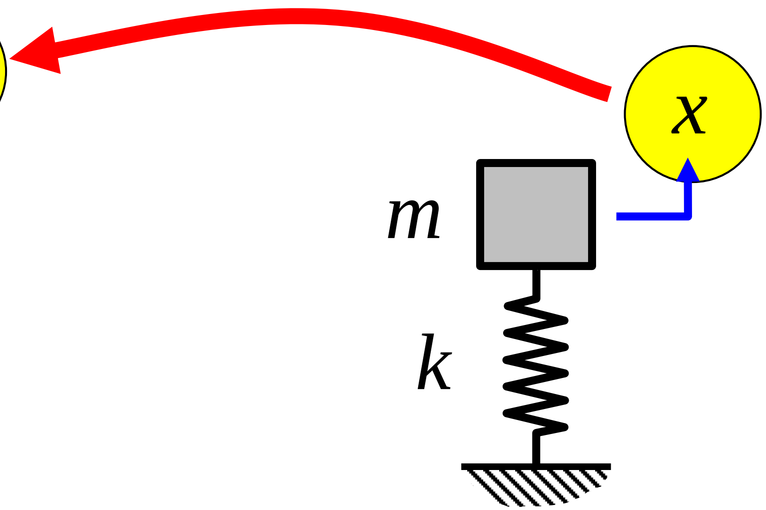


- Solving for *m* and *k* using estimates for X, Y and Z.
- What are your choices and how good will they be?

Lumped mass system

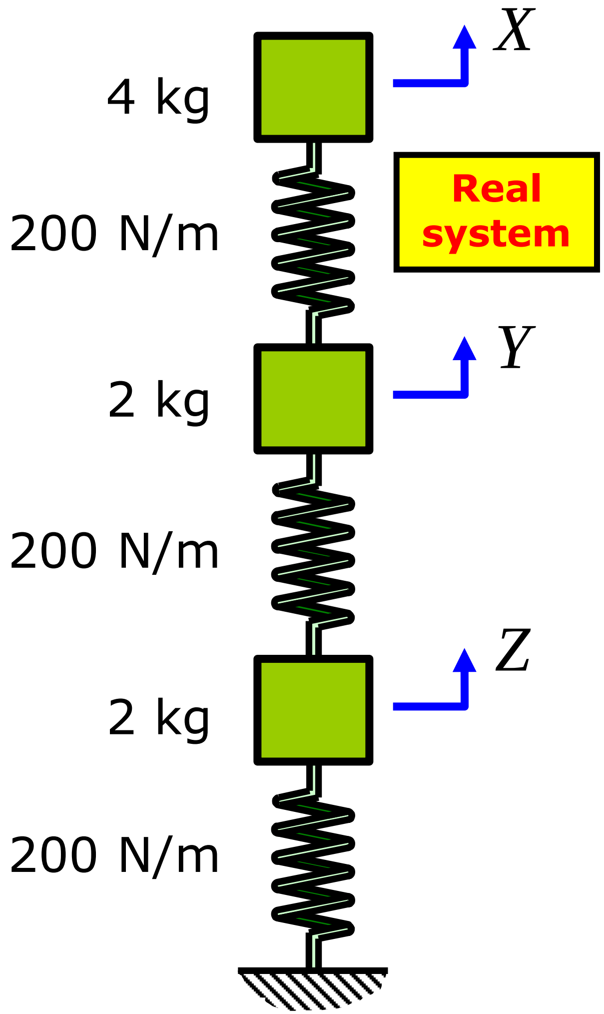
Objective:
To find an approximate single-degree-of-freedom model to analyse the motion of the top mass of a 3-degree-of-freedom system

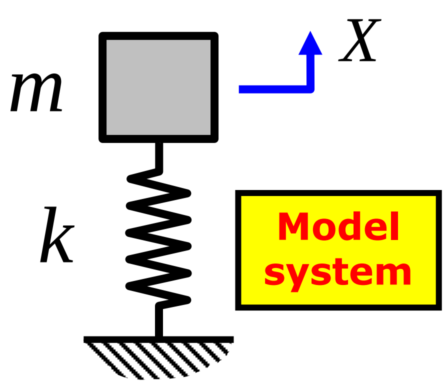




The displacement of the model mass will be linked to the top mass, since that's the point on the real system that we are interested in

The values for $\boldsymbol{m}$ and $\boldsymbol{k}$ are found by equating the kinetic and strain energies in the real and model systems





Equating max. **kinetic** energies in the real and model systems

$$\tfrac{1}{2}4(\omega X)^2 + \tfrac{1}{2}2(\omega Y)^2 + \tfrac{1}{2}2(\omega Z)^2$$

$$= \tfrac{1}{2}m(\omega X)^2$$

Hence $\boxed{m = \dfrac{4X^2 + 2Y^2 + 2Z^2}{X^2}}$

Equating max. **strain** energies in the real and model systems

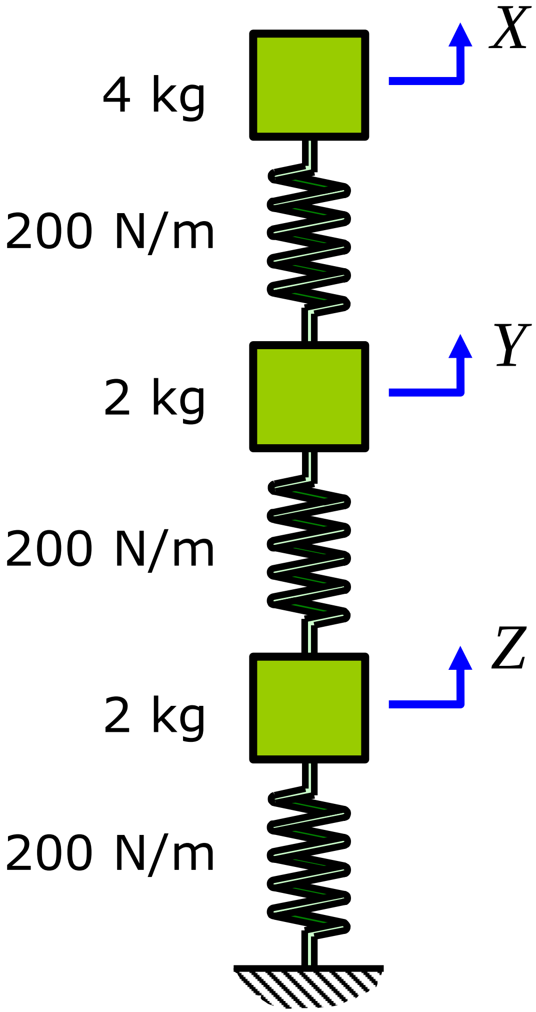
$$\tfrac{1}{2}200(X-Y)^2 + \tfrac{1}{2}200(Y-Z)^2 + \tfrac{1}{2}200(Z)^2$$

$$= \tfrac{1}{2}k(X)^2$$

Hence $\boxed{k = \dfrac{200\left[(X-Y)^2 + (Y-Z)^2 + Z^2\right]}{X^2}}$

$$m = \frac{4X^2 + 2Y^2 + 2Z^2}{X^2} \qquad k = \frac{200\left[(X-Y)^2 + (Y-Z)^2 + Z^2\right]}{X^2}$$

To get values for $\boldsymbol{m}$ and $\boldsymbol{k}$, we need an estimate for the mode shape



Characteristics:

- All masses move in phase with each other
- $\boldsymbol{X > Y > Z}$

Choice #1

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{Bmatrix} 3 \\ 2 \\ 1 \end{Bmatrix}$$

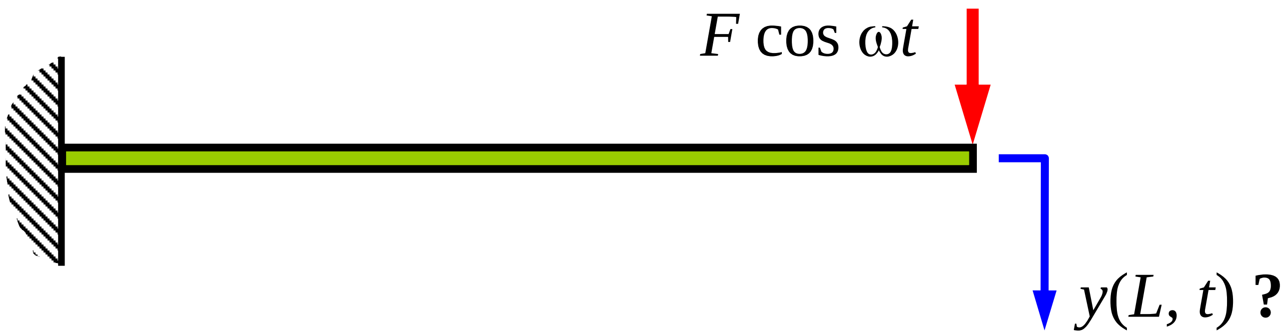
Since all springs have the same stiffness, we might guess that they all deflect by the same amount

$\boldsymbol{m}$ = 5.11 kg and $\boldsymbol{k}$ = 66.7 N/m

Other choices are given in the lecture notes

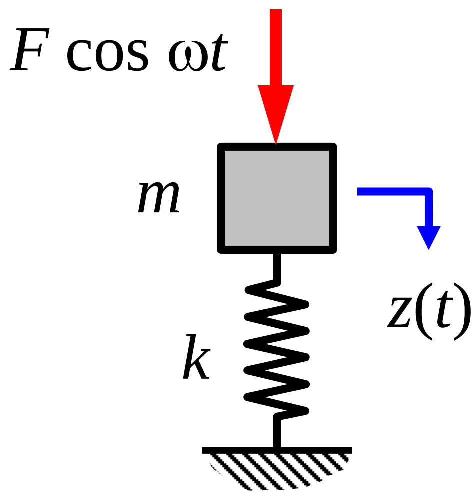
Single DOF approximations

- Cantilevered beam

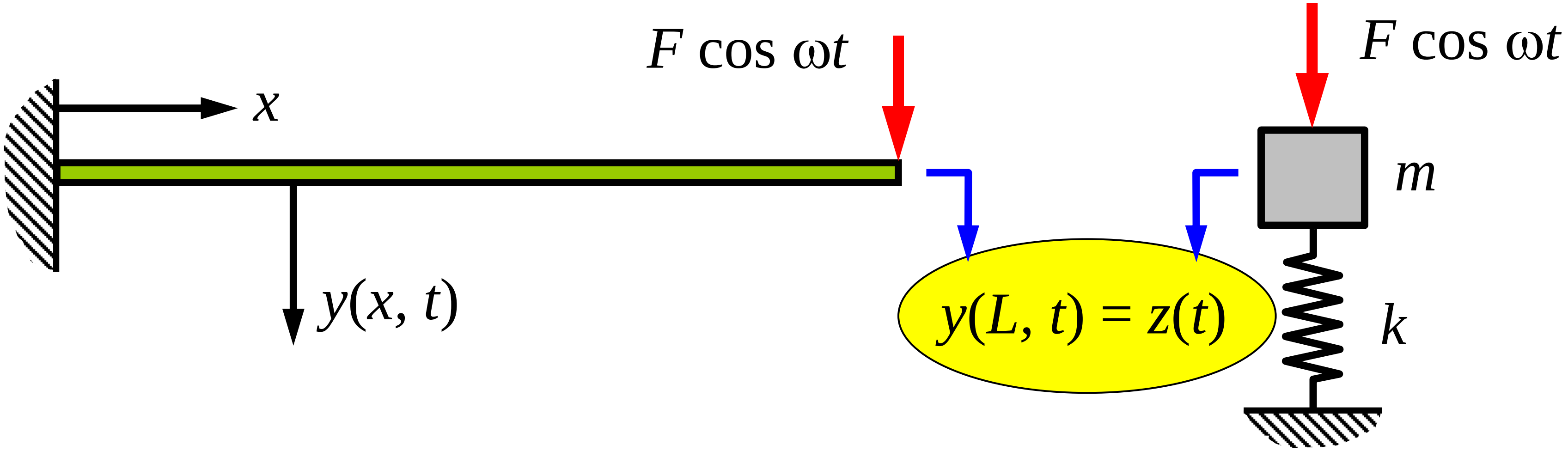


There are two stages

1. Set up the approximate model
2. Use it to do the steady-state response calculation



Stage 1: Set up the model



We must link the displacement of the model mass to that at the free end of the cantilever

In terms of the chosen displacement variables, $y(L, t) = z(t)$

For steady-state, sinusoidal vibration, the link can be written as

$$Z \cos\omega t = Y(L)\cos\omega t \quad \text{or} \quad Z = Y(L)$$

To proceed, we need to choose an expression for the deflected shape of the cantilever

Choice #1 $Y(x) = C x^2$

We must find the value of C that links the two systems

$$Z = Y(L) = C L^2$$

Hence $C = \dfrac{Z}{L^2}$ and $\boxed{Y(x) = \dfrac{Z}{L^2} x^2}$

This expression for $Y(x)$ is used to calculate the maximum kinetic and strain energies in the beam

Equating each with the equivalent expressions for the single-degree-of-freedom model gives the required mass and stiffness values

$$T_{\max} = \tfrac{1}{2}\,\omega^2 \int_0^L \rho A \left[Y(x)\right]^2 \mathrm{d}x = \tfrac{1}{2}\, m\,\omega^2 Z^2$$

Thus

$$\tfrac{1}{2}\,\omega^2 \int_0^L \rho A \left[\frac{Z}{L^2} x^2\right]^2 \mathrm{d}x = \tfrac{1}{2}\, m\,\omega^2 Z^2$$

$$\tfrac{1}{2}\,\omega^2\, \rho A \frac{Z^2}{L^4} L^5 = \tfrac{1}{2}\, m\,\omega^2 Z^2$$

Hence, the model mass is $\boxed{m = 0.2\,\rho A L}$

Equating the strain energies, gives the model stiffness $\boxed{k = 4\,\dfrac{EI}{L^3}}$

Applying the natural frequency test to the approximate model,

we find that $\omega_n = \sqrt{\dfrac{k}{m}} = \dfrac{4.47}{L^2}\sqrt{\dfrac{EI}{\rho A}}$

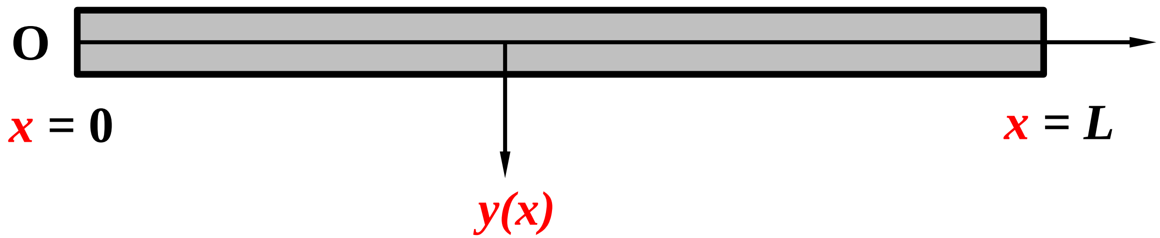
This is the same (poor) result obtained with Rayleigh's Method using this choice for $Y(x)$

For instance, from your previous courses you would have found that the stiffness of a uniform cantilever beam is

$$k = 3\,\frac{EI}{L^3}$$

Other choices are given in the lecture notes

Beam Vibration



- Given a generalized beam we wish to solve for
  - Natural Frequency ω_{nr}
    - Where r is the frequency number (1, 2, 3, …)
  - Mode shapes associated with specific values of ω_{nr}
    - Essentially we are looking for the vertical displacement, y, for any given point along the beam, x

- From previous experience we know then that we need to find a generalized equation

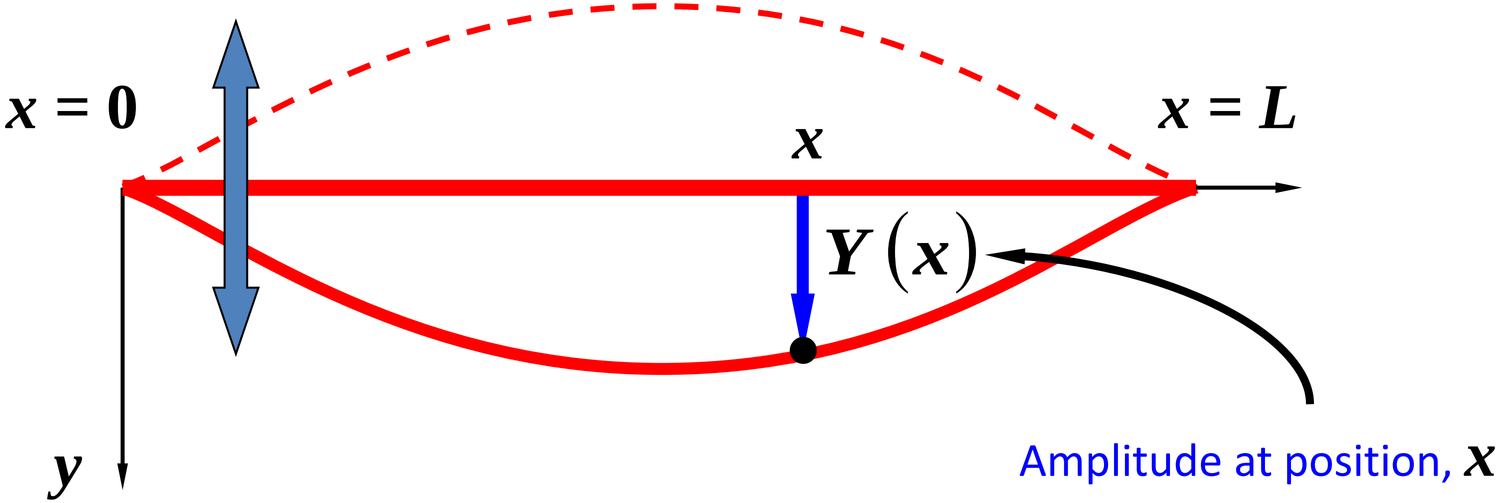
$$[Z]\{C\} = \{0\}$$

- Where $\det[Z] = 0$ will give us ω_{nr}

- Solving the solution vector $\{C\}$ at ω_{nr} will define the mode shapes

- To do this you need a generalized equation for vertical displacement, *y*, as a function of distance along the beam, *x*, and time, *t*.

- For free vibration at a natural frequency, the motion of each point on the beam will be sinusoidal, but the amplitude of vibration will vary along the length



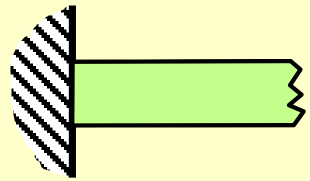
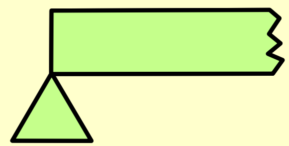
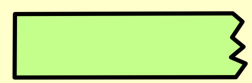
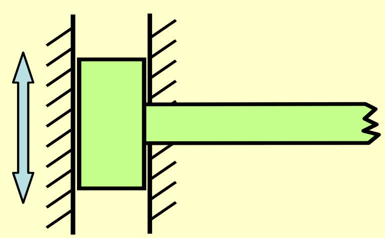
- Substitution of $y(x,t) = Y(x)\cos\omega t$ into $EI\dfrac{\partial^4 y}{\partial x^4} = -\rho A\dfrac{\partial^2 y}{\partial t^2}$

$$Y(x) = C_1\sin\lambda x + C_2\cos\lambda x + C_3\sinh\lambda x + C_4\cosh\lambda x \qquad (7)$$

$$Y(x) = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 \sinh \lambda x + C_4 \cosh \lambda x$$

- This results in a generalized equation for displacement of y at any given point along the beam, x, for a given frequency of vibration (contained in λ)

- **HOWEVER**, this contains 4 unknowns (C_1, C_2, C_3 and C_4) and you will therefore need a minimum of 4 equations to solve for them
  - Boundary conditions must be used!!!

Descriptive terms	Diagrammatic	Boundary conditions
Built-in clamped encastré		$y = 0$ $\quad \frac{\partial y}{\partial x} = 0$
Simple support hinged pinned		$y = 0$ $M = 0 \quad \therefore \quad \frac{\partial^2 y}{\partial x^2} = 0$
Free		$M = 0 \quad \therefore \quad \frac{\partial^2 y}{\partial x^2} = 0$ $S = 0 \quad \therefore \quad \frac{\partial^3 y}{\partial x^3} = 0$
Massless slider		$\frac{\partial y}{\partial x} = 0$ $S = 0 \quad \therefore \quad \frac{\partial^3 y}{\partial x^3} = 0$

You will therefore need to partially differentiate (6)

$$Y(x) = C_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \sinh\lambda x + C_4 \cosh\lambda x \quad \textbf{(6a)}$$

several times with depending on what boundary conditions you have

$$\frac{dY}{dX} = C_1 \lambda\cos\lambda x - C_2 \lambda\sin\lambda x + C_3 \lambda\cosh\lambda x + C_4 \lambda\sinh\lambda x \quad \textbf{(6b)}$$

$$\frac{d^2Y}{dx^2} = -C_1 \lambda^2 \sin\lambda x - C_2 \lambda^2 \cos\lambda x + C_3 \lambda^2 \sinh\lambda x + C_4 \lambda^2 \cosh\lambda x \quad \textbf{(6c)}$$

$$\frac{d^3Y}{dx^3} = -C_1 \lambda^3 \cos\lambda x + C_2 \lambda^3 \sin\lambda x + C_3 \lambda^3 \cosh\lambda x + C_4 \lambda^3 \sinh\lambda x \quad \textbf{(6d)}$$

In the exam you are expected to be able to form the generalized matrices ($[Z]\{C\}=\{0\}$) for any boundary conditions.

You will not be asked to solve for natural frequencies and/or mode shapes by hand.

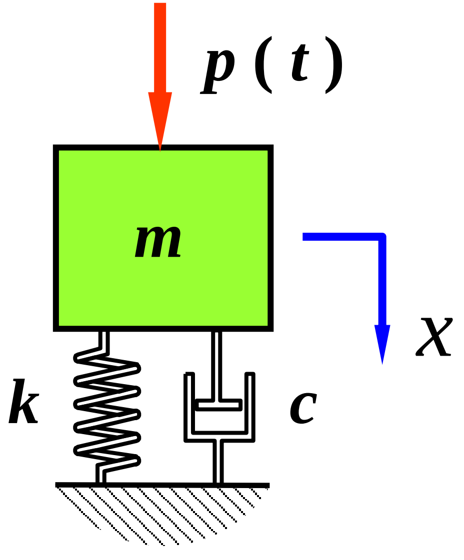
You are expected to be able to sketch basic mode shapes given boundary conditions, without solving for values.

Vibration Isolation

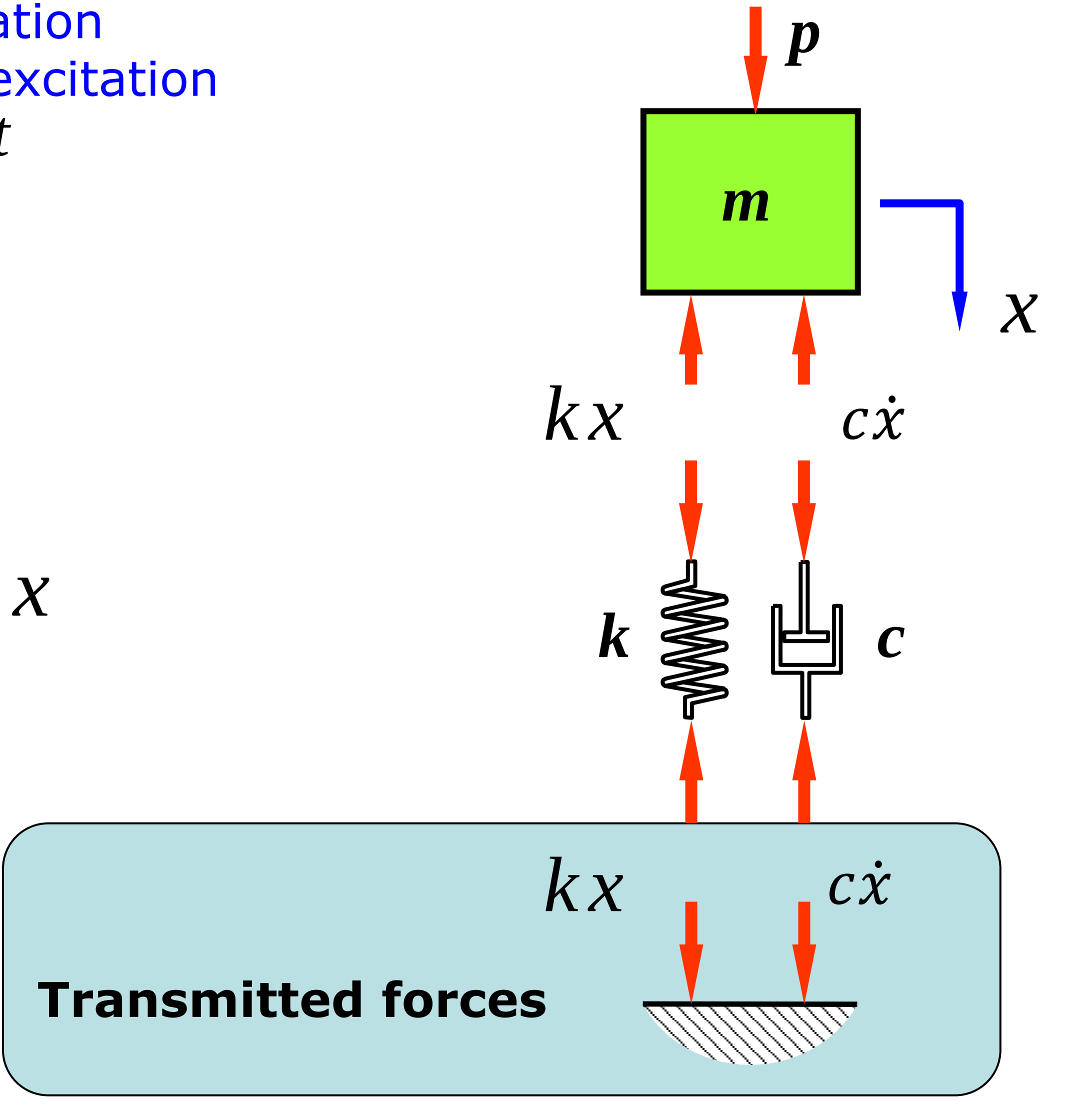
Case (a) Source of vibration within a device : **How much force is transmitted to the support?**

STEP 1: Dynamic model

Assume that the vibration source generates an excitation force, $p(t) = P\cos\omega t$



STEP 2: Free Body Diagram

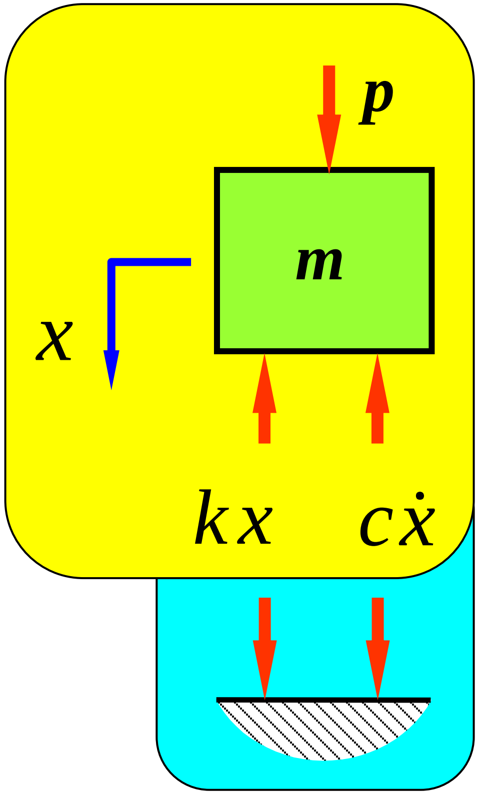


STEP 3: Equation of motion

For the device

$\downarrow x$

$$m\ddot{x} + c\dot{x} + kx = p \qquad (1)$$



Transmitted force $$q(t) = kx + c\dot{x} \qquad (2)$$

Substitutions:

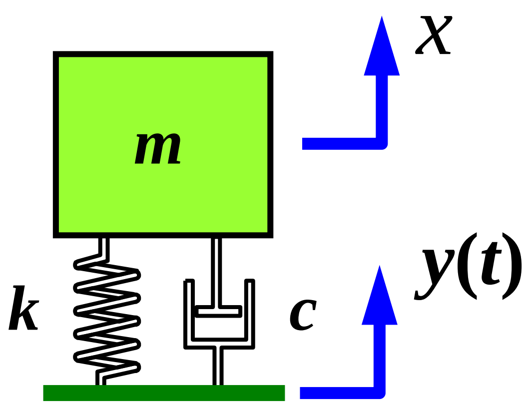
$$p(t) = P\,e^{i\omega t},\quad q(t) = Q^*\,e^{i\omega t}$$ and
$$x(t) = X^* e^{i\omega t}$$
$$\dot{x}(t) = i\omega X^* e^{i\omega t}$$
$$\ddot{x}(t) = -\omega^2 X^* e^{i\omega t}$$

$$T_F = \left|\frac{Q^*}{P}\right| = \sqrt{\frac{k^2 + c^2\omega^2}{\left(k - m\omega^2\right)^2 + c^2\omega^2}}$$

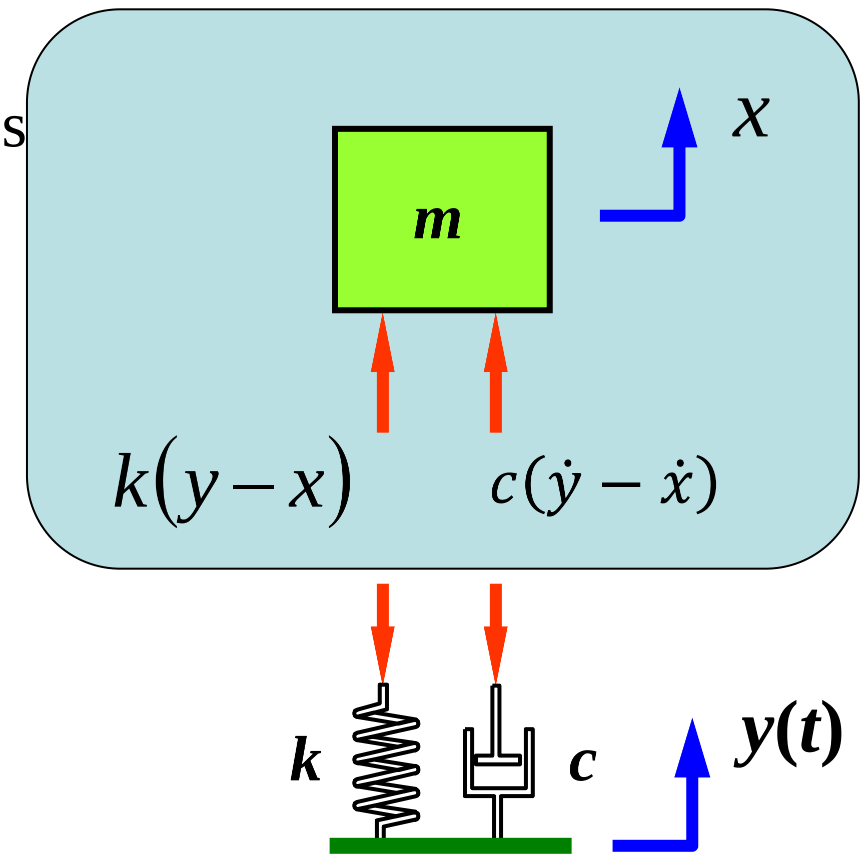
Case (b) Source of vibration from the support : **How much vibration is transmitted to the device?**

STEP 1: Dynamic model

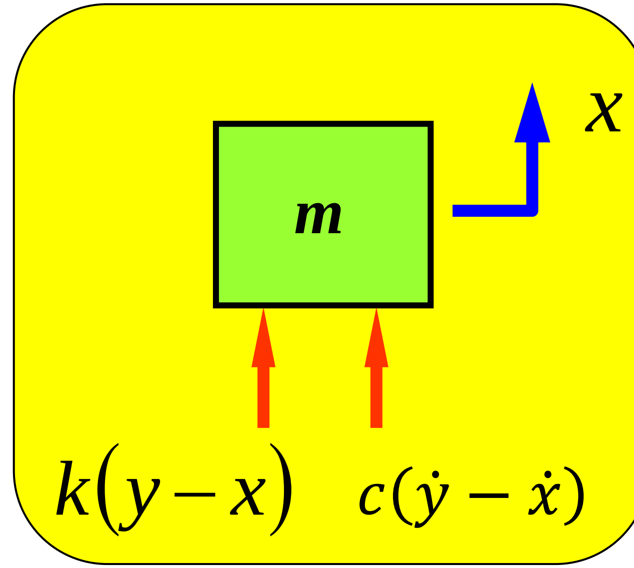
The support vibration is defined by the **displacement**, $y(t) = Y \cos$

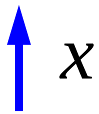


STEP 2: Free Body Diagram



STEP 3: Equation of motion





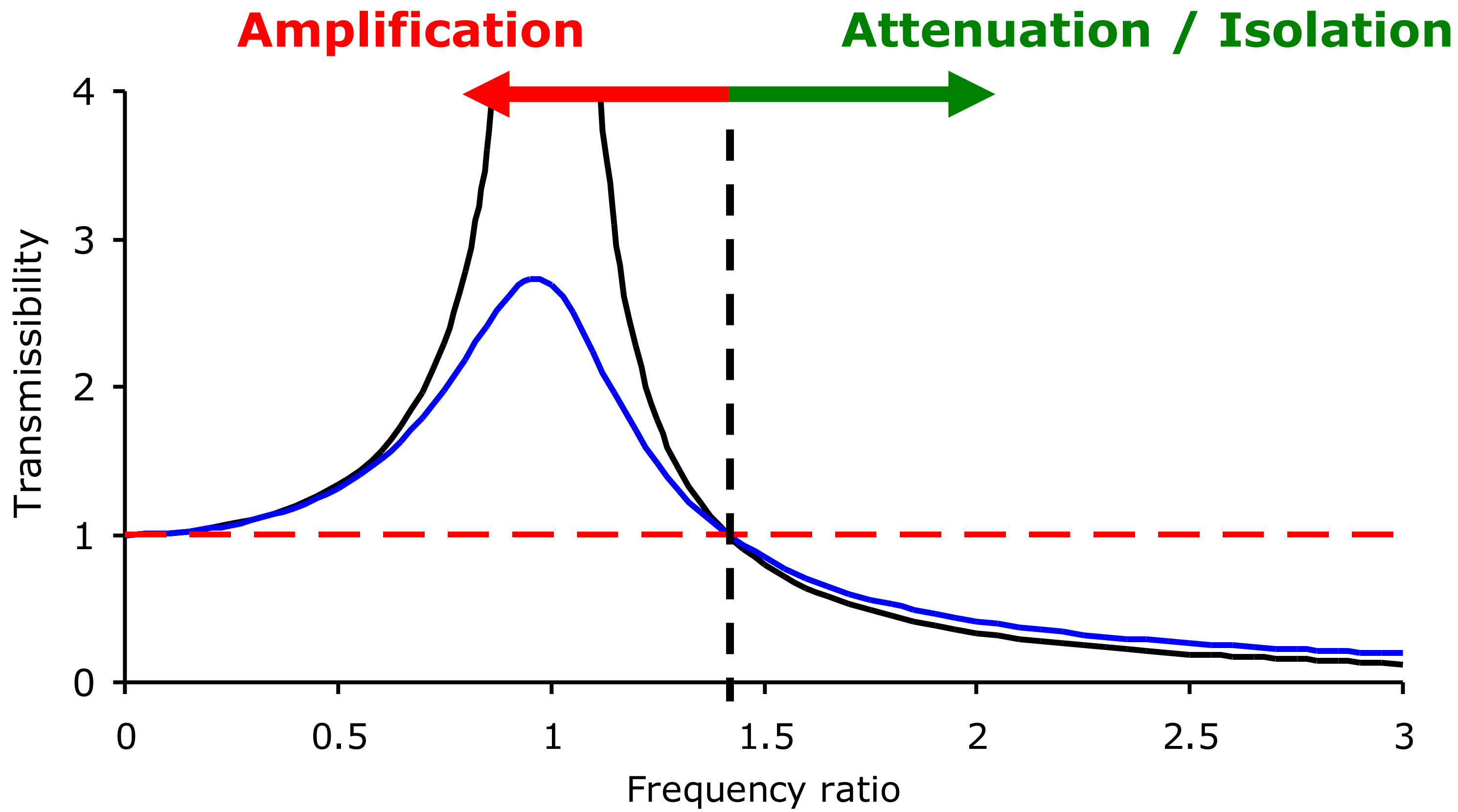
$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

Substitutions:
$$y(t) = Ye^{i\omega t}$$
$$\dot{y}(t) = i\omega Ye^{i\omega t}$$
and
$$x(t) = X^* e^{i\omega t}$$
$$\dot{x}(t) = i\omega X^* e^{i\omega t}$$
$$\ddot{x}(t) = -\omega^2 X^* e^{i\omega t}$$

We define **DISPLACEMENT TRANSMISSIBILITY** as

$$T_{\mathrm{D}} = \left|\frac{X^*}{Y}\right| = \sqrt{\frac{k^2 + c^2\omega^2}{\left(k - m\omega^2\right)^2 + c^2\omega^2}}$$

!!!NOTE!!!
Other physical systems will have different transmissibility expressions.
To be sure of your work it is best to derive $T_{D,F}$ every time.



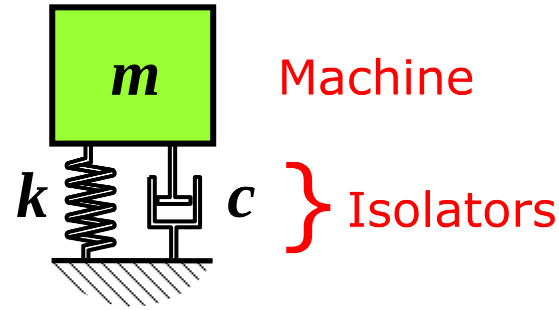
It's easy to show that T = 1 when $\omega/\omega_n = \sqrt{2}$

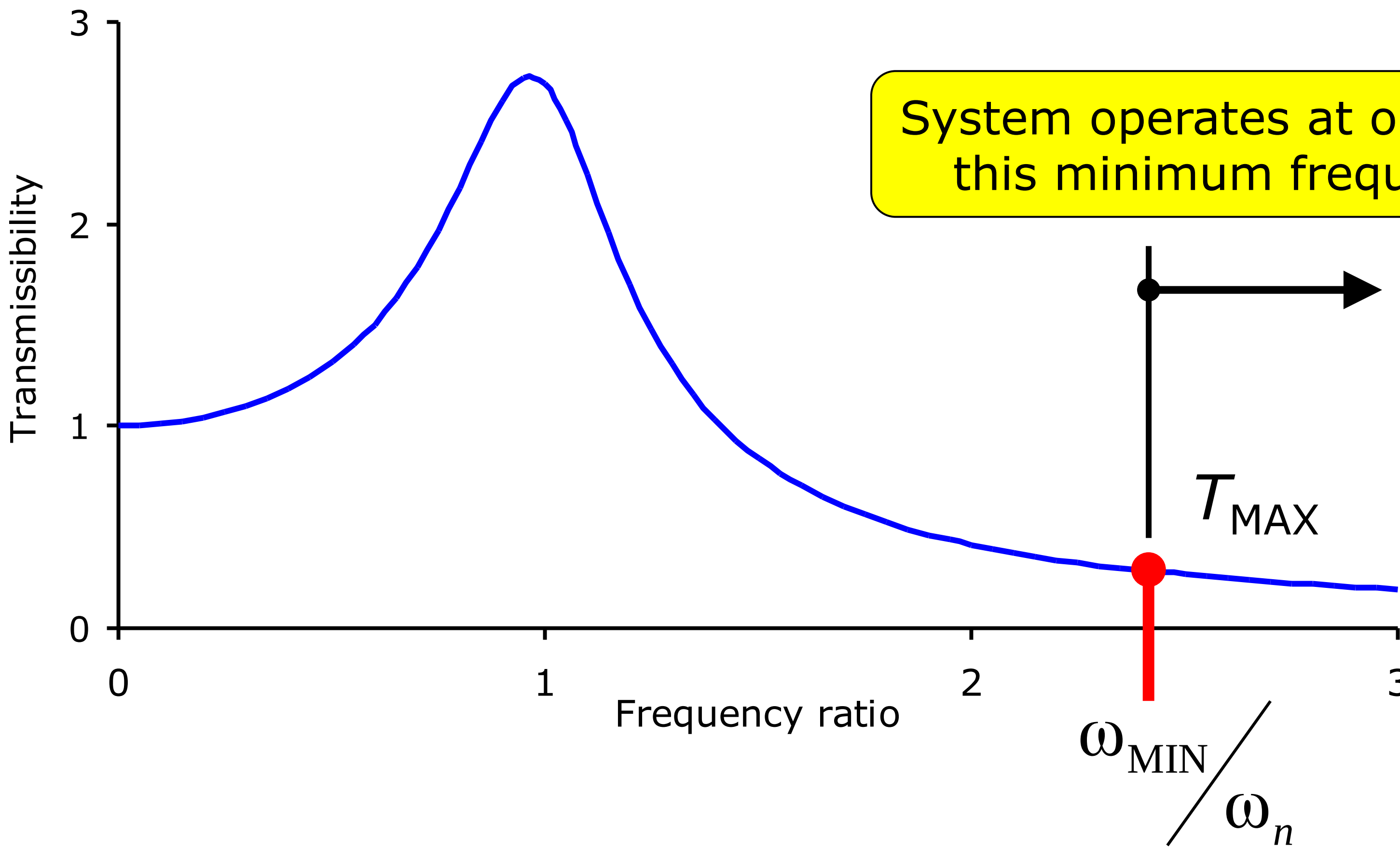
The aim in selecting isolators is to ensure that the system operates in the "isolation region"

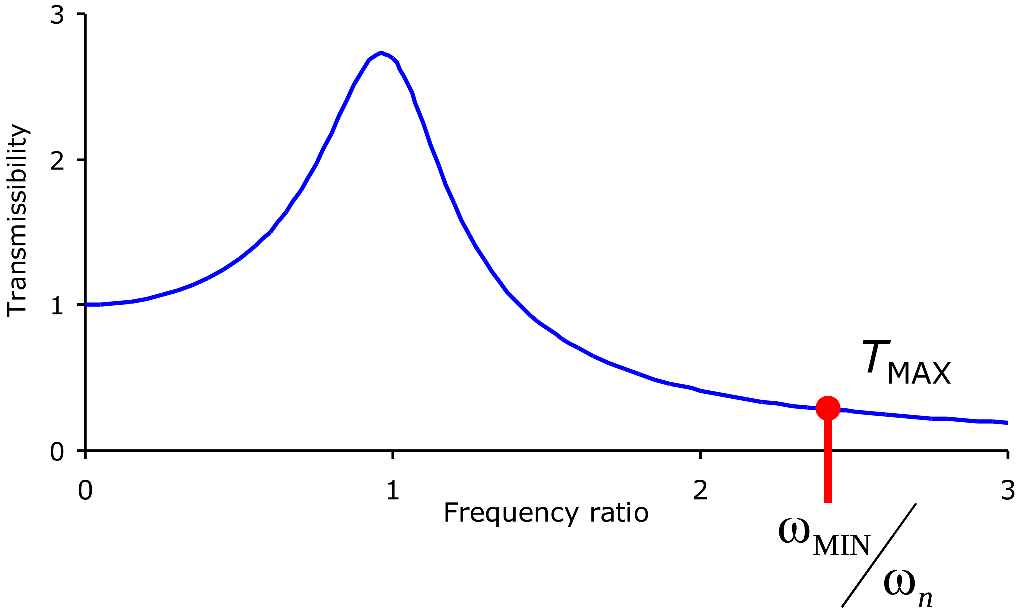
Design Approach for Isolator Selection

Two constraints for isolator selection:

- the lowest excitation frequency, ω_{MIN}
- the maximum allowable transmissibility, T_{MAX}







$$T = \frac{1}{\omega^2 / \omega_n^2 - 1}$$

If $T = T_{MAX}$ at $\omega = \omega_{MIN}$

$$\omega_n^2 = \frac{T_{MAX}\, \omega_{MIN}^2}{1 + T_{MAX}}$$

Since $\omega_n^2 = \dfrac{k}{m}$, the required isolator stiffness is

$$\boxed{k = m\omega_n^2 = \frac{m T_{MAX}\, \omega_{MIN}^2}{1 + T_{MAX}}} \qquad (1)$$

Equation (1) is the ***maximum*** stiffness consistent with the design constraints

There are also constraints imposed by static considerations

Manufacturers often express these constraints by specifying a ***maximum static deflection***

The actual static deflection, X_0, is given by

$$X_0 = \frac{m\,g}{k_{\mathrm{ISOLATOR}}} \qquad (2)$$

Alternatively, combining (1) and (2) gives

$$X_0 = \frac{g}{\omega^2_{\mathrm{MIN}}}\left(1 + \frac{1}{T_{\mathrm{MAX}}}\right) \qquad (3)$$

This is the ***minimum* static deflection** consistent with the design constraints

Examples

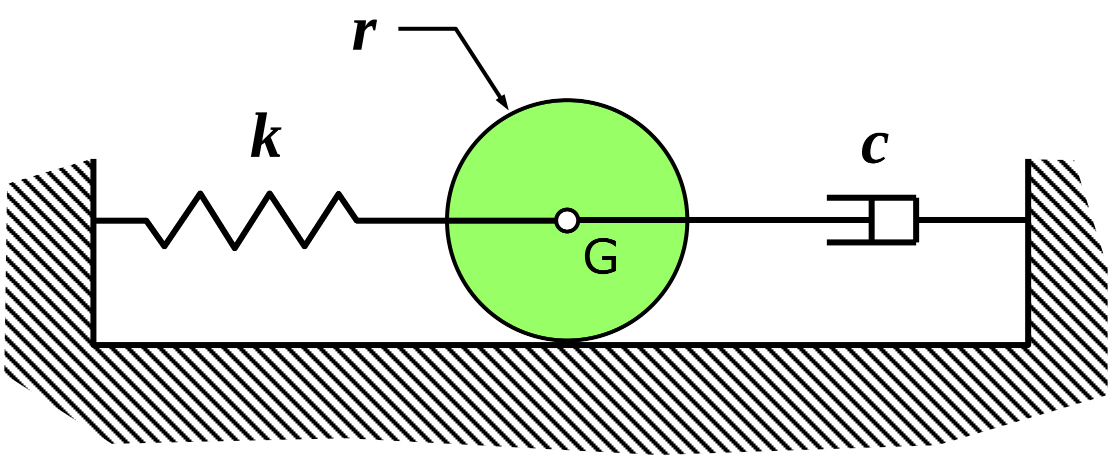
These are some further examples of what you might expect on the exam. In most cases the final solution has not been worked out as it is suggested that you try them yourself first.

1. Figure Q.1 shows a solid disc with a mass of 0.8 kg and a radius of 0.6 m, which can roll without slipping on a horizontal surface. It is restrained by a spring of stiffness 5 kN/m and by a viscous damper having a damping coefficient of 15 Ns/m.

   (a) Draw a fully annotated free body diagram for the disc and derive the equation of motion. Hence show that the undamped natural frequency and the damping ratio are approximately 10.3 Hz and 0.097 respectively. [10]

   (b) With the disc in its equilibrium position, it is given an initial angular velocity of 6 rad/s. Find the maximum angular displacement of the disc during the subsequent free vibration. [10]

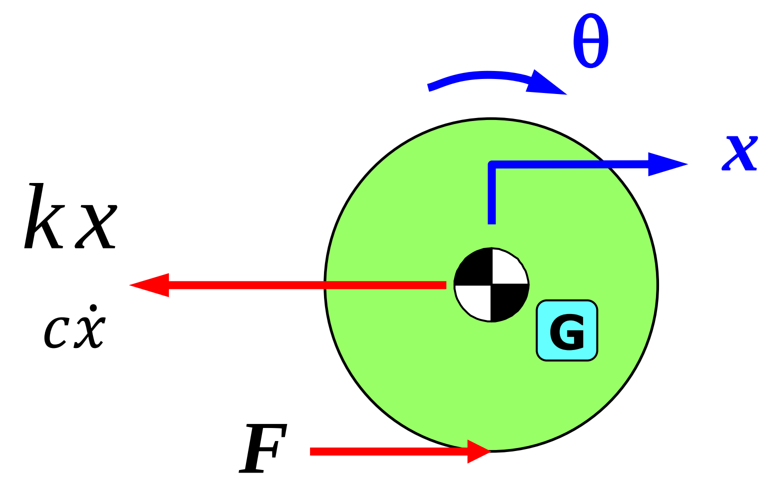
Figure Q.1



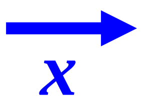
Free Body Diagram

Question asked for "a fully annotated free body diagram for the disc"

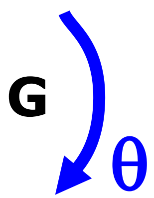
Students lost marks if they didn't draw one



Equations of motion



$$-kx - c\dot{x} + F = m\ddot{x}$$



$$-Fr = I_G\ddot{\theta}$$

For no slip at the ground, $x = r\theta \quad \dot{x} = r\dot{\theta} \quad \ddot{x} = r\ddot{\theta}$

Substitute for x and then eliminate F to give the equation of motion

Final equation of motion is $3m\ddot{\theta} + 2c\dot{\theta} + 2k\theta = 0$

Hence, $\omega_n = \sqrt{\dfrac{2k}{3m}} = 10.3\,\mathrm{Hz}$

$$\gamma = \frac{"C"}{2\sqrt{"K""M"}} = \frac{2c}{2\sqrt{2k\,.\,3m}} = 0.097$$ so we have "light" damping

Response expression

$$\theta(t) = e^{-\gamma\omega_n t}[\,B_1 \cos \Omega_n t + B_2 \sin \Omega_n t]$$

where $\Omega_n = \omega_n \sqrt{1-\gamma^2}$

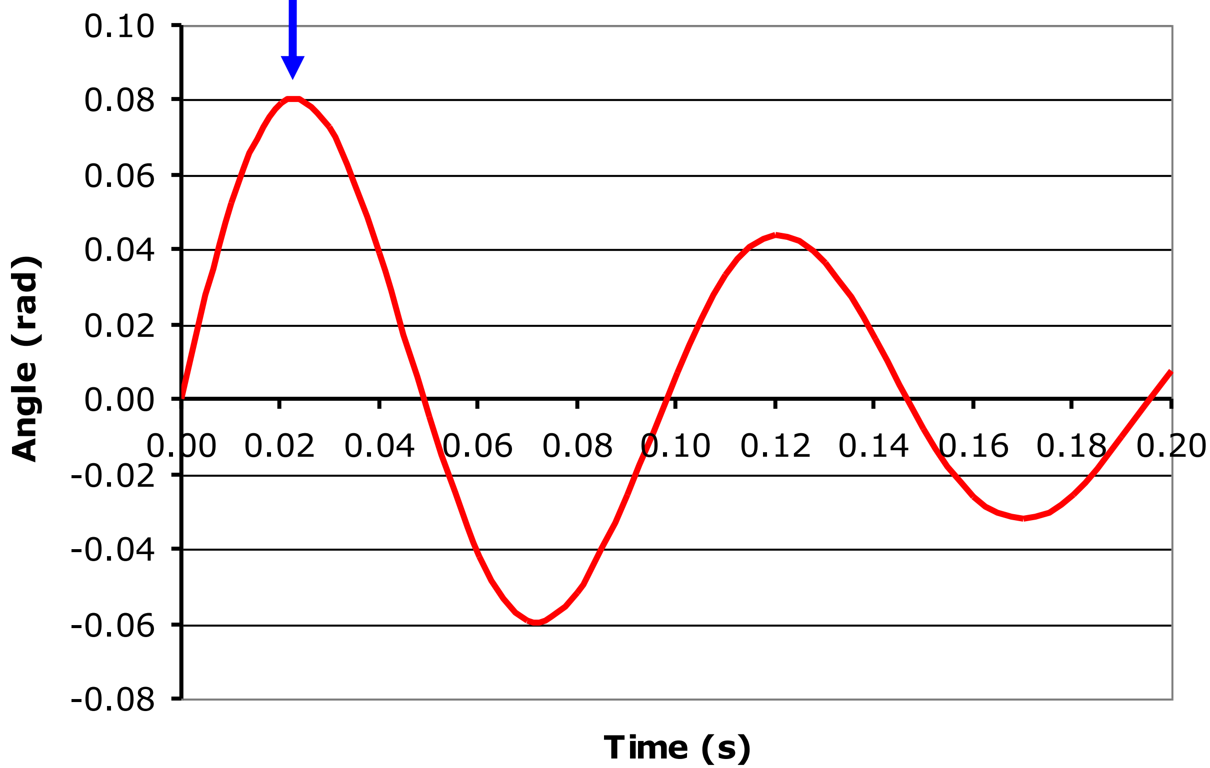
Initial conditions given: $\theta = 0$ and $\dot{\theta} = 6\,rad/s$ when $t = 0$

Use these to find the constants B_1 and B_2

Initial conditions: $\theta_0 = 0$ and $\dot{\theta} = 6\, rad/s$ when $t = 0$

This gives
$$\theta(t) = \frac{\ddot{\theta}_o}{\Omega_n} e^{-\gamma\omega_n t} \sin \Omega_n t$$

Max. angular displacement is when $\dot{\theta} = 6$



2. A rigid bar, AB, has length L and moment of inertia, I_O, about the central pivot at O. Identical masses (mass m) are connected at each end of the bar by springs K_1 and K_2, as shown in Figure Q.2.

   (a) Draw fully annotated free body diagrams for the system. [5]

   (b) Derive the equations of motion in matrix form. [10]

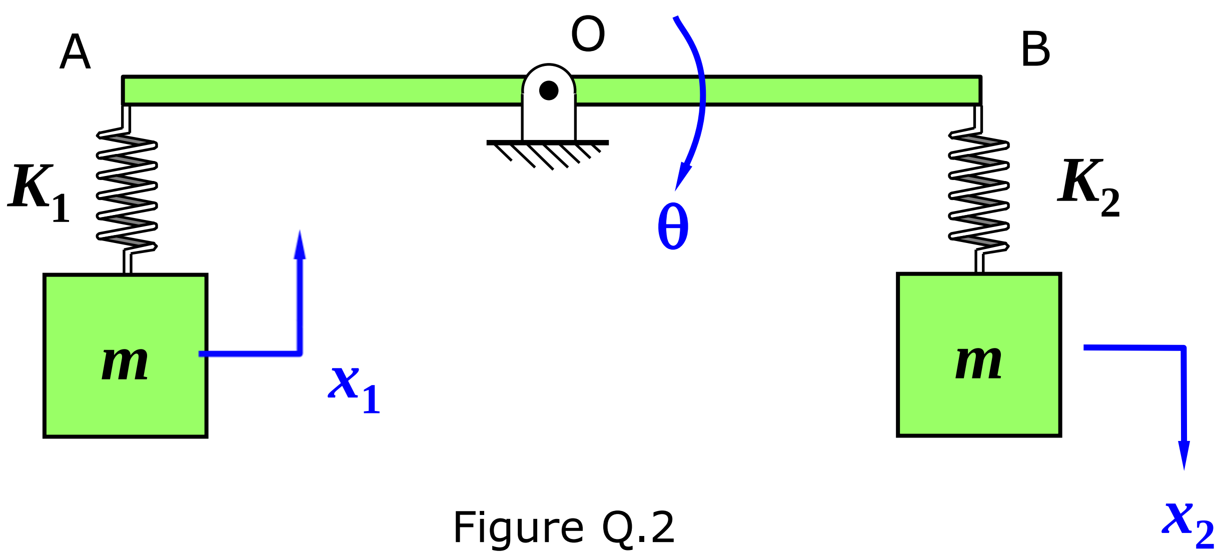
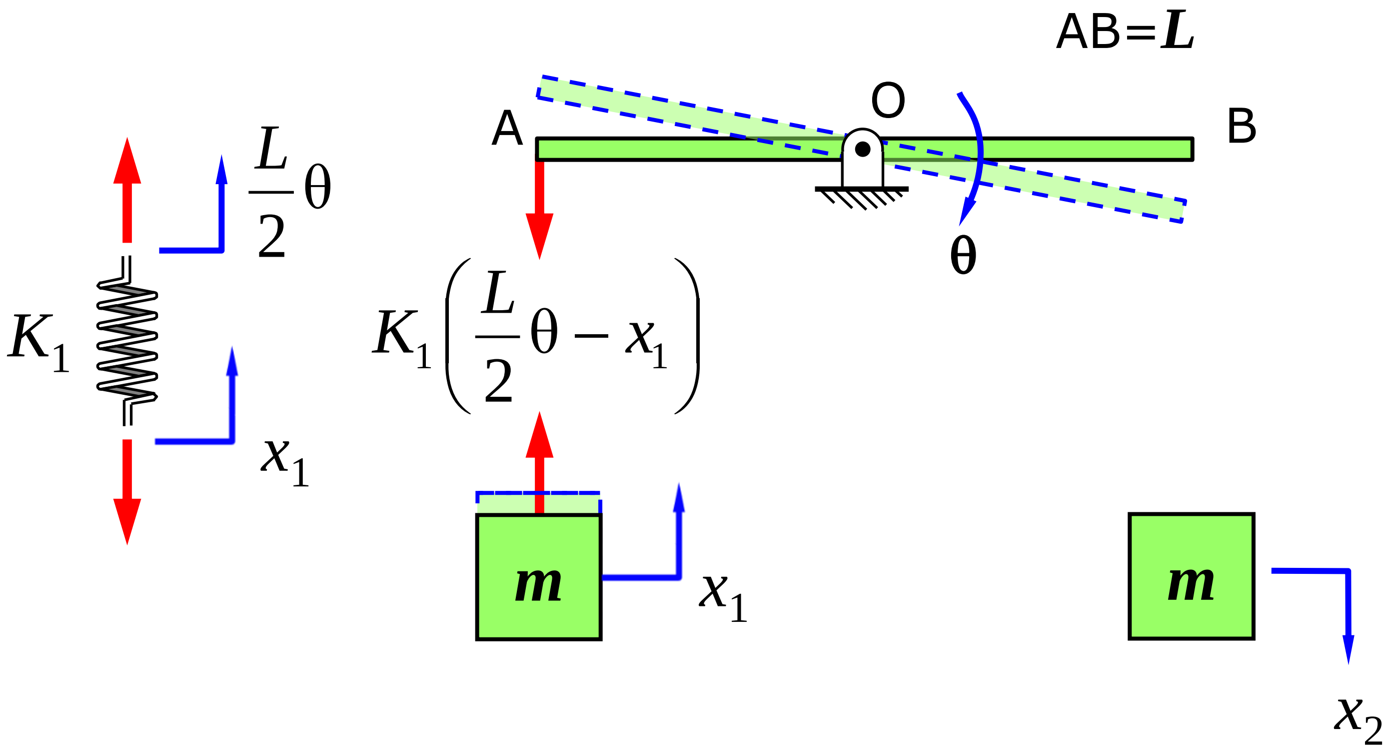


Figure Q.2

Free Body Diagrams

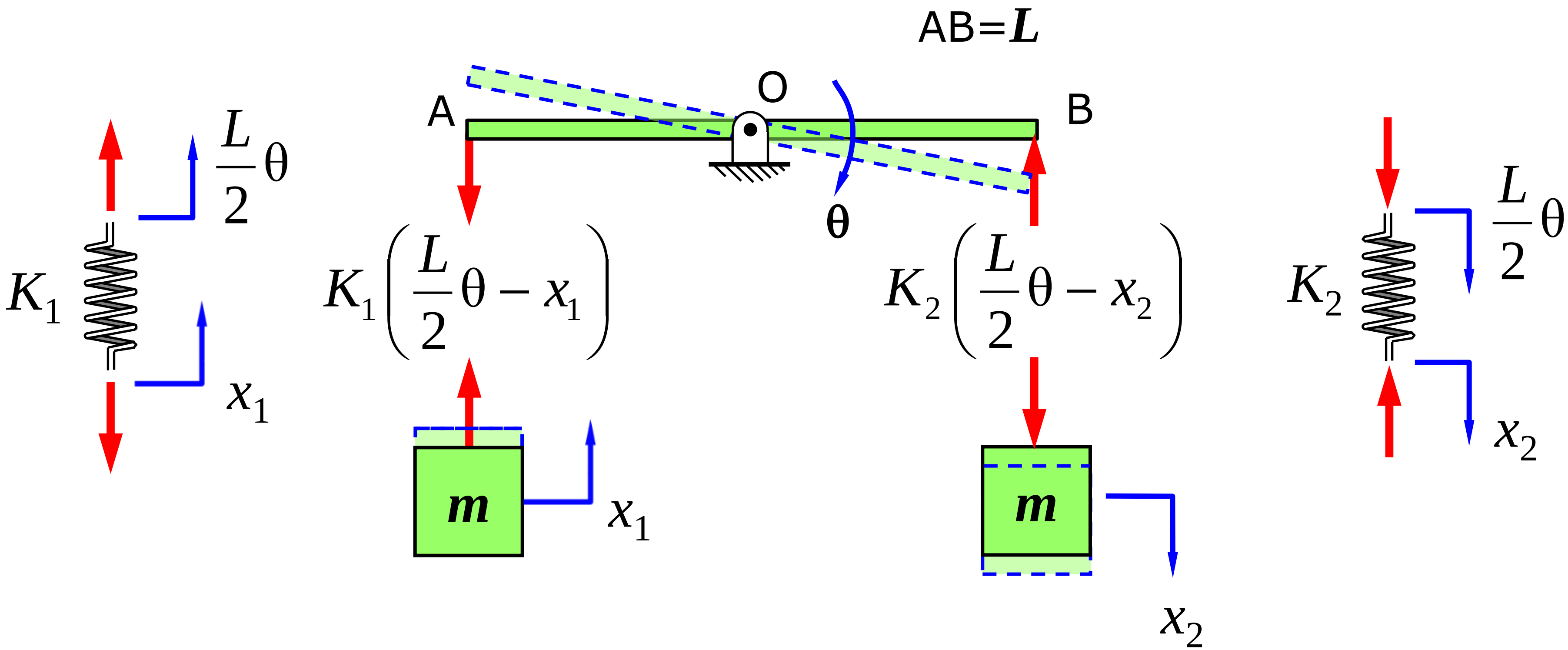


Change of length of spring K_1 is $\frac{L}{2}\theta - x_1$

Spring is in **TENSION**

Force is $K_1\left(\frac{L}{2}\theta - x_1\right)$

Free Body Diagrams

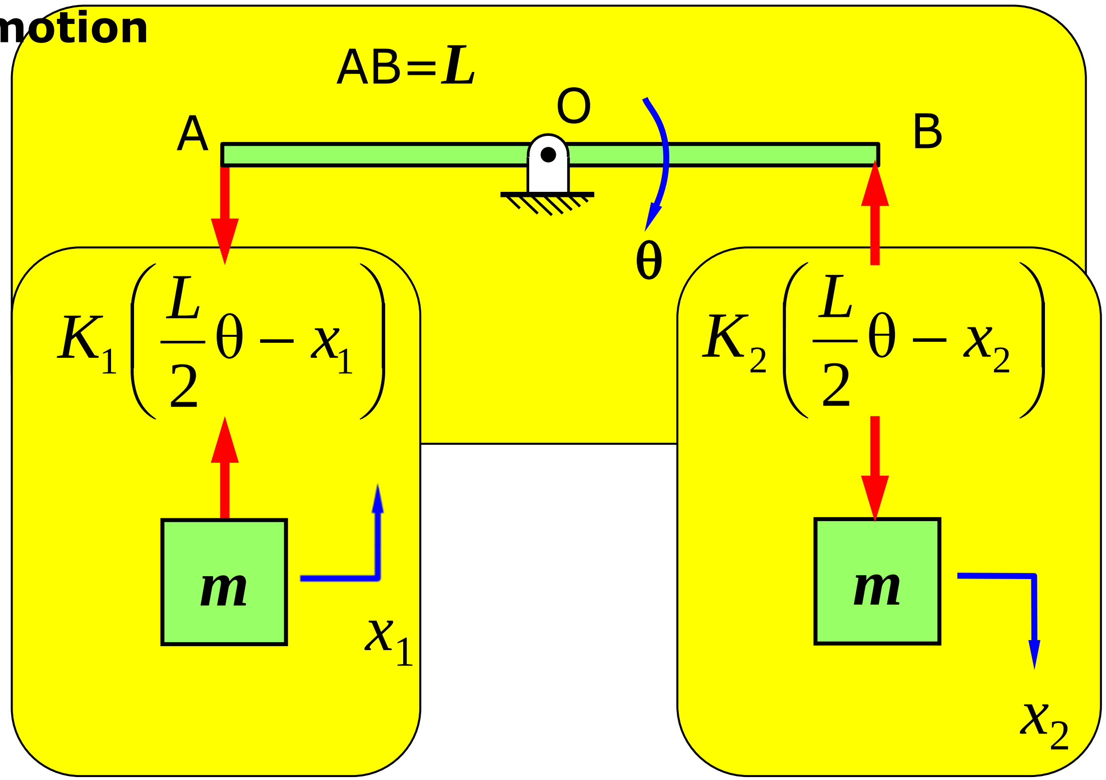


Cha is e of length of spring K_2 is $\frac{L}{2}\theta - x_2$

Spring is in **COMPRESSION**

Force is $K_2\left(\frac{L}{2}\theta - x_2\right)$

Equations of motion



x_1 ↑

$$+K_1\left(\frac{L}{2}\theta - x_1\right) = m\ddot{x}_1$$

x_2 ↓

$$+K_2\left(\frac{L}{2}\theta - x_2\right) = m\ddot{x}_2$$

O θ

$$-K_1\left(\frac{L}{2}\theta - x_1\right) \times \frac{L}{2} - K_2\left(\frac{L}{2}\theta - x_2\right) \times \frac{L}{2} = I_o\ddot{\theta}$$

Re-arrange into matrix form:

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_o \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} K_1 & 0 & -K_1 \dfrac{L}{2} \\ 0 & K_2 & -K_2 \dfrac{L}{2} \\ -K_1 \dfrac{L}{2} & -K_2 \dfrac{L}{2} & (K_1 + K_2) \dfrac{L^2}{4} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Remember to check that

- leading diagonals contain only positive terms
- matrices are symmetric

3. Figure Q.3 shows a rocker arm with moment of inertia, I_O, about the pivot at O. Rubber blocks at ends A and B can each be modelled as a spring (stiffness, k) in parallel with a viscous damper (damping coefficient, c). The base of the block at A is attached to a rigid foundation. The base of the block at B is attached to a follower, which is driven by a cam that gives the follower a sinusoidal displacement of amplitude, Y, and frequency ω.

(a) Draw a fully annotated free body diagram for the rocker arm. [6]

(b) Derive an expression for the steady-state amplitude of the displacement at A. [10]

(c) Derive an expression for the steady-state amplitude of the force transmitted to the foundation at A. [4]

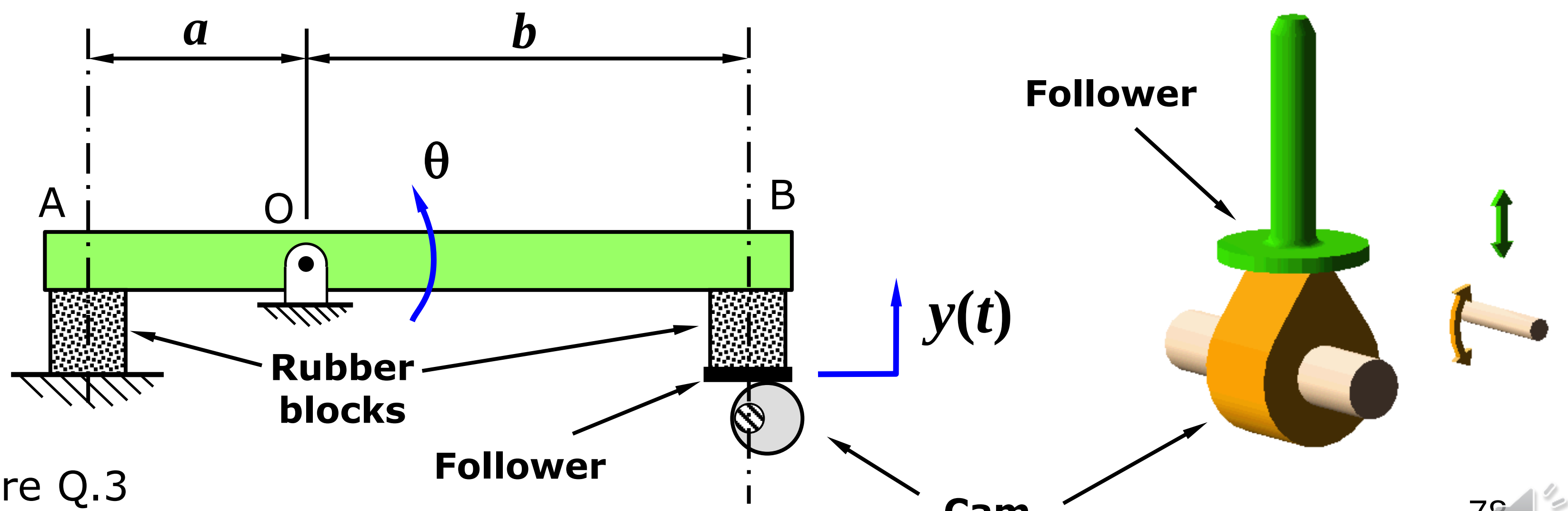
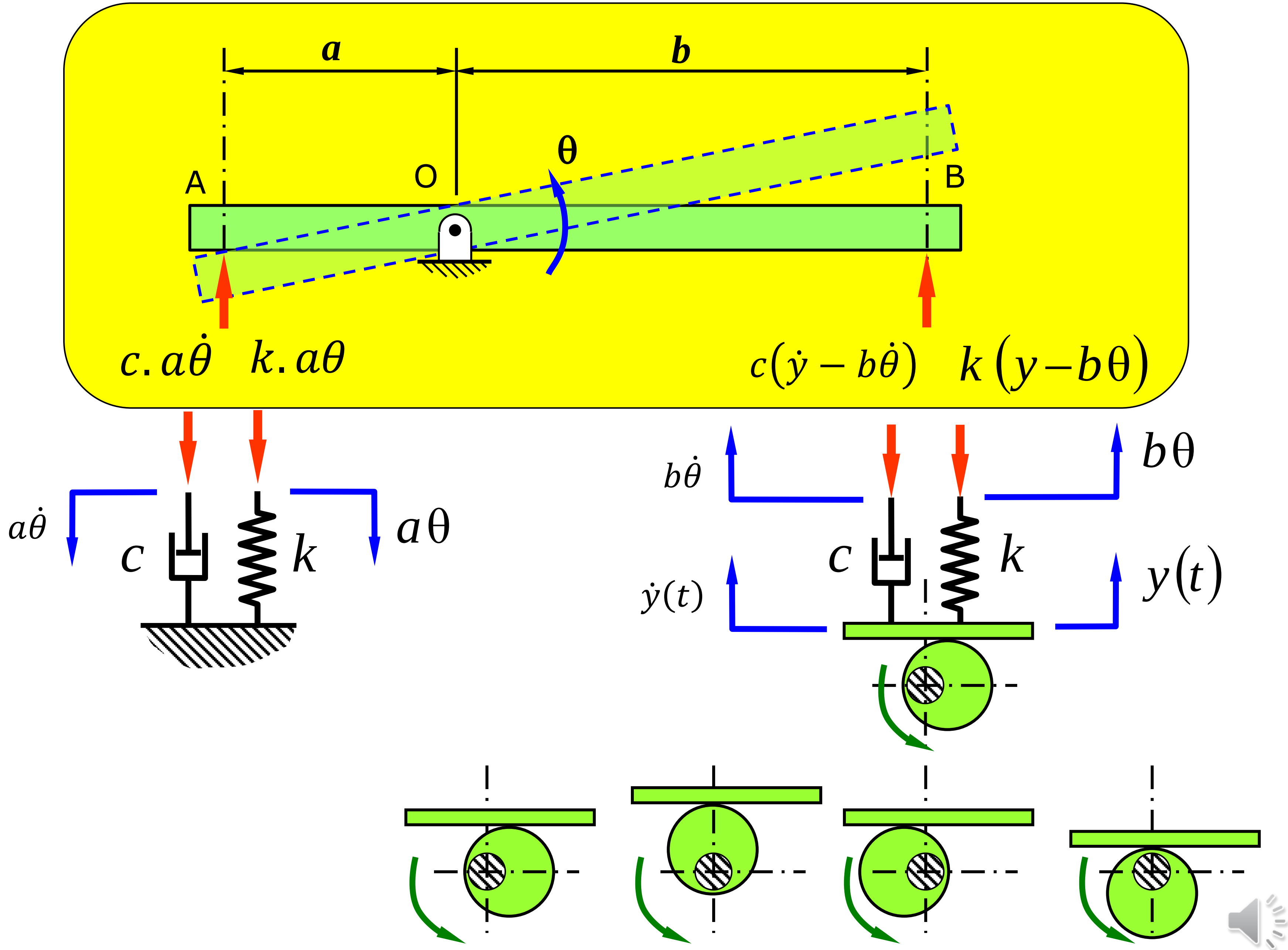
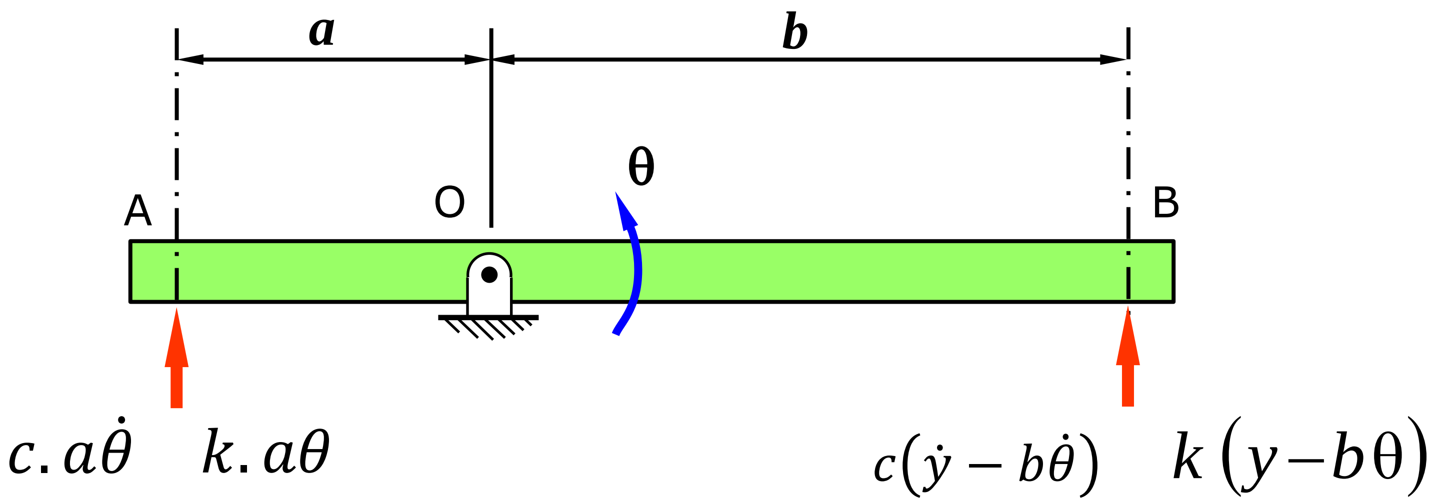


Figure Q.3

Free Body Diagram



Equation of motion



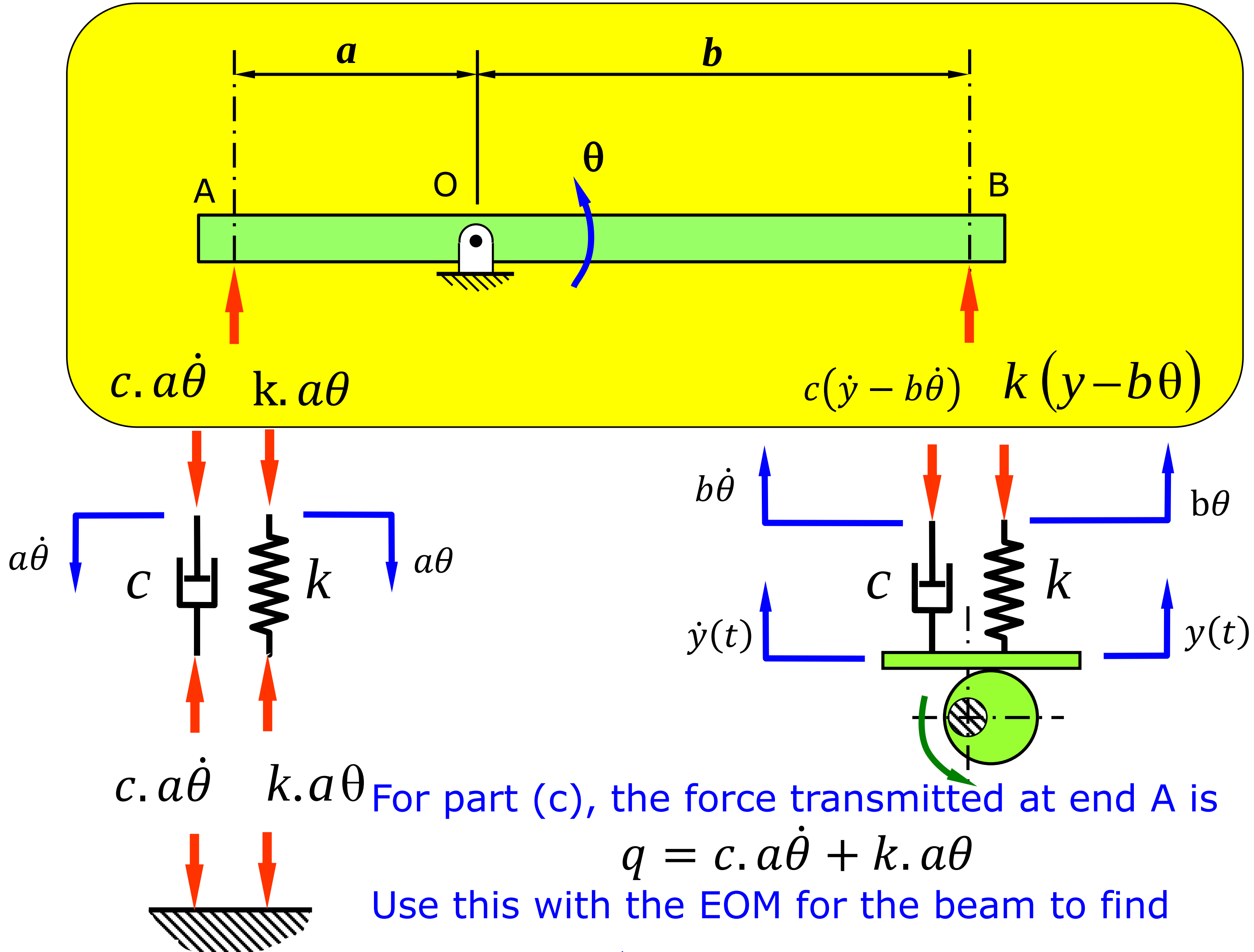
O θ $-ca\ddot{\theta} \times a$

or $I_o\ddot{\theta}$

Substitutions: $y(t) = Y\,\mathrm{e}^{\mathbf{i}\,\omega\,t}$ and $\theta(t) = \Theta^*\,\mathrm{e}^{\mathbf{i}\,\omega\,t}$

For part (b), the displacement at end A is $a\,\Theta^*$

Free Body Diagram



For part (c), the force transmitted at end A is

$$q = c.a\dot{\theta} + k.a\theta$$

Use this with the EOM for the beam to find

$T_f = \frac{Output}{Input} = \frac{Q^*}{Y}$. Note that the Θ^* terms will drop out.

5. Use Rayleigh's Method, with at least two estimates for the mode shape, to estimate the lowest natural frequency of the system shown in Figure Q.5. [10]

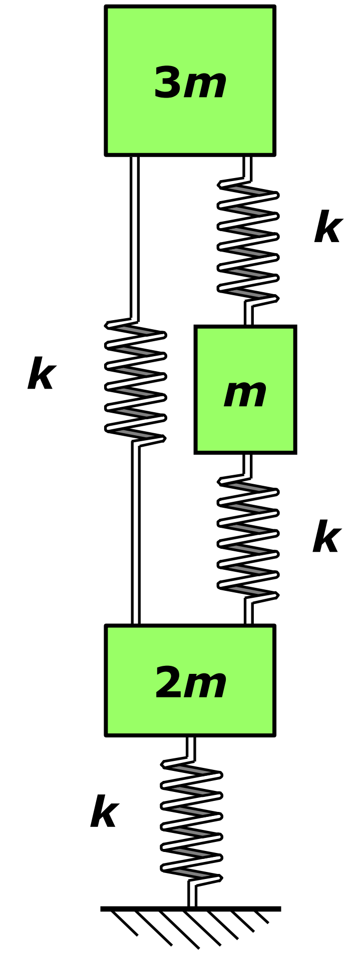
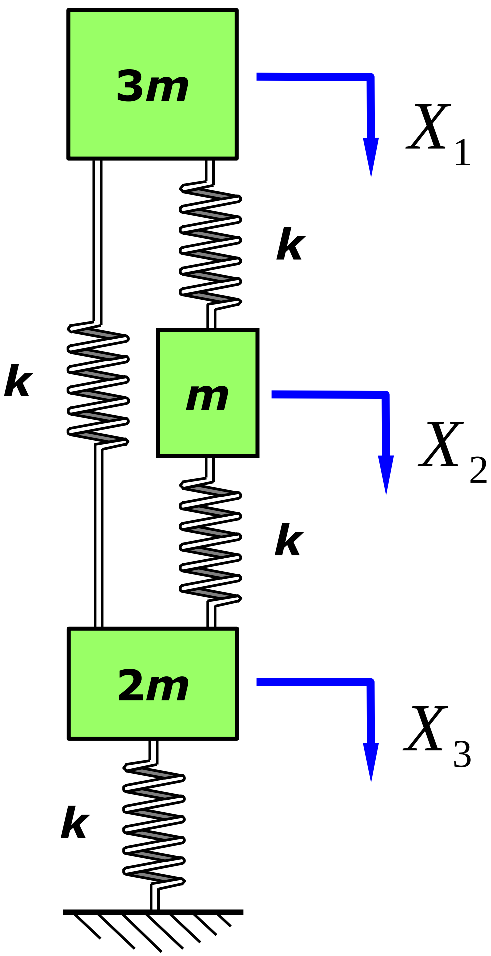


Figure Q.5

Rayleigh's Method



The maximum kinetic energy of mass ***j*** is

$$\left(T_{\max}\right)_{\text{Mass } j} = \tfrac{1}{2} m_j \left(\omega X_j\right)^2$$

Here

$$T_{\max} = \tfrac{1}{2}\omega^2\left(3m X_1^2 + m X_2^2 + 2m X_3^2\right)$$

The maximum strain energy in a spring is

$$\tfrac{1}{2} \times \text{Stiffness} \times \left(\text{Change of length}\right)^2$$

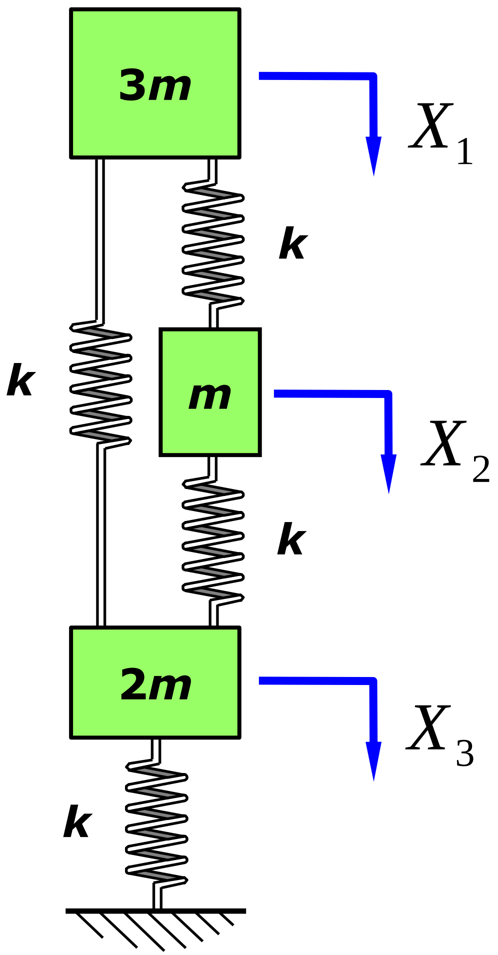
Here

$$U_{\max} = \tfrac{1}{2} k \Big\lfloor \left(X_1 - X_2\right)^2 + \left(X_2 - X_3\right)^2 + \left(X_1 - X_3\right)^2 + X_3^2 \Big\rfloor$$

Mode shape estimates

Try to estimate the static deflection shape

Here $X_1 > X_2 > X_3$



Guess #1 $\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 2 \\ 1 \end{Bmatrix} \Rightarrow \omega_n = 0.461\sqrt{\frac{k}{m}}$

Guess #2 $\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 3 \\ 2 \end{Bmatrix} \Rightarrow \omega_n = 0.392\sqrt{\frac{k}{m}}$

Guess #3 $\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 8 \\ 7 \\ 6 \end{Bmatrix} \Rightarrow \omega_n = 0.366\sqrt{\frac{k}{m}}$

Exact value is $\omega_n = 0.363\sqrt{\frac{k}{m}}$