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MMME2053
Mechanics of Solids

Revision

Lecture - 18/05/2023

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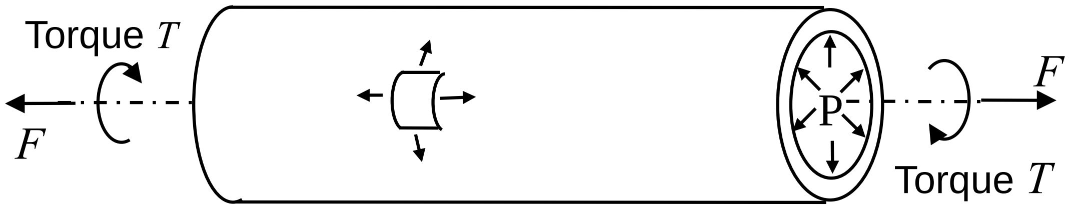
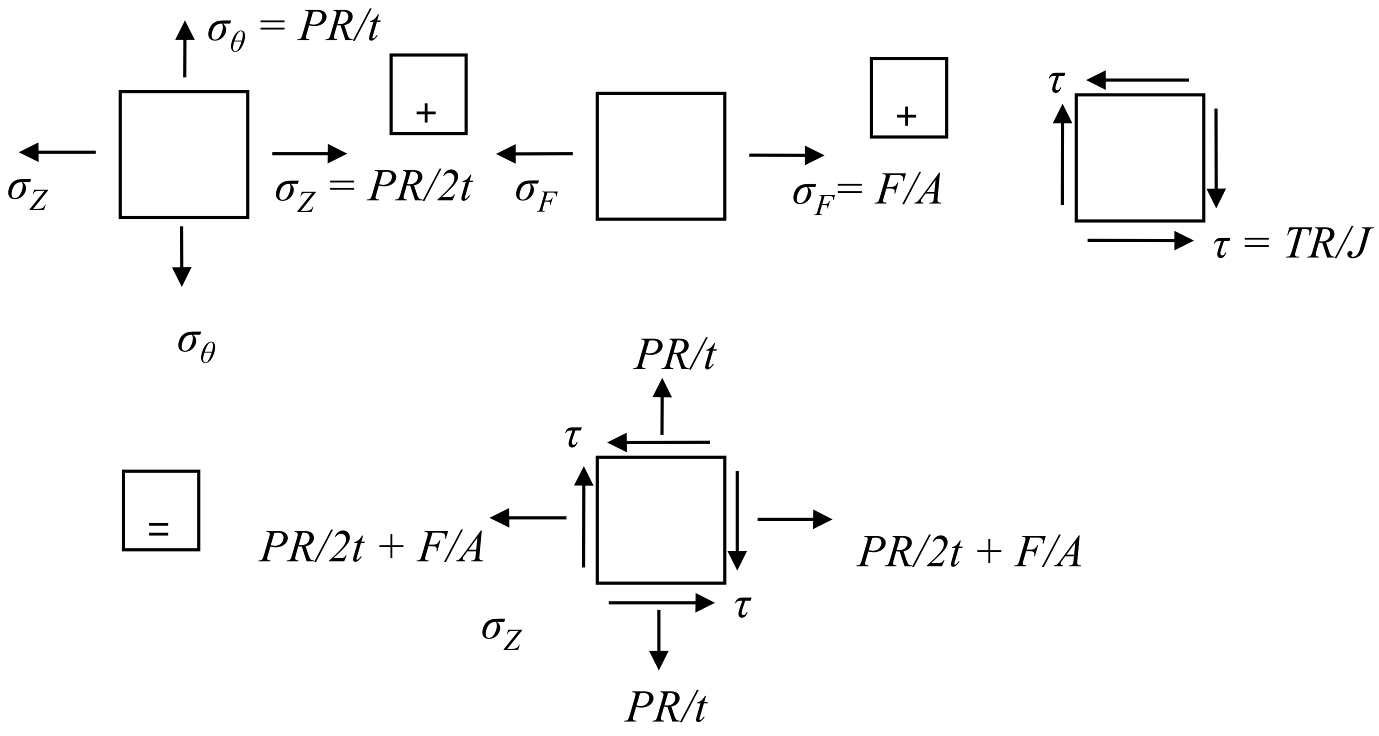
Superposition and Combined Loading

The Principal of Superposition states that:

$$\left(\begin{array}{c}\text{The total effect of } \underline{\textit{combined}} \\ \text{loads applied to a body}\end{array}\right) = \sum \left(\begin{array}{c}\text{The effects of the individual} \\ \text{loads applied } \underline{\textit{separately}}\end{array}\right)$$

Combined pressure, axial and torsional loading





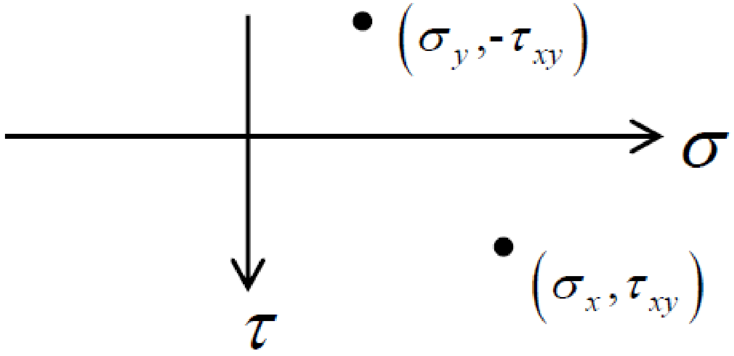
How to solve combined loading problem?

Methodology for analysing components or structures under combined loading:

1. Identify a 2D element at the location of interest in the component
2. Determine the stresses acting on the element arising from each individual load
3. Superpose the stresses from each individual load to obtain the combined stresses on the element
4. Use Mohr's circle to determine the principal stresses and the maximum shear stress on the element

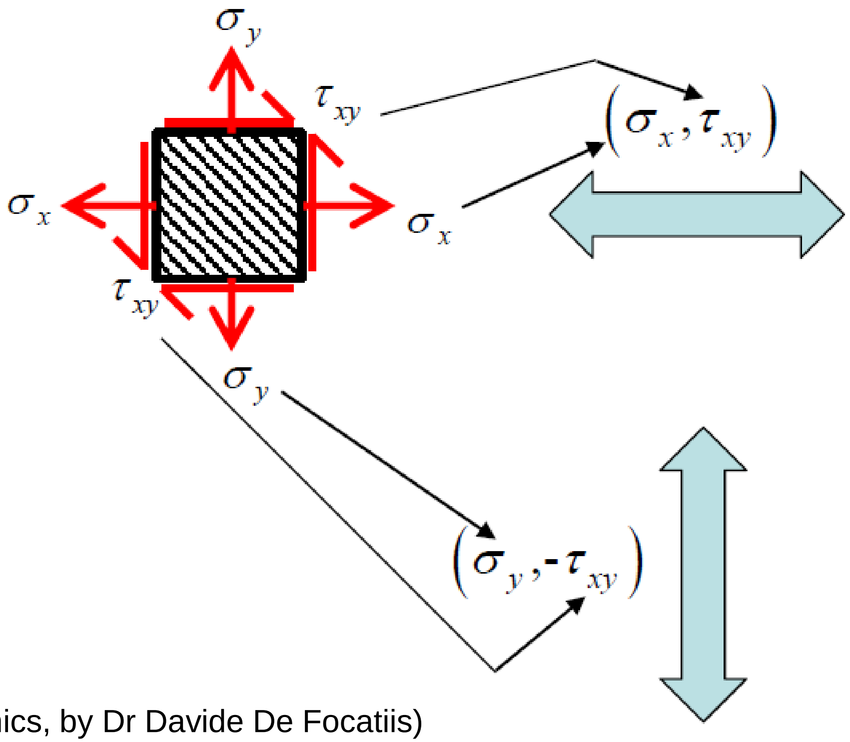
Rules for Mohr's circle

1. Draw normal stress axis horizontal and shear stress axis vertical (positive down):



2. Look in one direction of stress state and identify σ_x and $+\tau_{xy}$ (+ve anticlockwise). Plot this point on the diagram:

3. Look at right angles and identify σ_y and $-\tau_{xy}$. Add this point on the diagram:



Rules for Mohr's circle

4. Join these points to form the diameter of a circle, and sketch the circle

5. Find the centre of the circle

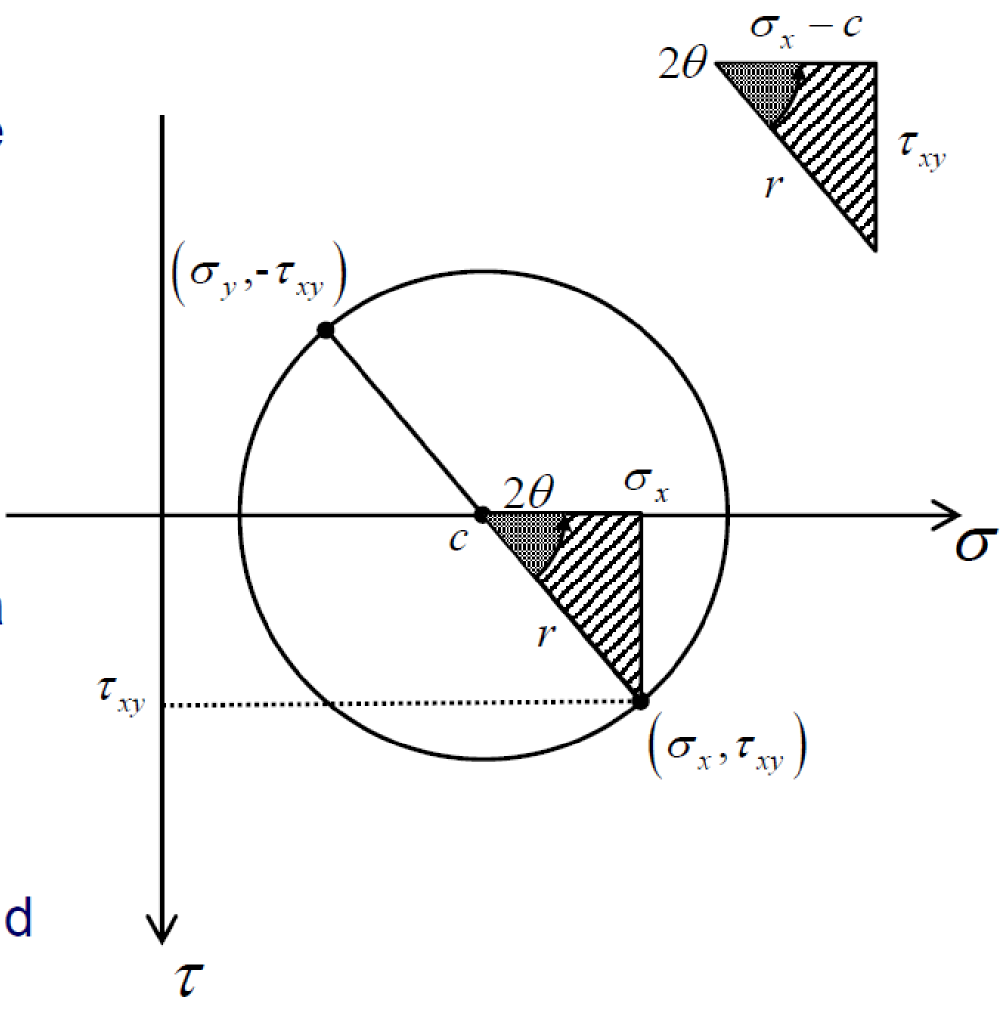
$$c = \frac{\sigma_x + \sigma_y}{2}$$

6. Find the radius of the circle using a right-angled triangle

$$r = \sqrt{(\sigma_x - c)^2 + \tau_{xy}^2}$$

7. Find the angle to the horizontal (and label this as 2θ)

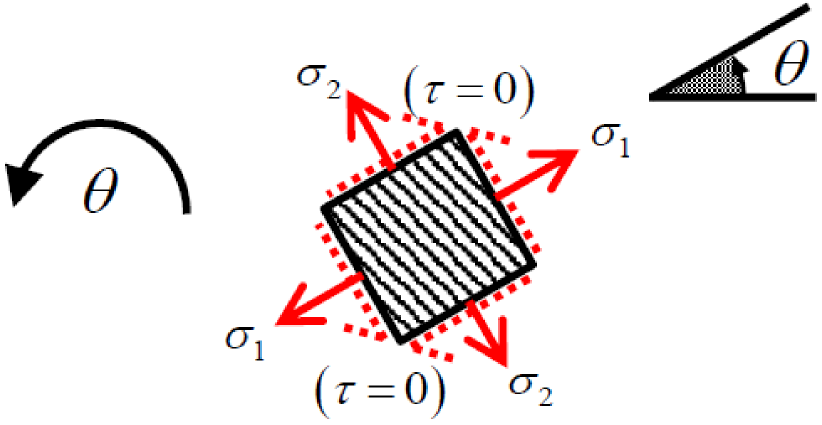
$$2\theta = \tan^{-1}\left(\frac{\tau_{xy}}{\sigma_x - c}\right)$$

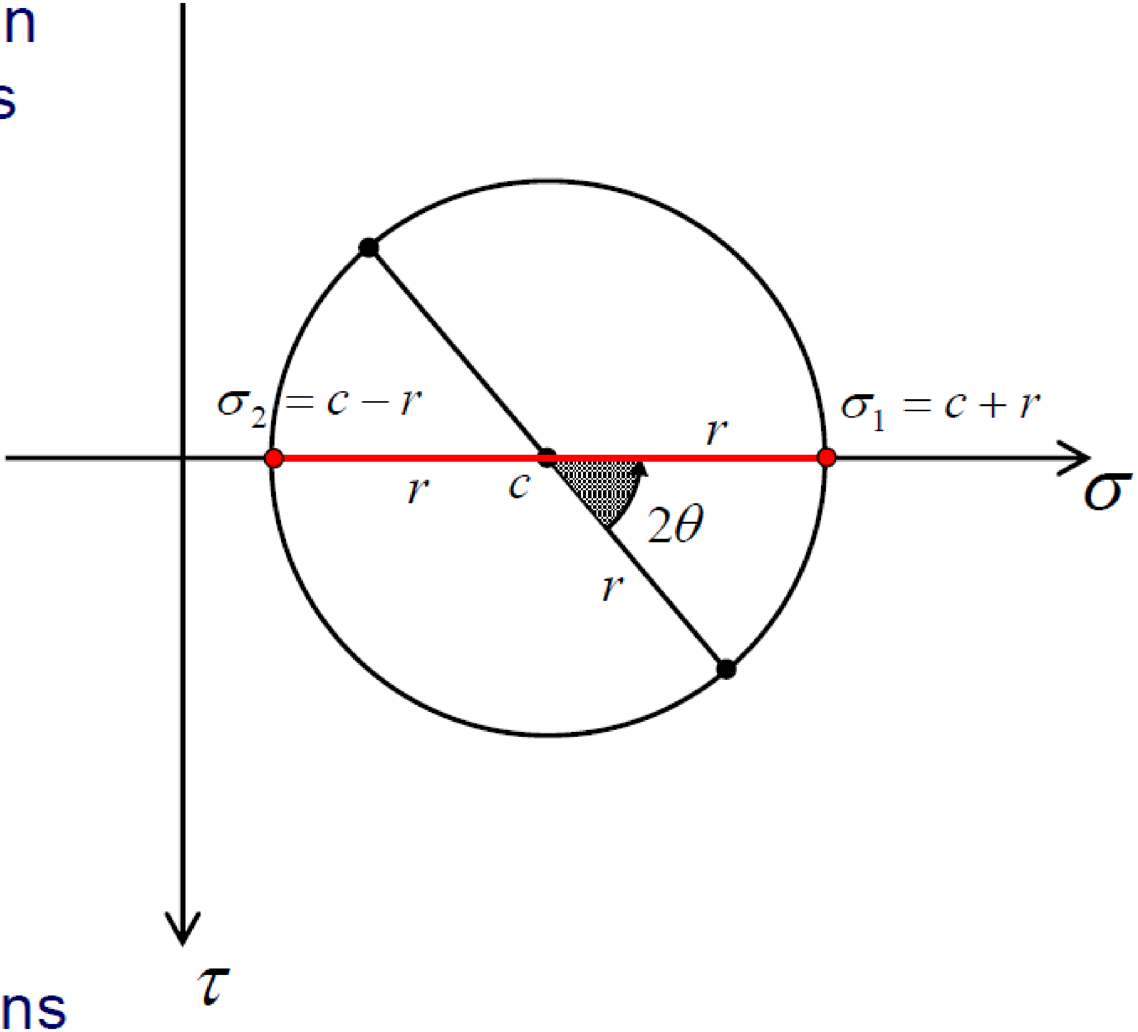


Rules for Mohr's circle

8. To find the principal stresses, rotate to the horizontal diameter (by 2θ) on the Mohr's circle, by θ on the stress element:

$$\sigma_1 = c + r \qquad \sigma_2 = c - r$$





Note that in the principal stress directions the shear stresses are always equal to zero.

Yield criteria

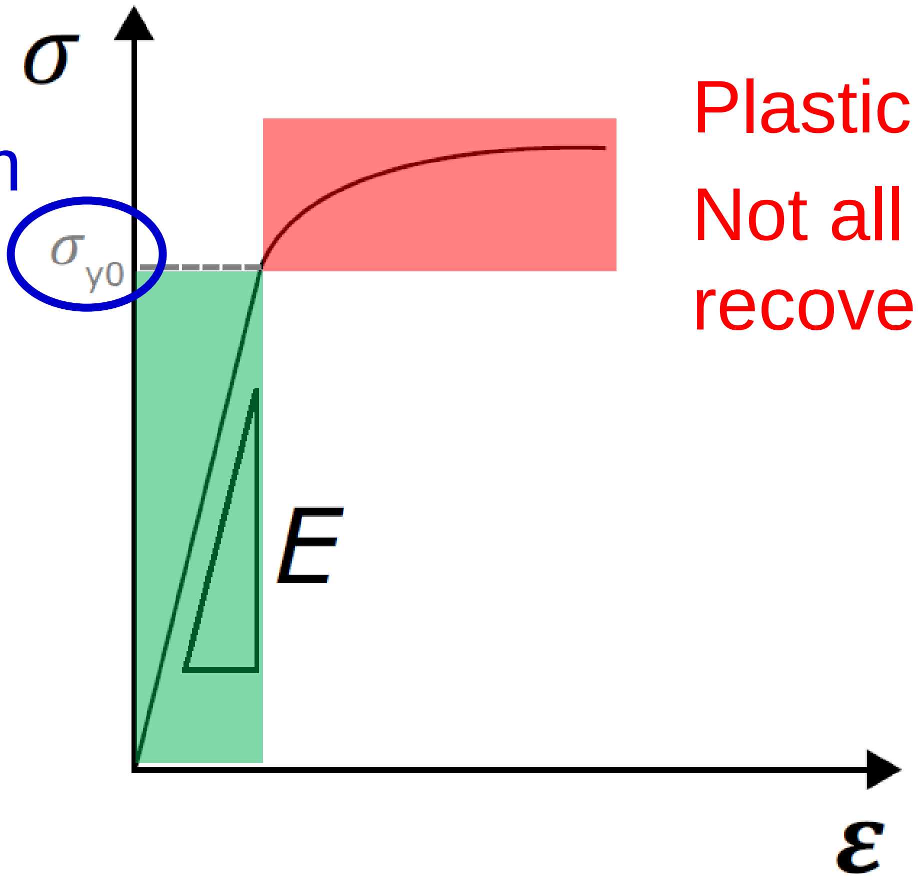
Yield point separates them

Plastic Region

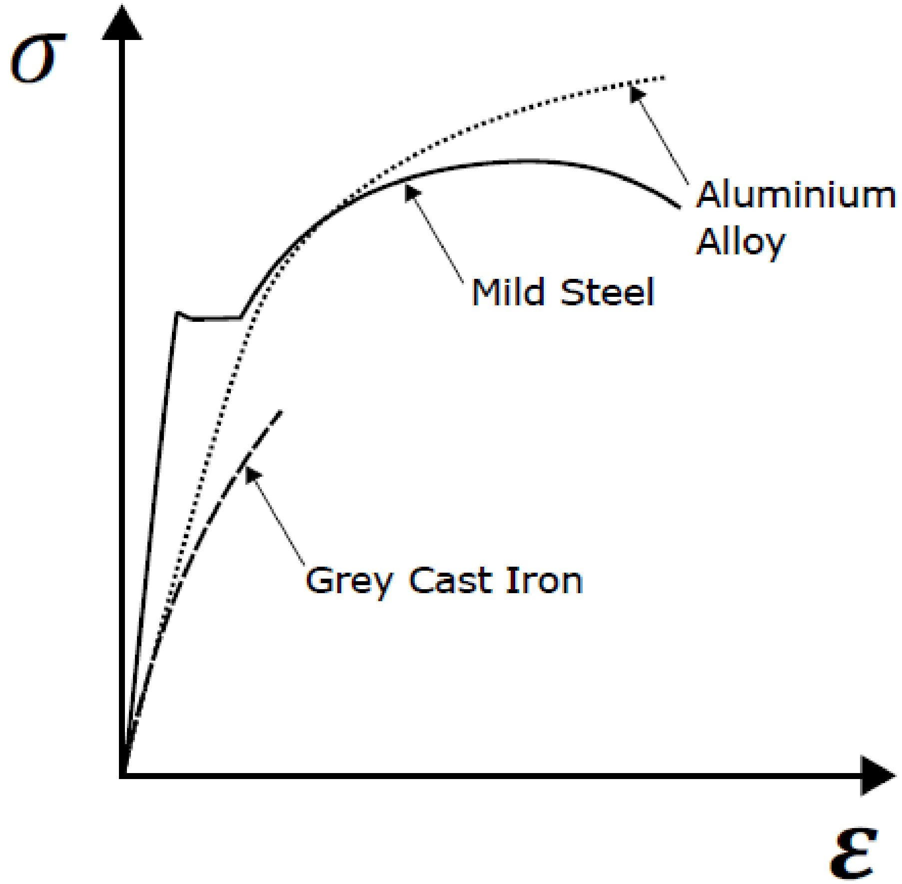
Not all deformation recovered

Elastic Region

All deformation recovered



Yield criteria



Ductile materials:

failure occurs at the onset of plastic deformation

Brittle materials:

failure occurs at fracture

The topic of “**Yield Criteria**” is limited to the prediction of the initiation of yielding in “ductile” materials.

Multiaxial Stress States

- Determining yield in a uniaxial system is relatively easy (depending on the material)
- For multiaxial stress states we need to employ **yield criteria** to determine the loading that leads to yield of a material
- Considering yield related only to ductile material

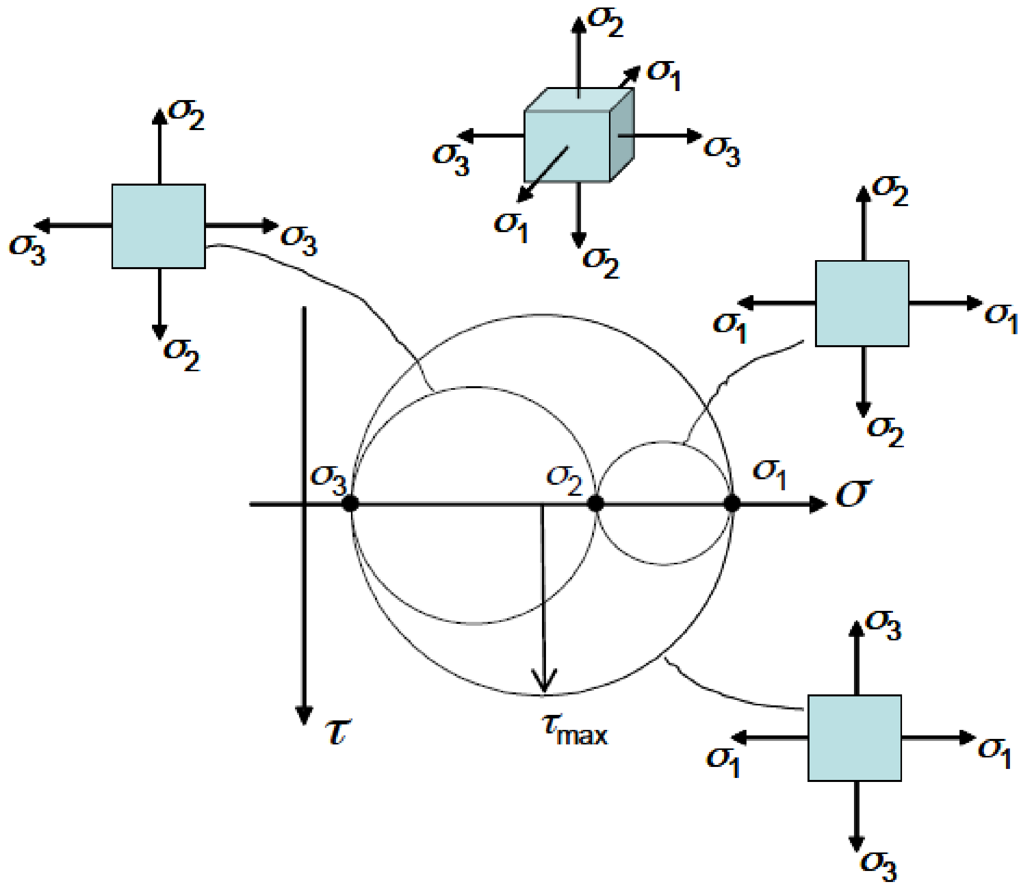
Multiaxial Stress States

- Two yield criteria that are generally considered for ductile materials:

    - The maximum shear stress (Tresca) criterion

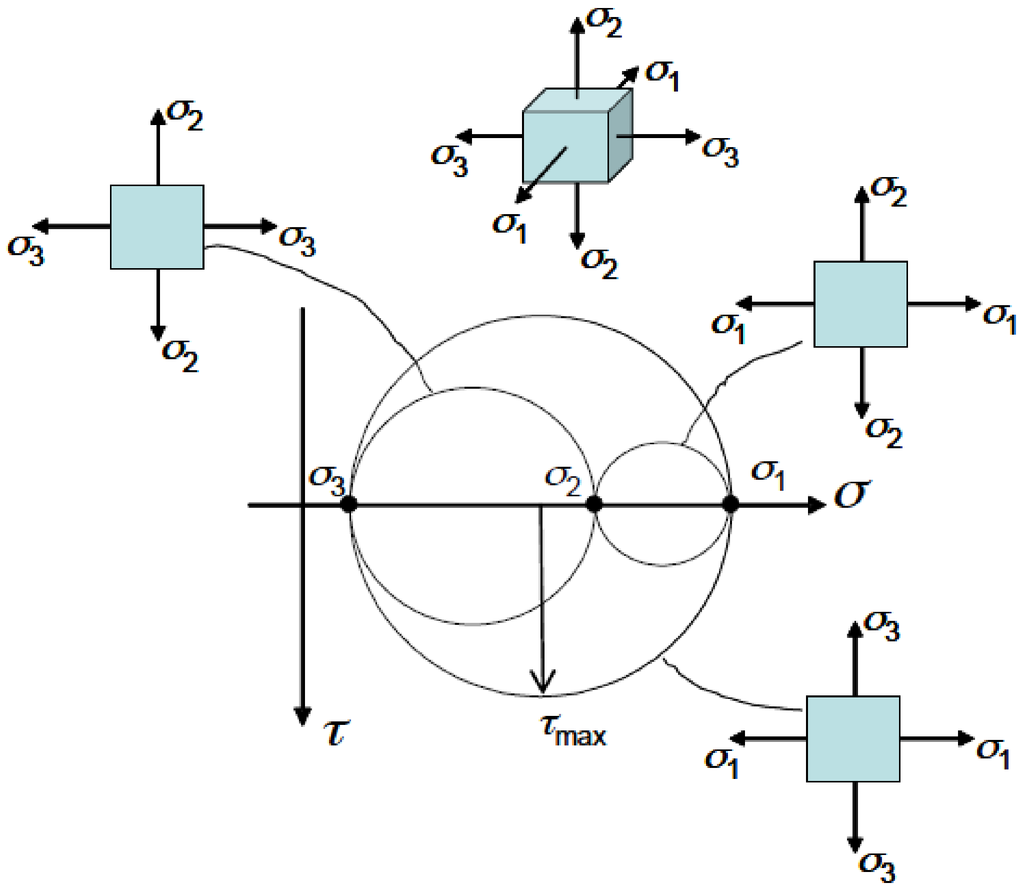
    - The maximum shear strain energy (von Mises) criterion

Mohr's Circle



- To determine the overall maximum shear stress in a stress state, it is necessary to consider **all** of the principal planes

Yield criteria



The **Tresca** (or maximum shear stress, τ_{max}) yield criterion states that the material will yield if:

$$\sigma_1 - \sigma_3 \geq \sigma_y \text{ for } \sigma_1 > \sigma_2 > \sigma_3$$

The **von Mises** (or maximum shear strain energy) yield criterion states that the material will yield if:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \geq 2{\sigma_y}^2$$

Factor of Safety

- all objects have a stress limit (e.g. yield strength) depending on the materials used

- for yielding, factor of safety is defined as the ratio of the maximum allowable stress to the yield criterion equivalent stress

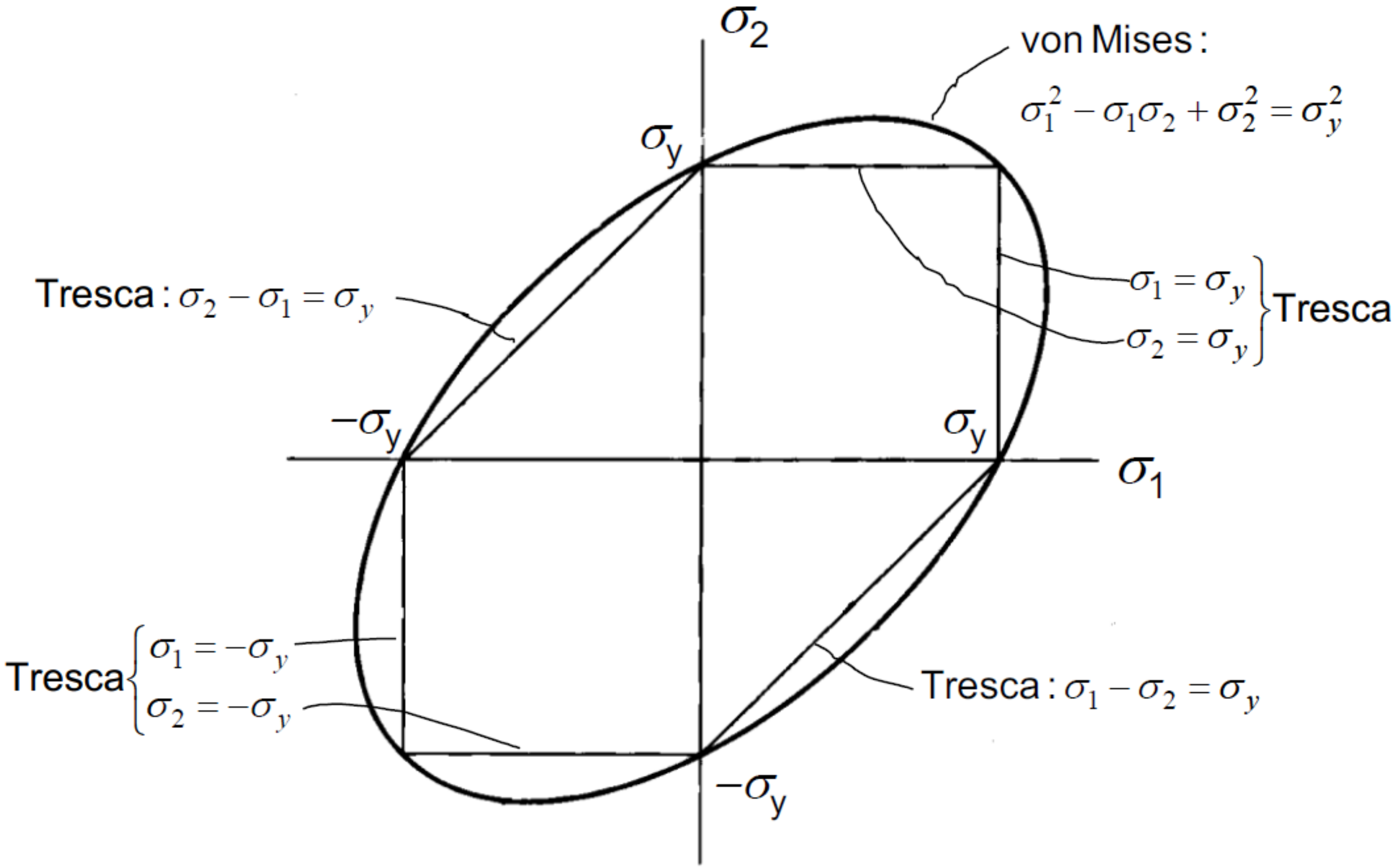
$$FoS = \frac{\sigma_y}{\sigma_{Tresca/vM}}$$

- FoS must be over 1 for the design to be acceptable

Two Dimensional Stress States

- The Treca and von Mises yield criterion in 2D ($\sigma_3 = 0$)

- von Mises - continuous



Three Dimensional Stress States

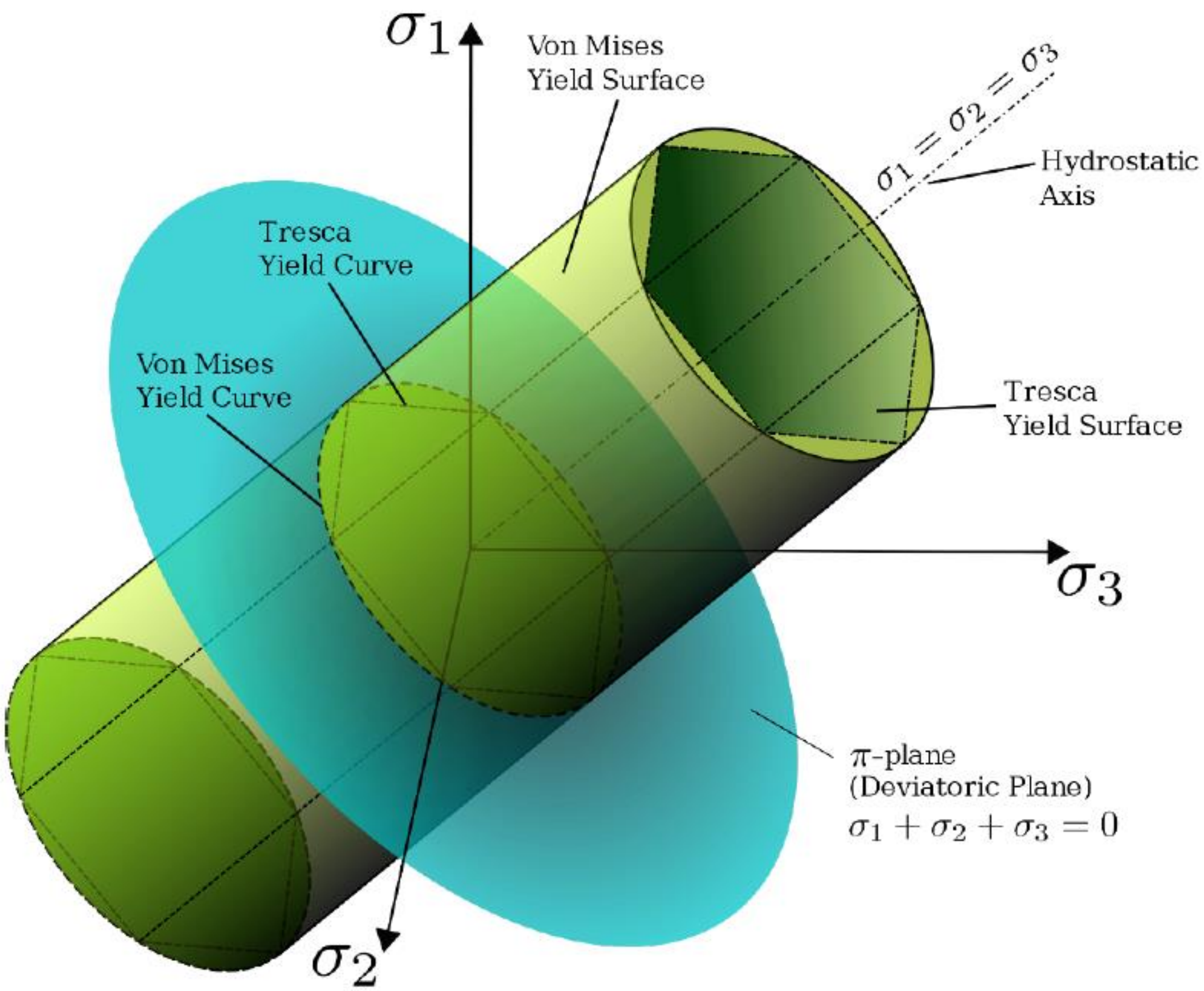


figure from Boresi, Schmidt and Sidebottom, “Advanced Mechanics of Materials”, 5th Ed, Wiley & Sons, 1993

Three Dimensional Stress States

- We can therefore decompose any stress state into two components, the hydrostatic stress, parallel to the hydrostatic axis and the deviatoric stress component, perpendicular to it.

- The deviatoric stress component is therefore given by $(\sigma_1 - \sigma_h\ ,\ \sigma_2 - \sigma_h\ ,\ \sigma_3 - \sigma_h)$

- Only this component is important in determining yield

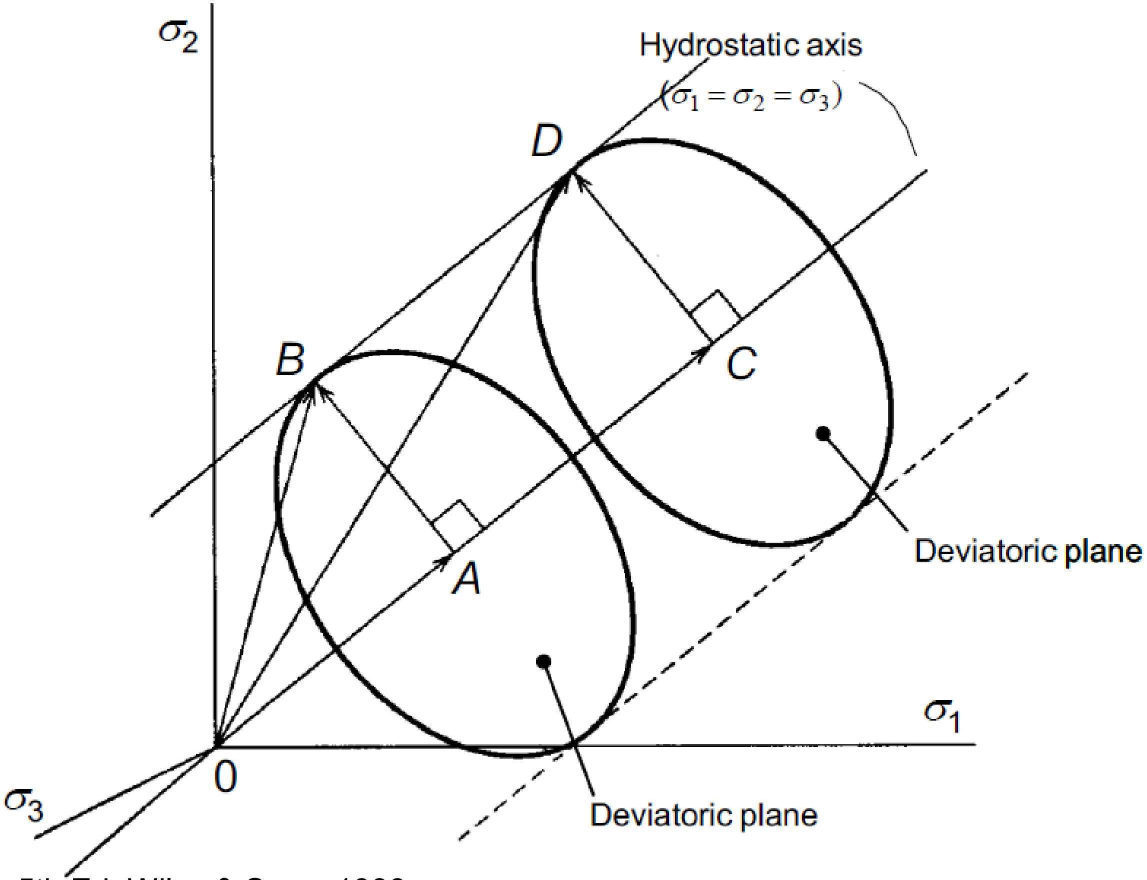


figure from Boresi, Schmidt and Sidebottom, “Advanced Mechanics of Materials”, 5th Ed, Wiley & Sons, 1993

Thermal Stress & Strain

Changes of temperature in a body cause expansion/contraction.

This phenomenon is quantified by the coefficient of thermal expansion, α.

Some typical values of thermal expansion coefficient for some common engineering materials are presented in Table 1.

For isotropic materials, α is the same for all directions.

Thermal Stress & Strain

- For a bar of length *l*, subjected to a temperature change ΔT, the change in length $\delta l_{thermal}$ due to the temperature change is given by:

$$\delta l_{thermal} = l\alpha\Delta T$$

- The thermal strain due to this length change can be determined as follows:

$$\varepsilon_{thermal} = \frac{\delta l_{thermal}}{l} = \frac{l\alpha\Delta T}{l} = \alpha\Delta T$$

Thermal Stress & Strain

- Using the principle of superposition, which states that:

$$\begin{bmatrix} The\ total\ effects\ of\ combined \\ loads\ applied\ to\ a\ body \end{bmatrix} = \sum \begin{bmatrix} The\ effects\ of\ the\ individual \\ loads\ applied\ separately \end{bmatrix}$$

- Thermal extensions can simply be added to elastic (mechanical) extensions to give the total extension by:

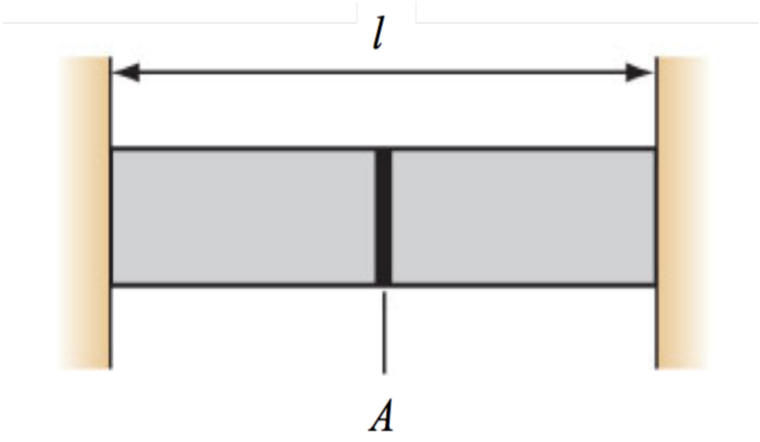
$$\delta l_{total} = \delta l_{elastic} + \delta l_{thermal}$$

- For our uniaxial bar:

$$\delta l_{total} = \frac{FL}{AE} + l\alpha\Delta T$$

Resistive Heating of a Bar

- The bar shown below is subjected to a temperature rise of *ΔT* and restricted from expanding by constraints at each end.



- Since the bar cannot extend:

$$\delta l_{total} = \delta l_{elastic} + \delta l_{thermal} = 0$$

Resistive Heating of a Bar

- Or:

$$\delta l_{total} = \frac{FL}{AE} + l\alpha\Delta T = 0$$

- Cancelling through *l* and rearranging for the reaction force, *F*, gives:

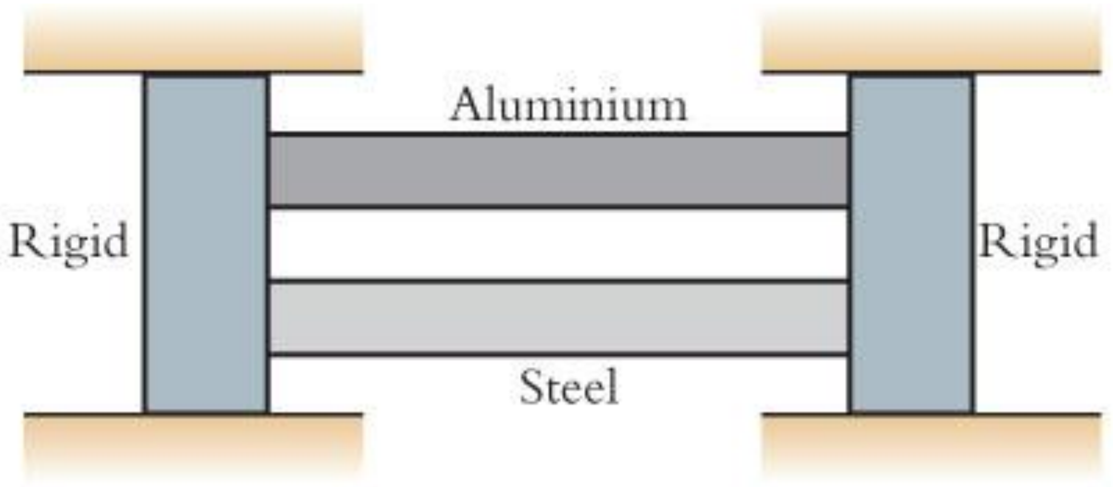
$$F = -AE\alpha\Delta T$$

- And we can determine the stress using:

$$\sigma = \frac{F}{A} = -E\alpha\Delta T$$

Compound Bar Assembly

The compound bar assembly



is subjected to a temperature change ΔT, will the bars be in tension or compression?

Compound Bar Assembly

- We can consider this intuitively

- The extension of the bars must be equal:

$$\delta l_{steel} = \delta l_{alu}$$

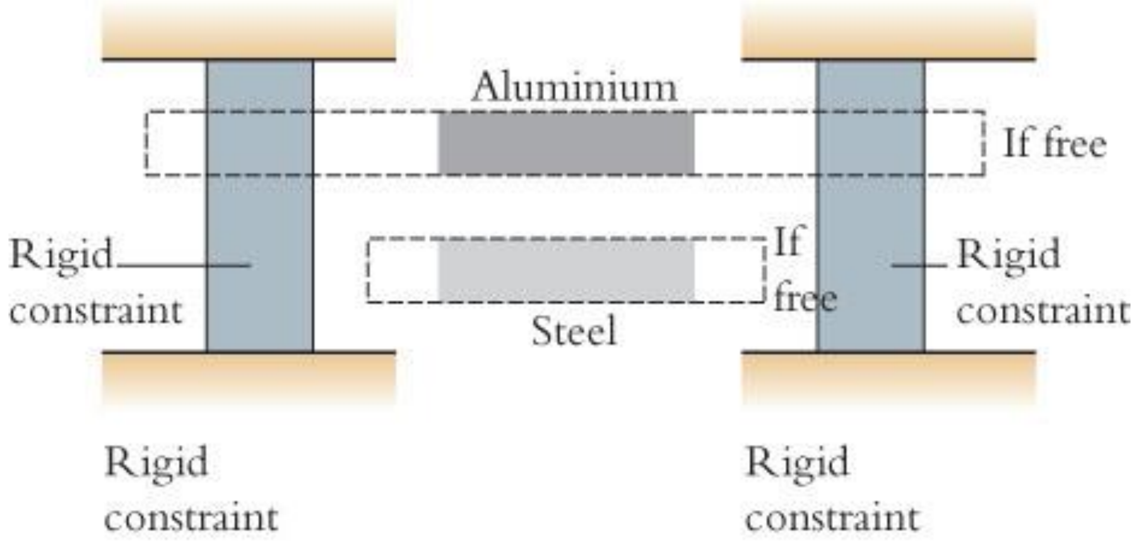
Material	Coefficient of Thermal Expansion, α, [°C^{-1}]
Concrete	10×10^{-6}
Steel	11×10^{-6}
Aluminium	23×10^{-6}
Nylon	144×10^{-6}
Rubber	162×10^{-6}

$$\alpha_{alu} > \alpha_{steel}$$

Compound Bar Assembly

- The aluminium bar will want to extend more than the steel bar but is constrained from doing so due to the rigid end blocks attached to the steel bar. This means that the aluminium bar will be in compression.

- The reverse is true of the steel bar, it wants to extend less than the aluminium bar but the rigid end blocks attached the aluminium bar forces it to extend further, therefore the steel bar is in tension.



Compound Bar Assembly

- Considering the problem analytically, the change in lengths are given by:

$$\frac{F_{steel}l}{A_{steel}E_{steel}} + l\alpha_{steel}\Delta T = \frac{F_{alu}l}{A_{alu}E_{alu}} + l\alpha_{alu}\Delta T$$

- Considering equilibrium using the FBD:

$$F_{steel} = -F_{alu}$$

Compound Bar Assembly

- Substituting in for F_{steel} and rearranging:

$$l\Delta T(\alpha_{steel} - \alpha_{alu}) = F_{alu} l \left[\frac{1}{A_{alu}E_{alu}} + \frac{1}{A_{steel}E_{steel}}\right]$$

- Therefore:

$$\sigma_{alu} = \frac{F_{alu}}{A_{alu}} = \frac{\Delta T(\alpha_{steel} - \alpha_{alu})}{\left[\frac{1}{E_{alu}} + \frac{A_{alu}}{A_{steel}E_{steel}}\right]}$$

- As $\alpha_{steel} < \alpha_{alu}$ this means that $\sigma_{alu} < 0$, i.e. **the aluminium bar is in compression**.

Compound Bar Assembly

- From the force equilibrium:

$$A_{alu}\sigma_{alu} = -A_{steel}\sigma_{steel}$$

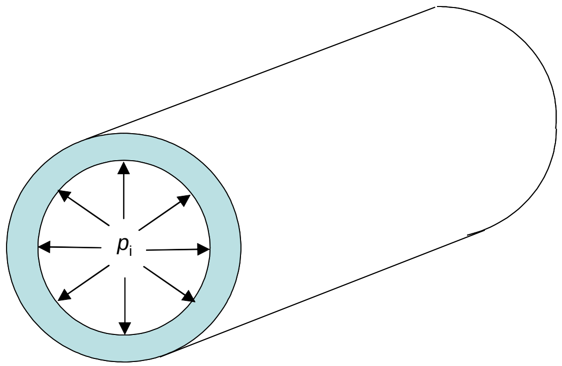
- Then:

$$\sigma_{steel} = -\frac{A_{alu}\sigma_{alu}}{A_{steel}} = -\frac{A_{alu}}{A_{steel}} \frac{\Delta T(\alpha_{steel} - \alpha_{alu})}{\left[\frac{1}{E_{alu}} + \frac{A_{steel}}{A_{steel}E_{steel}}\right]}$$

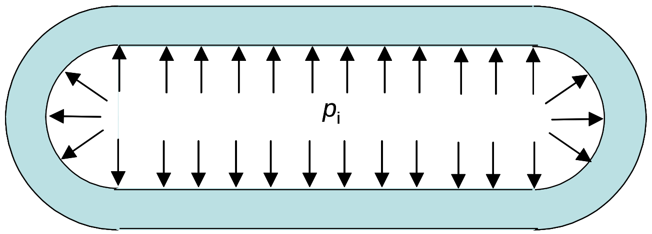
- As $\alpha_{steel} < \alpha_{alu}$ this means that $\sigma_{steel} > 0$, i.e. **the steel bar is in tension**.

Thick Cylinders

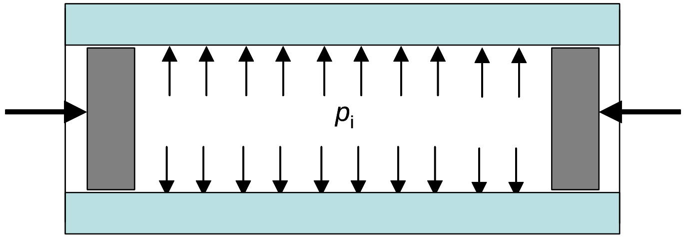
- Examples



Closed ends



With pistons



Thin Cylinders Recap

- For an internally pressurised thin cylinder:
  - – the variations of the stresses through the wall thickness are negligible (assumption)
  - – problem is ***statically determinate***, i.e. expressions for the stresses can be obtained by consideration of equilibrium alone
- Thin cylinders are normally those with **t/R < 1/10**, where t is the wall thickness and R is the mean radius (to centre of wall)

Thick Cylinders

- Thick cylinders differ from thin cylinders in that the variation of stress through the wall thickness is significant when subjected to internal and/or external pressure whereas for thin cylinders, the variation of stress is negligible.

Analysis of Thick Cylinders

- Thick cylinder problems are ***statically indeterminate***
  - In order to obtain a solution, it is necessary to consider:
    - Equilibrium
    - Compatibility
    - Material behaviour (stress-strain relationship)

Assumptions

i. Plane transverse sections remain plane (this is true remote from the ends)

ii. Deformations are small

iii. The material is linear elastic, homogenous and isotropic

Analysis of Thick Cylinders

- The hoop and radial stresses at any point (radius, *r*) in the wall cross-section of a thick cylinder can be determined using ***Lame's equations***:

$$\sigma_\theta = A + \frac{B}{r^2} \qquad \text{hoop stress}$$

$$\sigma_r = A - \frac{B}{r^2} \qquad \text{radial stress}$$

- Where *A* and *B* are *Lame's constants* (constants of integration)
- The derivation of Lame's equations is given in the lecture notes (p4-7).

Analysis of Thick Cylinders

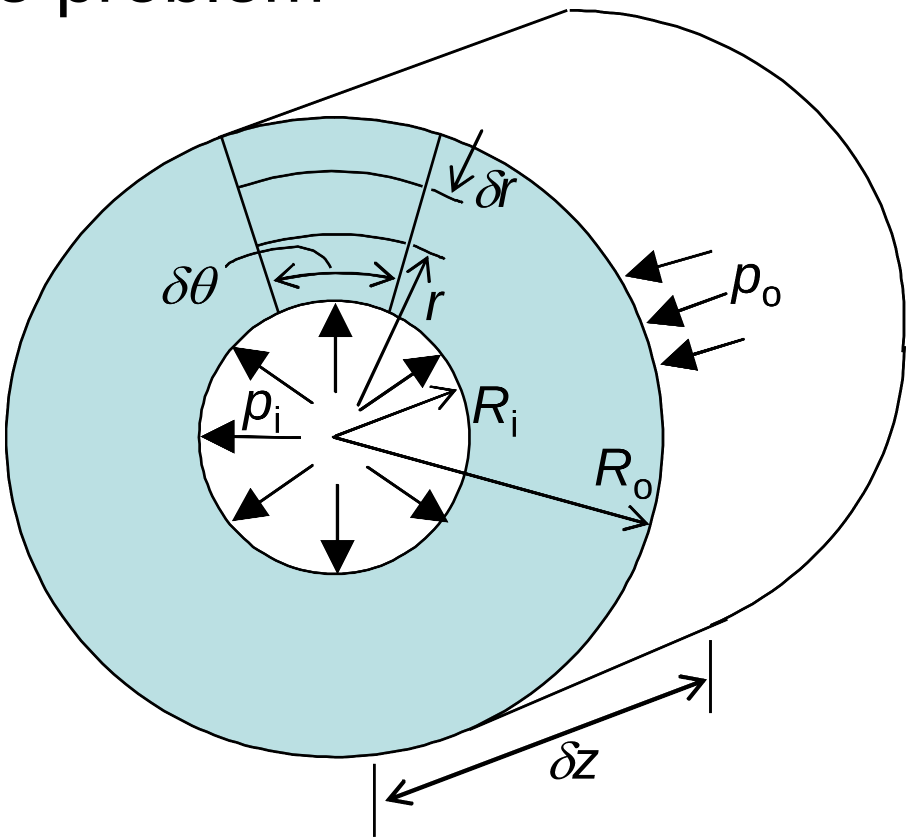
- We can obtain the values of Lame's constants from the boundary conditions of the problem

- At $r = R_i$ (inner radius)

$$\boldsymbol{\sigma_r = -p_i}$$

- At $r = R_o$ (outer radius)

$$\boldsymbol{\sigma_r = -p_o}$$



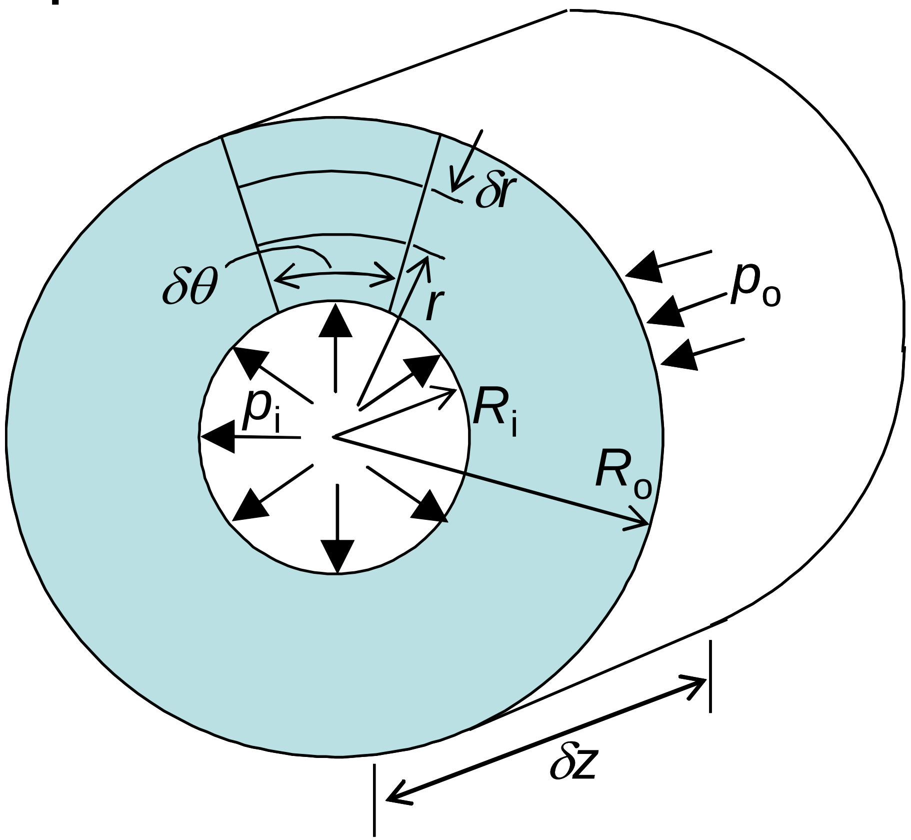
Analysis of Thick Cylinders

- We can obtain the values of Lame's constants from the boundary conditions of the problem

- Therefore:

$$-p_i = A - \frac{B}{R_i^2}$$

$$-p_o = A - \frac{B}{R_o^2}$$



- And *A* and *B* can be determined (two unknowns, two equations)

Analysis of Thick Cylinders

- For a cylinder with closed ends, simple axial equilibrium leads us to:

$$\sigma_z \pi (R_o^2 - R_i^2) = \pi R_i^2 p_i - \pi R_o^2 p_o$$

$$\sigma_z = \frac{R_i^2 p_i - R_o^2 p_o}{(R_o^2 - R_i^2)}$$ axial stress

- While for a cylinder with pistons, no axial load is transferred to the cylinder:

$$\sigma_z = 0$$

Analysis of Thick Cylinders

- For a <u>solid</u> cylinder, $R_i = 0$:

$$\boldsymbol{\sigma_r = A - \frac{B}{0^2} = \infty}$$

- Unless B = 0. So in this case, as the stresses cannot be infinite, the radial and hoop stresses are equal and are constant

$$\boldsymbol{\sigma_r = \sigma_\theta = A}$$

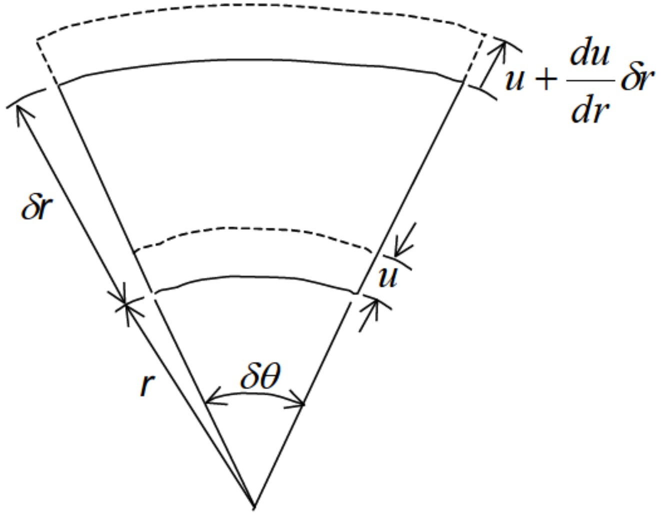
Analysis of Thick Cylinders

- Displacements can be obtained using (generalised Hooke's law):

hoop strain $$\varepsilon_\theta = \frac{(r+u)\delta\theta - r\delta\theta}{r\delta\theta} = \frac{u}{r} = \frac{1}{E}\left(\sigma_\theta - \nu(\sigma_r + \sigma_z)\right)$$

axial strain $$\varepsilon_z = \frac{\Delta l}{l} = \frac{1}{E}\left(\sigma_z - \nu(\sigma_r + \sigma_\theta)\right) = \text{constant}$$

radial strain $$\varepsilon_r = \frac{\left(u + \frac{du}{dr}\delta r\right) - u}{\delta r} = \frac{du}{dr} = \frac{1}{E}\left(\sigma_r - \nu(\sigma_\theta + \sigma_z)\right)$$



(u: increase in radius; ν: Poisson's ratio)

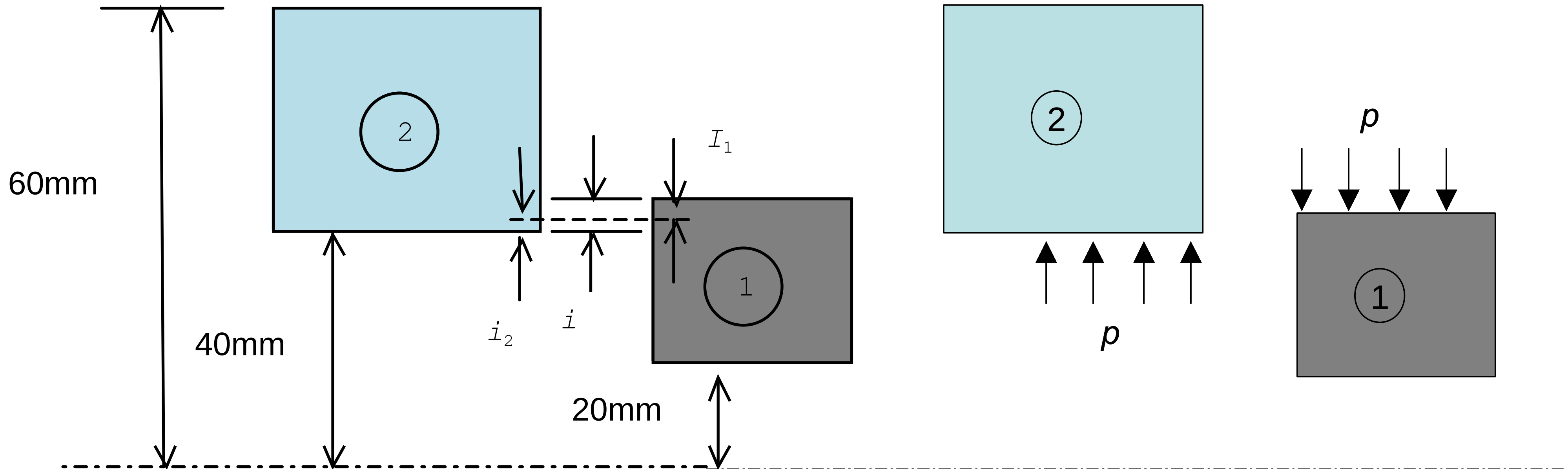
Shrink fitting a cylinder onto another

A pair of mild steel cylinders (E = 200 GPa) of equal length have the following dimensions:

- 40mm bore and 80.06mm outside diameter
- 80mm bore and 120mm outside diameter

(i.e. there is a diametral interference of 0.06mm)

The larger cylinder is heated, placed around and allowed to shrink onto the smaller cylinder. Calculate the stresses after assembly.



Shrink fitting a cylinder onto another

Assumptions

i. After assembly, the radial interference pressure, p, will be the same on both cylinders, i.e. Cylinder 1 will have an external pressure, p, and Cylinder 2 will have an internal pressure, p, as indicated in the figure.

ii. The decrease in the outside radius of Cylinder 1, i_1, plus the increase in the inside radius of Cylinder 2, i_2, will be equal to the radial interference, i.e. $i = i_1 + i_2$

iii. Axial stresses are assumed to be zero (or negligible)

Analysis of Rotating Discs

- Rotating components such as flywheels and turbine discs can be regarded as **thick cylinders with body forces**, as well as possible pressure loads and as such represent an extension of the thick cylinder theory.

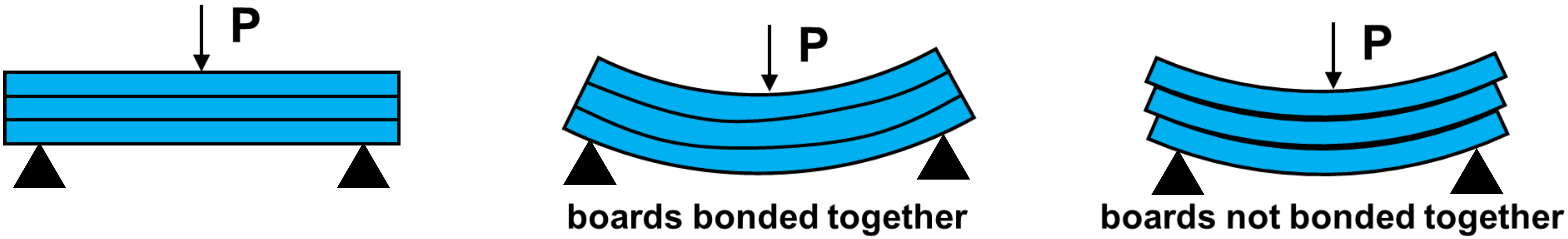
- Derivation is included in the notes

$$\sigma_r = A - \frac{B}{r^2} - \frac{\rho\omega^2(3+\nu)}{8}r^2$$

$$\sigma_\theta = A + \frac{B}{r^2} - \frac{\rho\omega^2(1+3\nu)}{8}r^2$$

Shear Stresses in Beams

- As the beam span to depth ratio reduces, i.e. if the beam is shorter and thicker, shear stresses become more important and should be calculated in any design evaluation
- This can be important for laminated beams, e.g. plywood or composite beams, where the transverse shear can cause failure between individual layers (plies) making up the beam



Shear Stresses in Beams

- The expression can also be written in **discrete** form as:

$$\boldsymbol{\tau = \frac{SA\bar{y}}{Iz}}$$

- where A is the area of the part of the cross-section <u>outside</u> the position at which $\boldsymbol{\tau}$ is determined, and y is the distance of the centroid of this area from the neutral axis.
- May see it in some texts as: $$\boldsymbol{\tau = \frac{SQ}{Iz}}$$
- Where Q represents $\boldsymbol{A\bar{y}}$ but is generally more applicable for complex sections with changes in cross-sectional area through the depth of the beam. We can calculate Q for each sub-area of the section and sum them together.

$$\int_A y\,dA = \sum_i y_i A_i = \bar{y}A$$

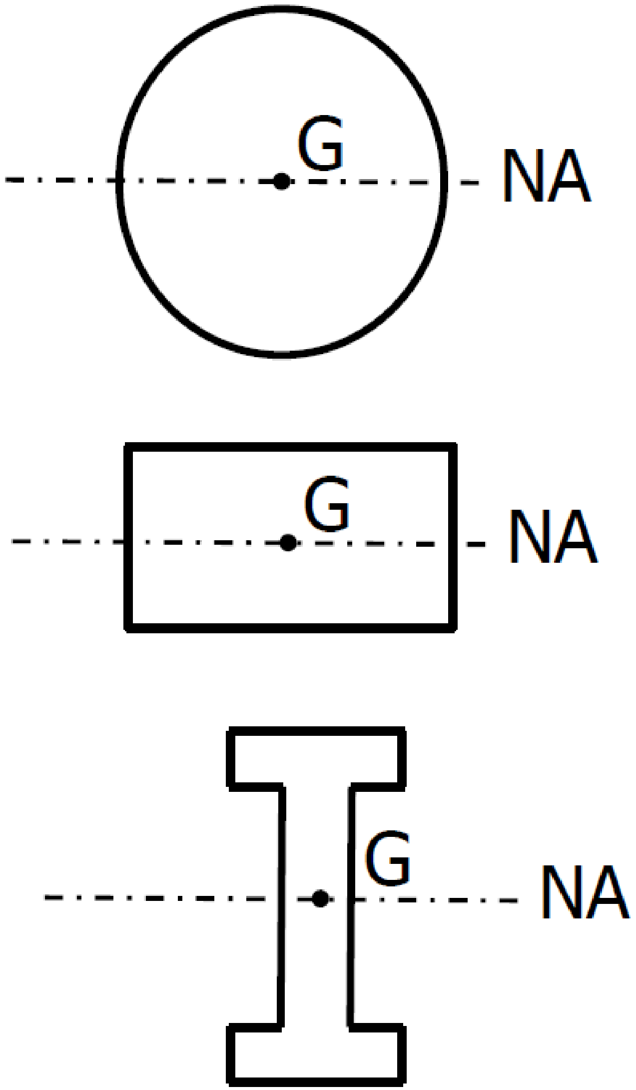
Neutral Axis

- The neutral axis goes through the centroid of the section – this is the same as the centre of mass *G* (for constant density)

- For symmetric sections it is in the middle
- For non-symmetric sections, easiest way to find it:

1. Take moment of area about a convenient axis
2. Equate this to the total moment of the area at the centroid about the axis

$$\int_A y\,dA = \bar{y}A$$

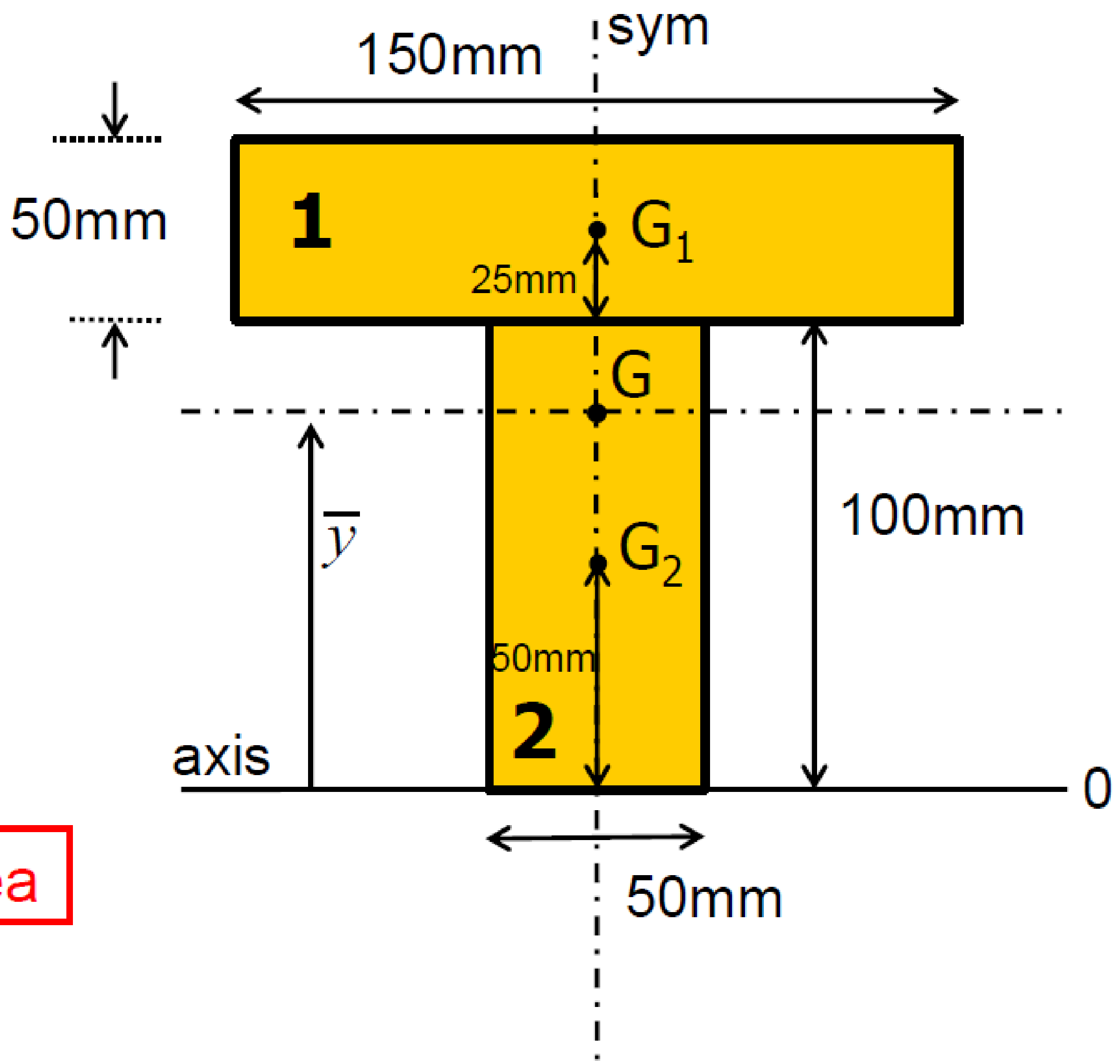


(slides from MMM1028 Statics and Dynamics, by Dr Davide De Focatiis)

Neutral Axis

- T-section

1. Take sum of moment of each area about a convenient axis



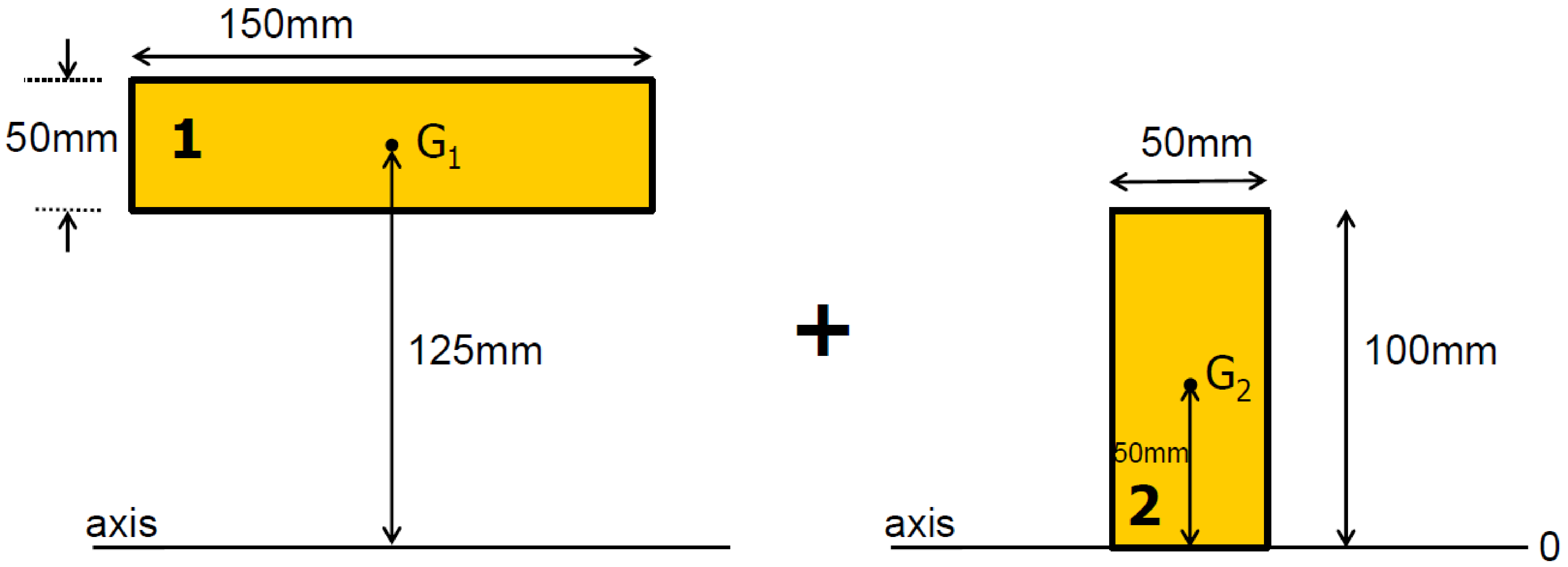
$$\int_A y\,dA = \sum_i y_i A_i$$

*i*th distance of area centroid to axis

*i*th area

(slides from MMM1028 Statics and Dynamics, by Dr Davide De Focatiis)

Neutral Axis



$$y_1A_1 = \left[(125\text{mm})\times(50\text{mm}\times150\text{mm})\right]$$

$$y_2A_2 = \left[(50\text{mm})\times(100\text{mm}\times50\text{mm})\right]$$

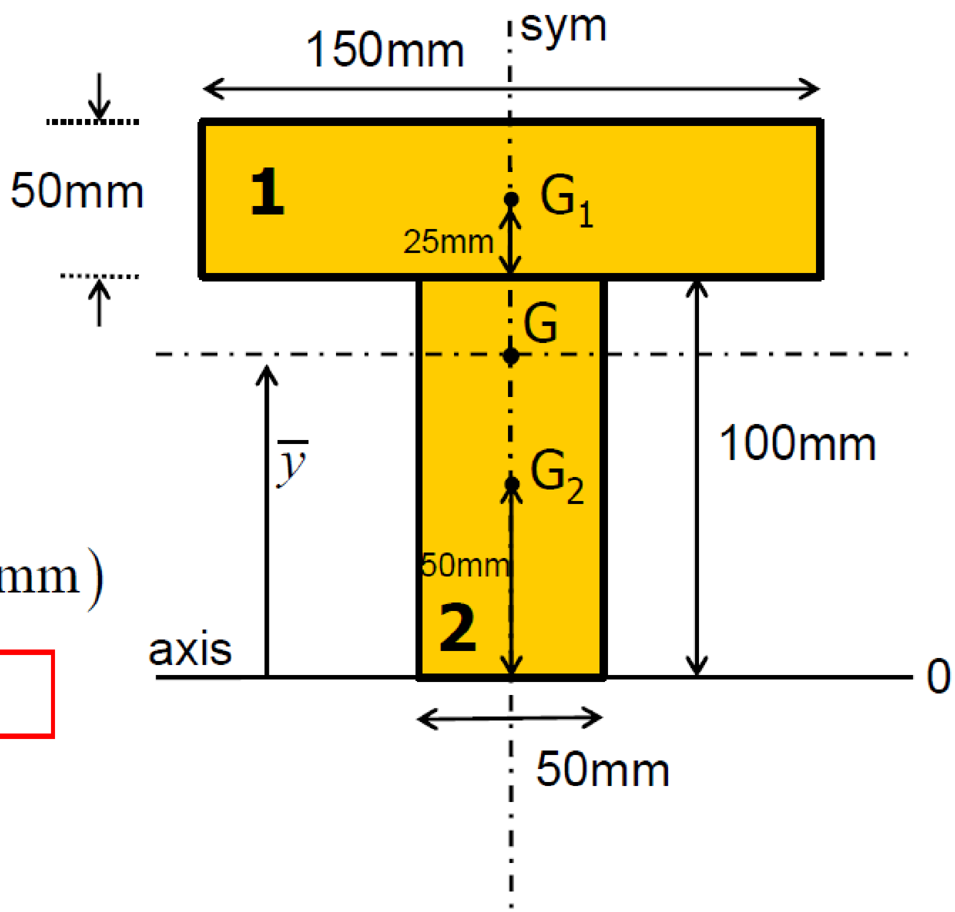
$$y_1A_1 + y_2A_2 = 1.1875\times10^6\ \text{mm}^3$$

(slides from MMM1028 Statics and Dynamics, by Dr Davide De Focatiis)

Neutral Axis

2. Equate this to the moment of the total area at the section centroid about the same axis

$$\int_A y\,dA = \sum_i y_i A_i = \bar{y}A$$



$$A = (50\text{mm}\times 150\text{mm}) + (100\text{mm}\times 50\text{mm})$$

$$= 1.25\times 10^4 \text{ mm}^2$$

Total area

$$\bar{y} = \frac{\sum_i y_i A_i}{A} = \frac{1.1875\times 10^6 \text{ mm}^3}{1.25\times 10^4 \text{ mm}^2}$$

$$= 95 \text{ mm}$$

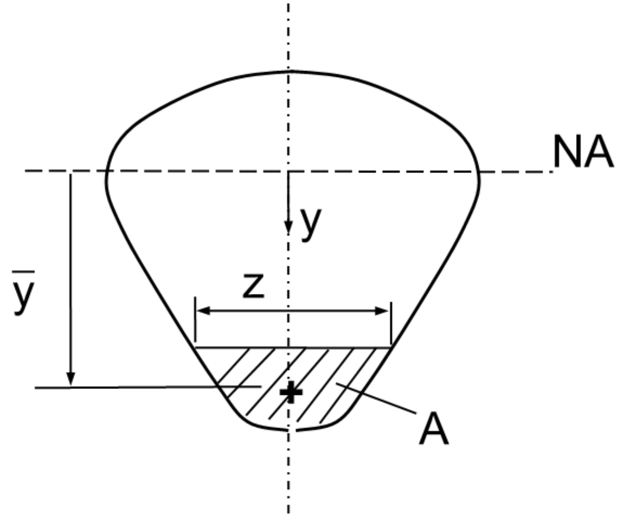
Position of the neutral axis relative to the base

(slides from MMM1028 Statics and Dynamics, by Dr Davide De Focatiis)

Shear Stresses in Beams

- For a general beam cross section:

$$\tau = \frac{SA\bar{y}}{Iz}$$



$\boldsymbol{\tau}$: the shear stress in the member at the point located a distance y from the N.A.

S: the shear force

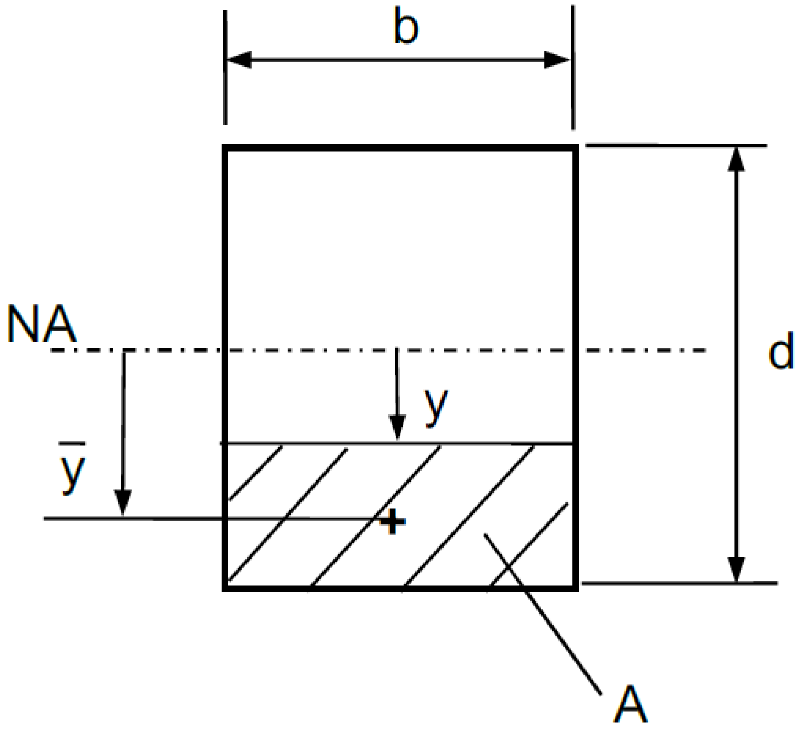
I: the second moment of area

z: the width/thickness of the cross section

A: the area of the top (or bottom) portion of the cross section, above (or below) the section plane where z is measured

$\bar{\boldsymbol{y}}$: the distance from the neutral axis to the centroid of *A*

Shear Stress Distribution in a Rectangular Section

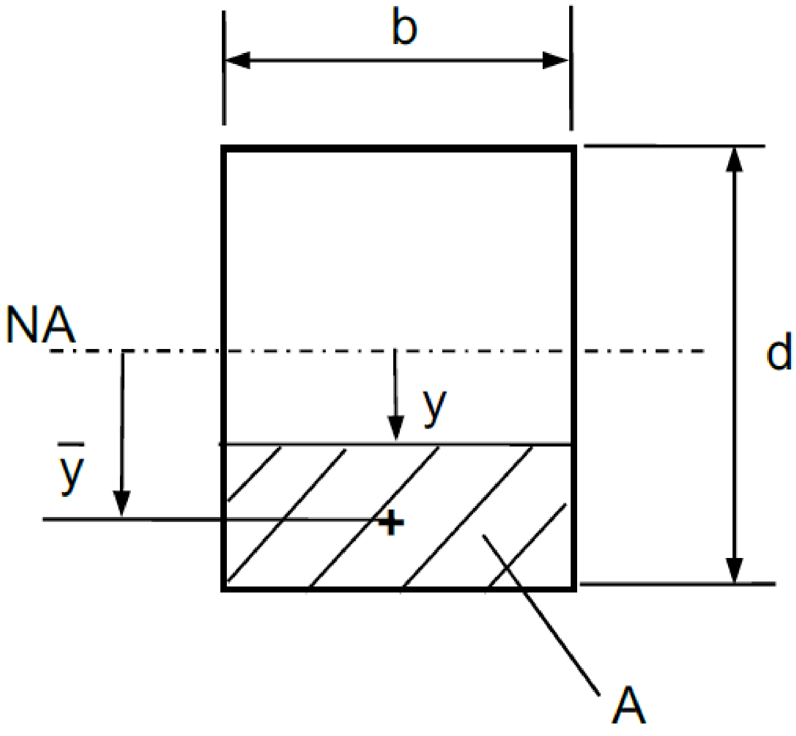


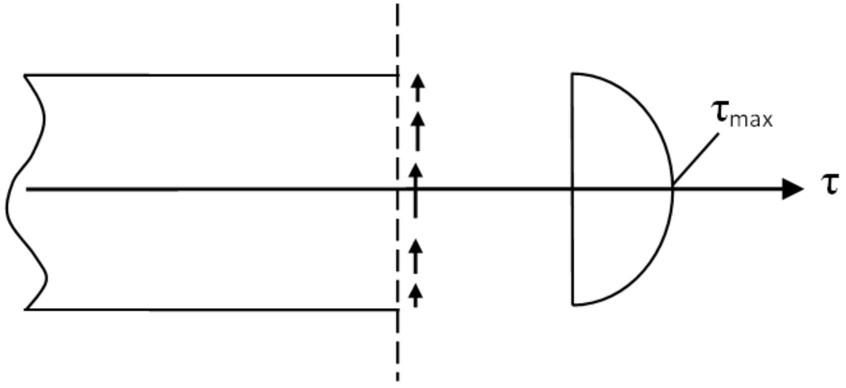
$$A = \left(\frac{d}{2} - y\right) b \qquad \bar{y} = \left(\frac{d}{2} + y\right)\frac{1}{2}$$

$$\boldsymbol{\tau = \frac{SA\bar{y}}{Iz}}$$

$$\tau = \frac{S}{\left(\frac{bd^3}{12}\right) b}\left(\frac{d}{2} - y\right) b \left(\frac{d}{2} + y\right)\frac{1}{2} \quad \longrightarrow \quad \tau = \frac{6S}{bd^3}\left[\left(\frac{d}{2}\right)^2 - y^2\right]$$

Shear Stress Distribution in a Rectangular Section





$$\tau = \frac{6S}{bd^3}\left[\left(\frac{d}{2}\right)^2 - y^2\right]$$

- Which gives a parabolic distribution (varies with y^2). At the top and bottom of the section, $\tau = 0$.

- At the neutral axis (y = 0):

$$\tau = \frac{6S}{bd^3}\frac{d^2}{4} = 1.5\frac{S}{bd}$$

Important Points

- The general formulae for shear stresses in beams in both integral and discrete forms are:

$$\tau = \frac{S}{Iz}\int_A y\,dA \qquad \tau = \frac{SA\bar{y}}{Iz}$$

where S is the shear force on the section, I is the second moment of area, y is the position from the N.A. at which you wish to determine the shear stress, z is the thickness of the section at that location, A is the area outside that location, and $\bar{y}$ is the distance from the N.A. to the centroid of that area.

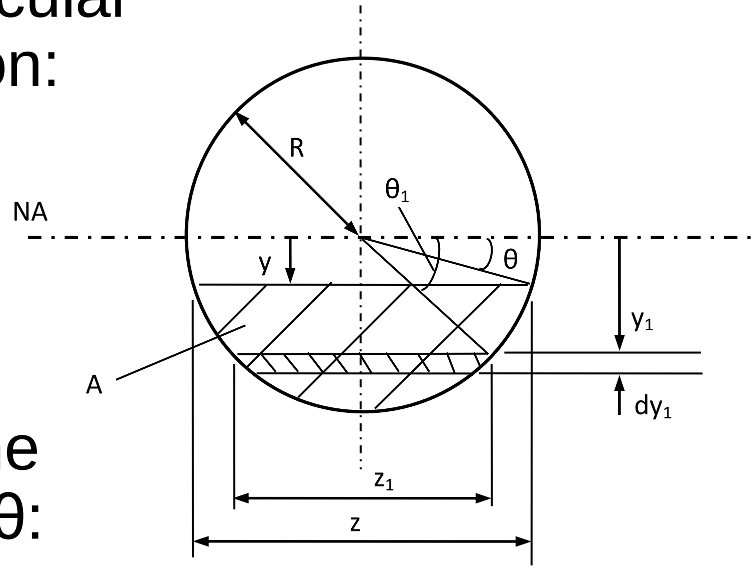
- For **rectangular** cross sections, the distribution of shear stresses through the depth of the section is **parabolic** (varies with y^2)

- At the free surfaces, the shear stress is 0

Shear Stress Distribution in a Circular Beam

- To calculate the transverse shear stress distribution in a circular cross section, we use the integral form of the shear equation:

$$\tau = \frac{S}{Iz}\int_A y\,dA$$



- Because of the circular shape, it is convenient to change the variables y and z in this equation to polar variables, R and θ:

$$y_1 = R\sin\theta_1$$

$$dy_1 = d(R\sin\theta_1) = R\cos\theta_1 d\theta_1$$

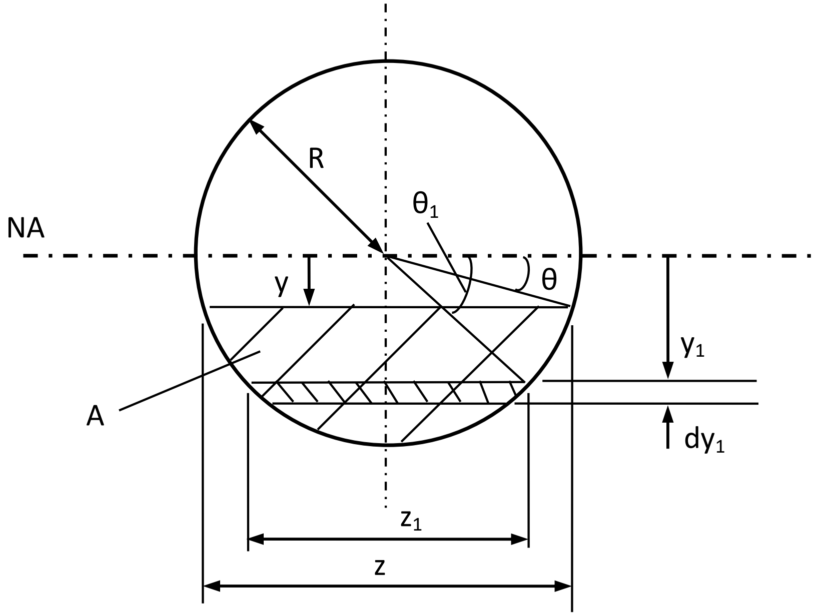
$$z_1 = 2R\cos\theta_1$$

$$z = 2R\cos\theta$$

Shear Stress Distribution in a Circular Beam

- As: $\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{y}{R}\right)^2$

$$\tau = \frac{4S}{3\pi R^2}\cos^2\theta = \frac{4S}{3\pi R^2}\left[1 - \left(\frac{y}{R}\right)^2\right]$$



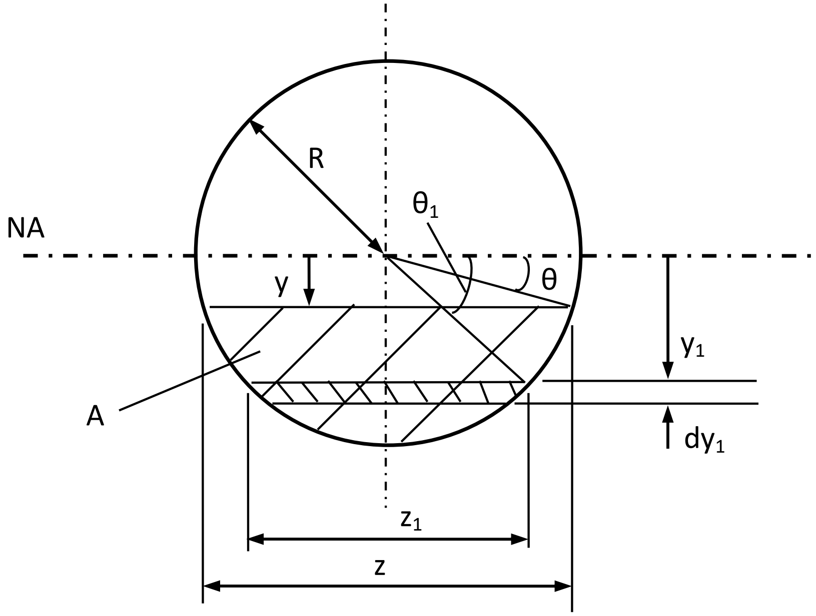
- Again a parabolic distribution
- With a maximum value of τ at the N.A.(y = 0):

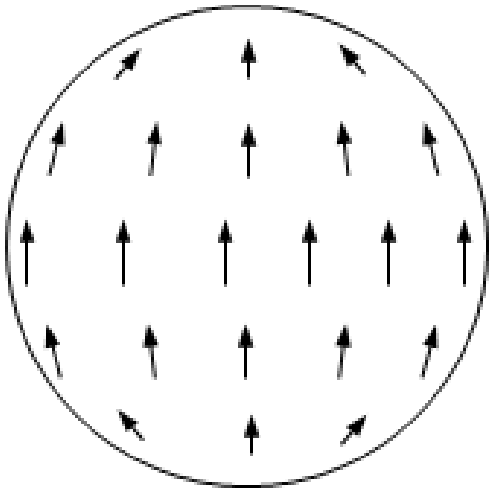
$$\tau_{max} = \frac{4S}{3\pi R^2} = \frac{4}{3}\cdot\frac{S}{A} = \frac{4}{3}\tau_{av}$$

Shear Stress Distribution in a Circular Beam

$$\tau = \frac{4S}{3\pi R^2}\left[1 - \left(\frac{y}{R}\right)^2\right]$$

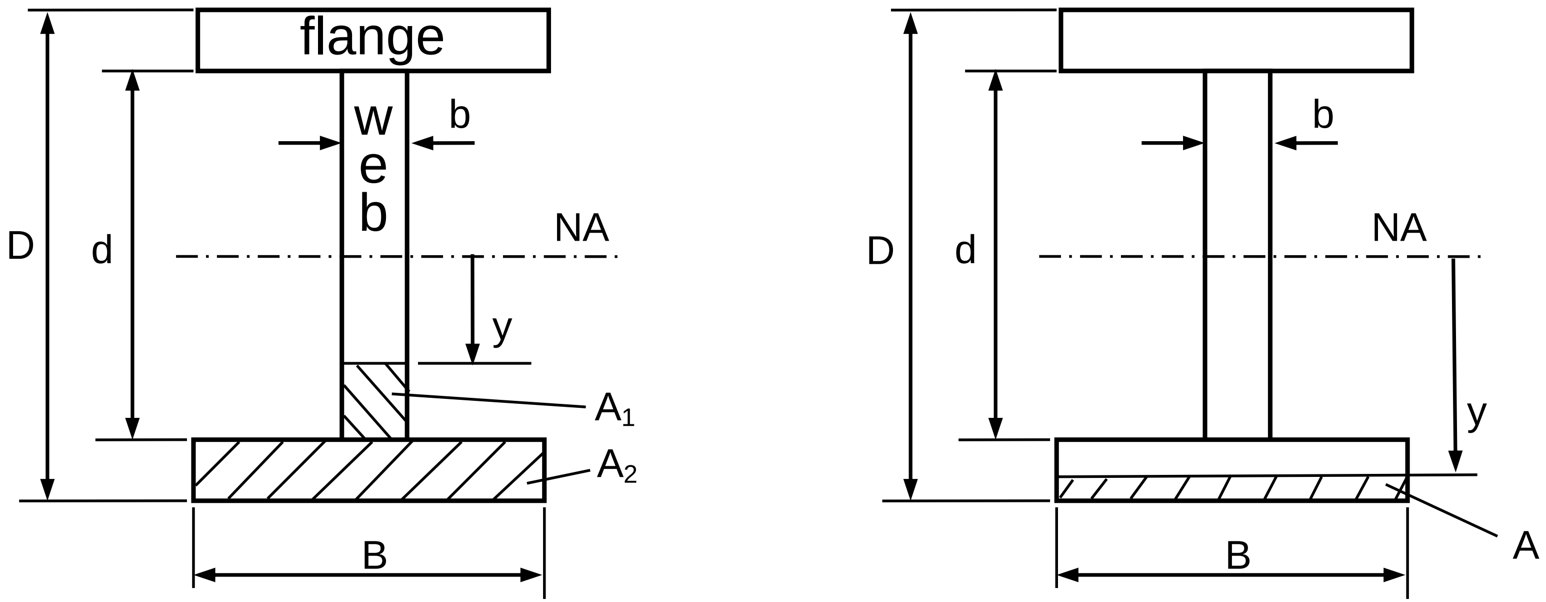
- In this case, τ must vary across the width of the section.
- For a circular cross section, we can no longer assume that the shear stresses act parallel to the y axis
- At the free surface, the shear stress must be zero. Therefore, the complementary shear on the cross-section, <u>normal to the boundary</u>, is also zero. Thus, shear must be tangential to the boundary as drawn.





Shear Stress Distribution in an I-Beam

- Transverse shear stress down the centre line
- Need to consider web and flange sections separately using the discrete formula



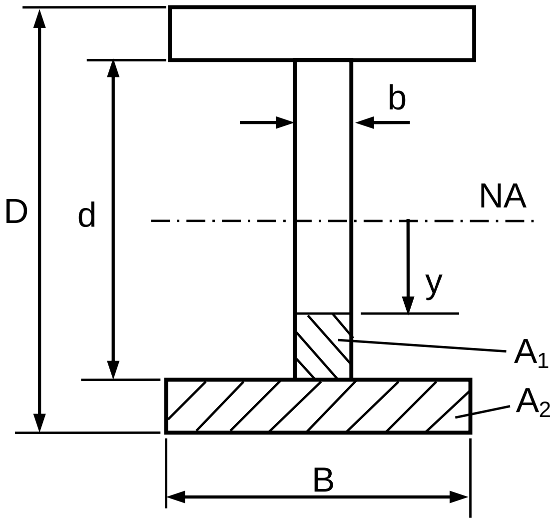
Shear Stress Distribution in an I-Beam

- In the web:

$$\tau = \frac{SA\bar{y}}{Iz}$$

- However we have two discrete areas to consider:

$$\tau = \frac{SA\bar{y}}{Iz} = \frac{S}{Iz}[A_1\bar{y}_1 + A_2\bar{y}_2]$$



- So:

$$\tau = \frac{S}{Ib}\left[\left(\frac{d}{2} - y\right) b \frac{1}{2}\left(\frac{d}{2} + y\right) + B\left(\frac{D}{2} - \frac{d}{2}\right)\frac{1}{2}\left(\frac{D}{2} + \frac{d}{2}\right)\right] = \frac{S}{Ib}\left[\frac{b}{2}\left(\frac{d^2}{4} - y^2\right) + \frac{B}{2}\left(\frac{D^2}{4} - \frac{d^2}{4}\right)\right]$$

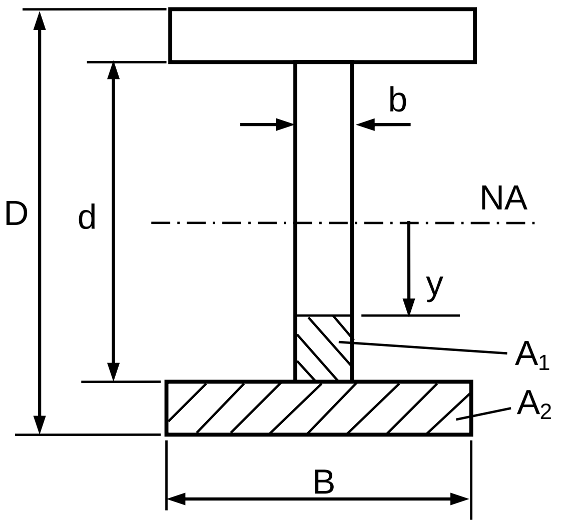
- And:

$$I = \frac{BD^3}{12} - \frac{(B-b)d^3}{12}$$

Shear Stress Distribution in an I-Beam

- The maximum τ at $y = 0$

$$\tau_{max} = \frac{S}{Ib}\left[\frac{BD^2}{8} - \frac{(B-b)d^2}{8}\right]$$



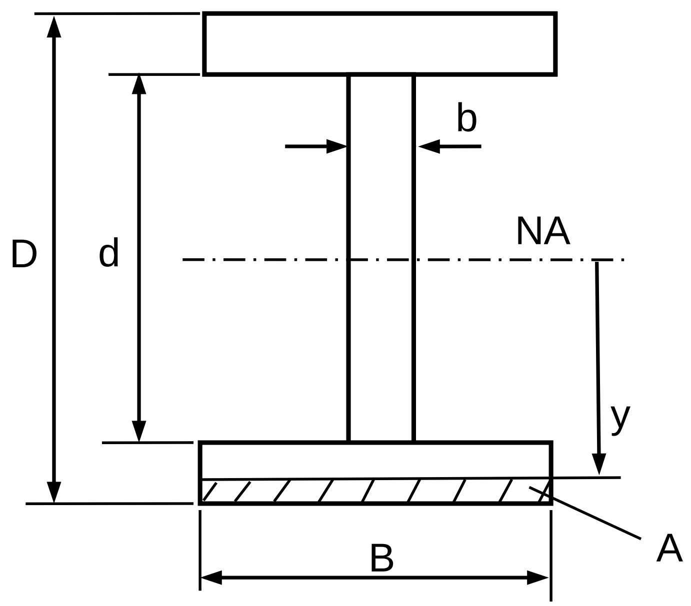
- And at the bottom and top of the web $y = \pm\frac{d}{2}$

$$\tau = \frac{S}{Ib}\frac{B}{8}(D^2 - d^2)$$

Shear Stress Distribution in an I-Beam

- In the flange: $\tau = \dfrac{S}{Iz} A\bar{y}$

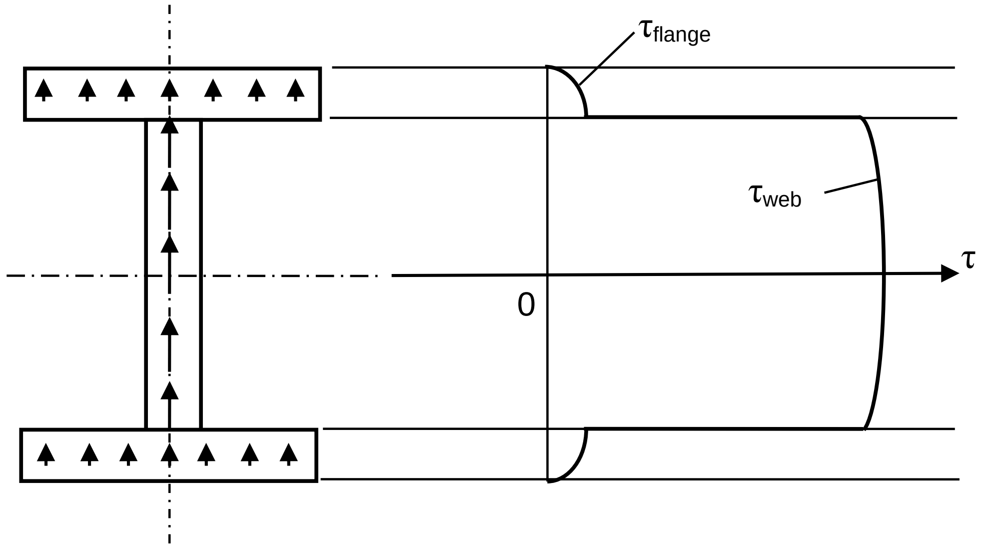
$$\tau = \frac{S}{IB}\left[B\left(\frac{D}{2} - y\right)\frac{1}{2}\left(\frac{D}{2} + y\right)\right] = \frac{S}{2I}\left(\frac{D^2}{4} - y^2\right)$$



- at $y = \pm\frac{D}{2}$, $\tau = 0$
- at $y = \pm\frac{d}{2}$: $\tau = \dfrac{S}{8I}(D^2 - d^2)$
- Compare with previous expression at this location: $\tau = \dfrac{S}{Ib}\dfrac{B}{8}(D^2 - d^2)$
- Step change in τ at this location due to change in section width

Shear Stress Distribution in an I-Beam

- The shear stress distribution down the centre line of the section is shown and illustrates the step change at the junction of the web and the flange
- The shear in the flanges is small compared to the web where it is approximately uniform with vertical position
- Because of the small shear in the flanges, the average shear stress in the web is ≈ S / bd, i.e. the shear force divided by the area of the web
- This distribution only applies down the centre line of the web.

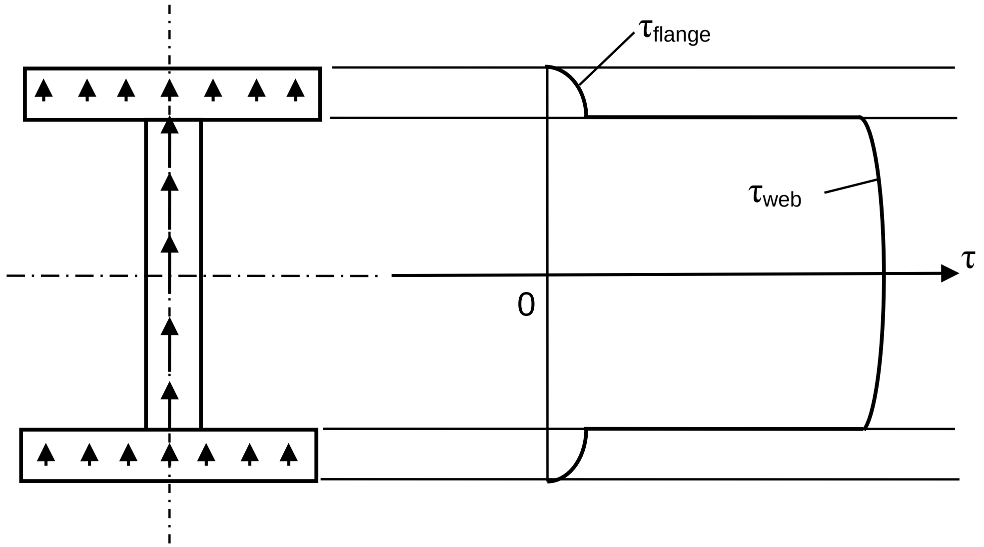


Shear Stress Distribution in an I-Beam

- The shear stresses in the flanges are small and non-uniform across the width. This must be the case as they must be zero at the top and bottom surfaces (i.e. free surfaces) of the flanges.

- There are more significant shear stresses in the flanges which act parallel to the flanges i.e. horizontally

- We will address these next…



Shear Stress Distribution in an I-Beam

$$\tau = -\frac{1}{Iz}\frac{\delta M}{\delta x}\int_A ydA$$

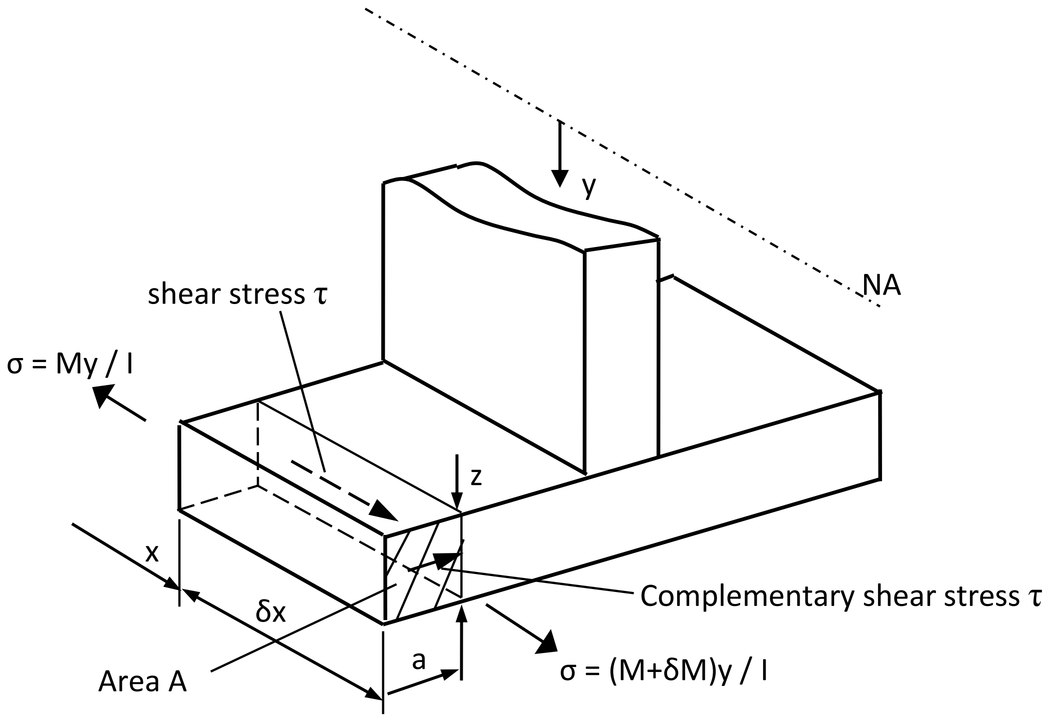
- In the limit

$$\lim_{\delta x \to 0}\frac{\delta M}{\delta x} = \frac{dM}{dx} = -S$$

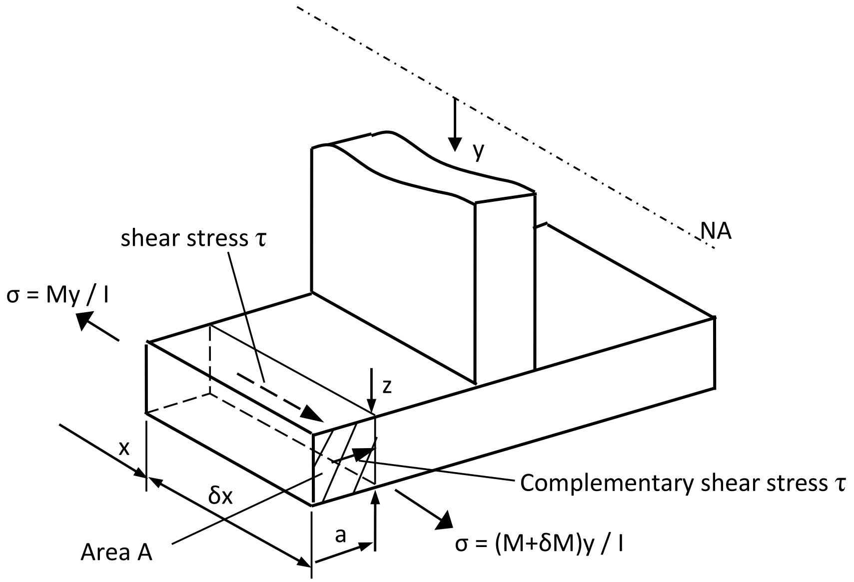
- Which gives:

$$\tau = \frac{S}{Iz}\int_A ydA = \frac{S}{Iz}A\bar{y}$$

- The same expression as for the vertical shear stress – handy!



Shear Stress Distribution in an I-Beam



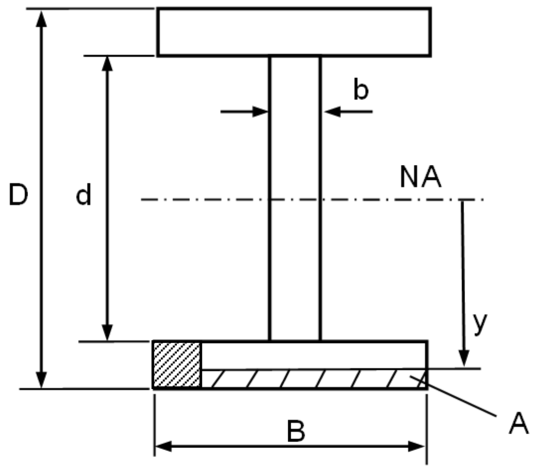
- *A*, $\bar{y}$ and *z* are slightly different as shown below. At a distance ***a*** from the edge of the flange, the horizontal shear stress is given by:

$$\tau = \frac{S}{Iz}(az)\frac{1}{2}\left(\frac{D}{2}+\frac{d}{2}\right) = \frac{Sa}{4I}(D+d)$$

- τ therefore varies linearly with **a** from zero at the flange edge to a maximum value at the flange centre (**a** = B/2):

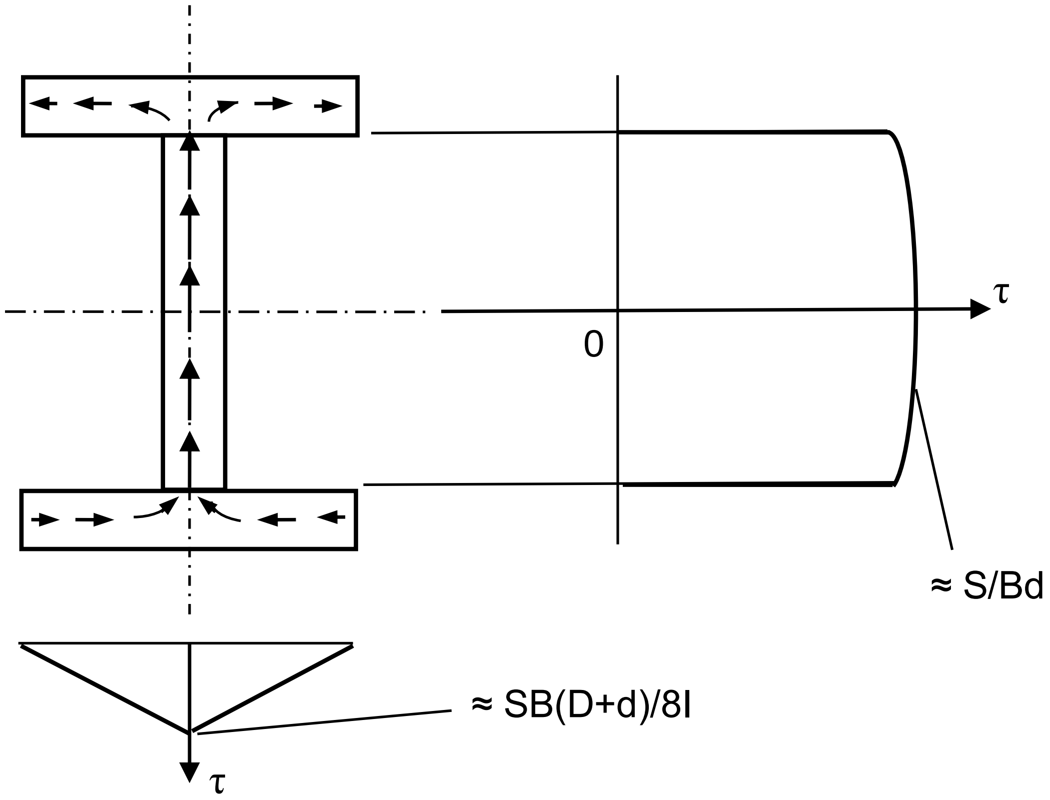
$$\tau_{max} = \frac{SB}{8I}(D+d)$$

- τ is also parallel to the flange, i.e. horizontal.



Shear Stress Distribution in an I-Beam

- We can now draw the dominant shear stresses in both the flange and the web



- The critical stress position is likely to be at the joint of the web and flange where both the shear and bending stresses are high

Shear Centre - Introduction

- What is the Shear Centre?

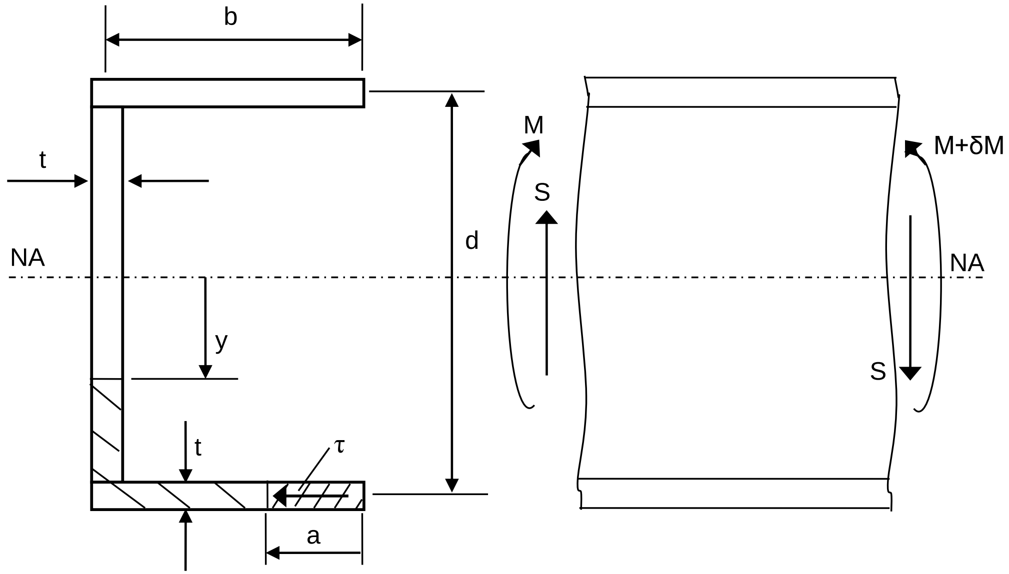
- The shear centre is the point through which the resultant of the shear stresses act

- The shear centre is important for beam sections which have <u>low torsional rigidity</u>, i.e. can twist easily, such as <u>thin-walled</u> sections. For such beams, if the resultant of the applied transverse loads do not act through the shear centre, they can cause twisting of the beam

- For <u>solid</u> sections, it is not usually important to determine the shear centre, because such sections usually have a considerable torsional rigidity and twist very little due to bending loads

Shear Centre - Thin Walled Channel Section

- Consider the shear stress distribution in a symmetric, thin walled channel section bending in the plane of the web



- For the flange at distance ***a*** from the edge, the horizontal shear stresses are:

$$\tau = \frac{S}{Iz}A\bar{y} = \frac{S}{It}(at)\cdot\left(\frac{d}{2}\right) = \frac{S\cdot d\cdot a}{2I}$$

(As analysed previously for the flange in an I-section)

- For the web at distance **y** from the N.A., the transverse shear stresses are:

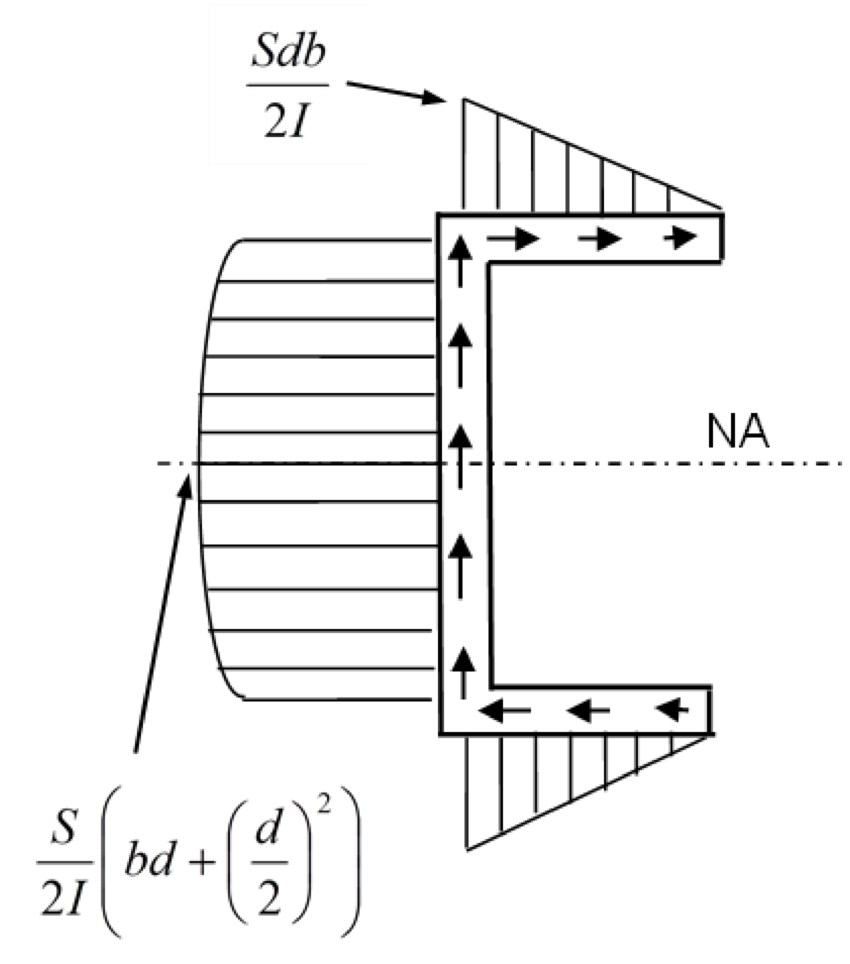
$$\tau = \frac{S}{Iz}A\bar{y} = \frac{S}{It}\left[bt\frac{d}{2}+\left(\frac{d}{2}-y\right)t\left(\frac{d}{2}+y\right)\frac{1}{2}\right] = \frac{S}{2I}\left(bd+\left(\frac{d}{2}\right)^2-y^2\right)$$

Shear Centre - Thin Walled Channel Section

- We can now draw the shear stress distribution in the web and flanges, as shown:

- The shear stress in the upper flange is in the opposite sense to that in the lower flange, i.e. there is no horizontal resultant from the two components.

- There are no shear stresses on the free surfaces, the shear stresses act along the walls, i.e. horizontal in the flanges and vertical in the web.



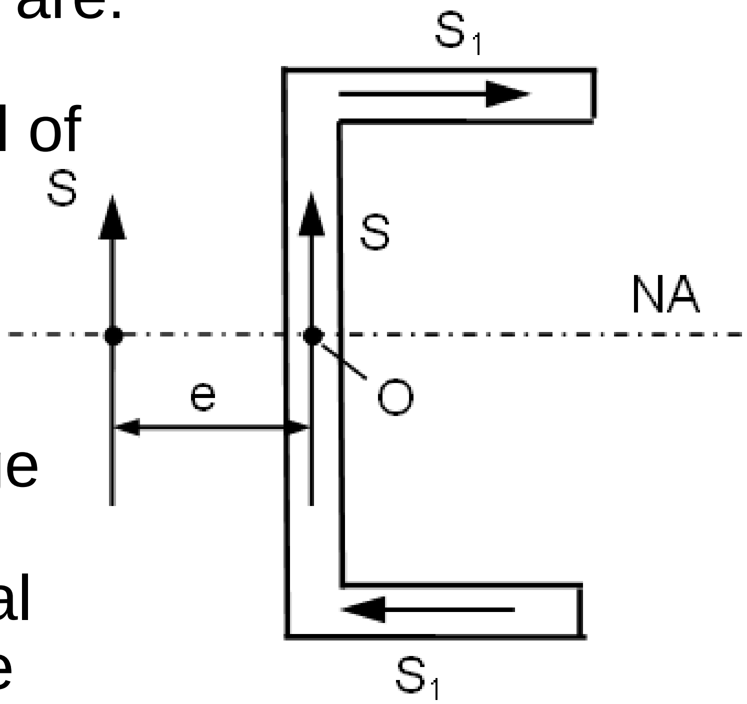
Shear Centre - Thin Walled Channel Section

- The resultant forces arising from this shear stress distribution are:
  - the total shear force in the lower flange, S_1, is the integral of the shear stresses in this flange:

$$S_1 = \int_0^b \tau t da = \int_0^b \frac{sda}{2I} t da = \frac{Sdtb^2}{4I}$$

  - an equal and opposite shear force acts in the upper flange
  - the shear force in the web is approximately S, i.e. the total vertical shear load (assuming thin flanges carry negligible vertical shear load)
  - if we take moments about O in the web:

$$S \cdot e = 2S_1 \frac{d}{2} \quad \Rightarrow \quad e = \frac{S_1 d}{S} = \frac{d^2 t b^2}{4I}$$



Shear Centre - Thin Walled Channel Section

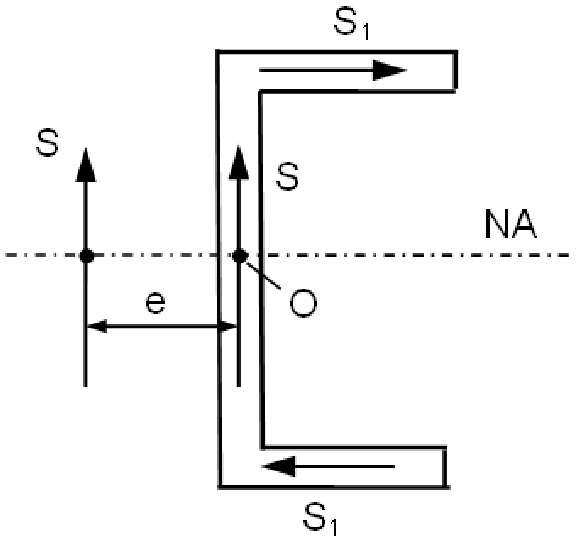
- actual twist is anti-clockwise, because reactive internal "equilibrium" forces cause the twisting

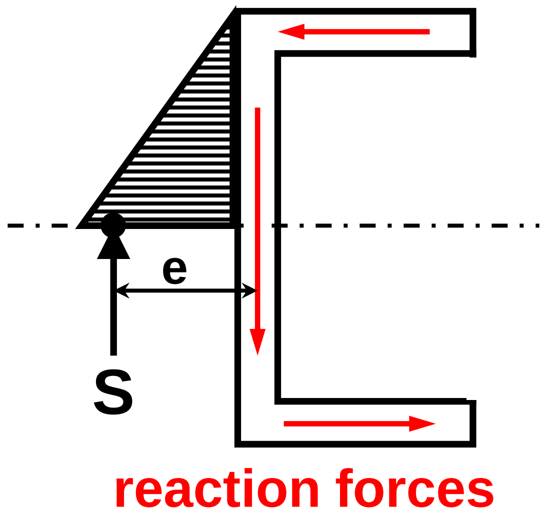
- in order to prevent this twisting and therefore cancel the unbalanced moment, it is necessary to apply **S** at a point located an eccentric distance **e** from the web

- this point so located is called the **shear centre**; when **S** is applied at this point, the beam will bend without twisting

- the shear centre will always lie on an axis of symmetry of a member's cross-sectional area; location of the shear centre is only a function of the geometry of the cross section, and does not depend upon the applied loading

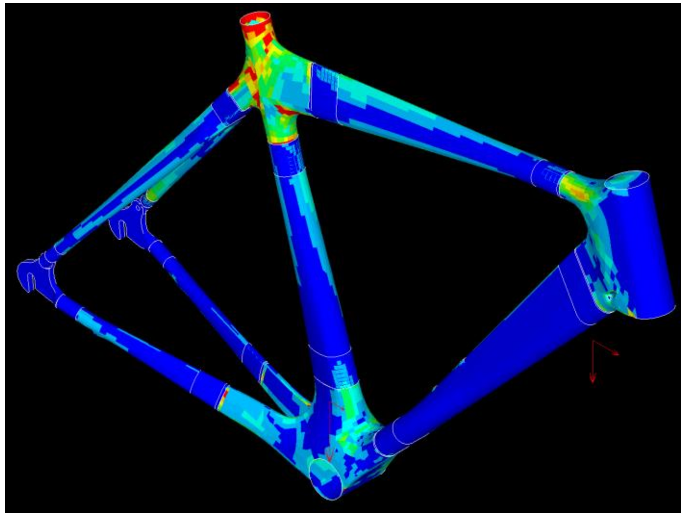
$$e = \frac{d^2tb^2}{4I}$$



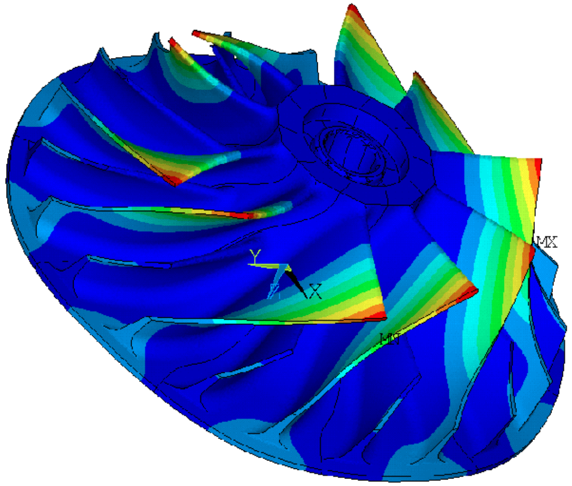


Finite Element Analysis

- **What is the Finite Element Method (FEM)?**
- FEM is a numerical technique for finding approximate solutions to partial differential equations
- These are often structural mechanics problems (and this is the focus here) but the method is also commonly applied to thermal, fluid, dynamic, electrical, magnetic and acoustic problems among others (and even combinations of these)



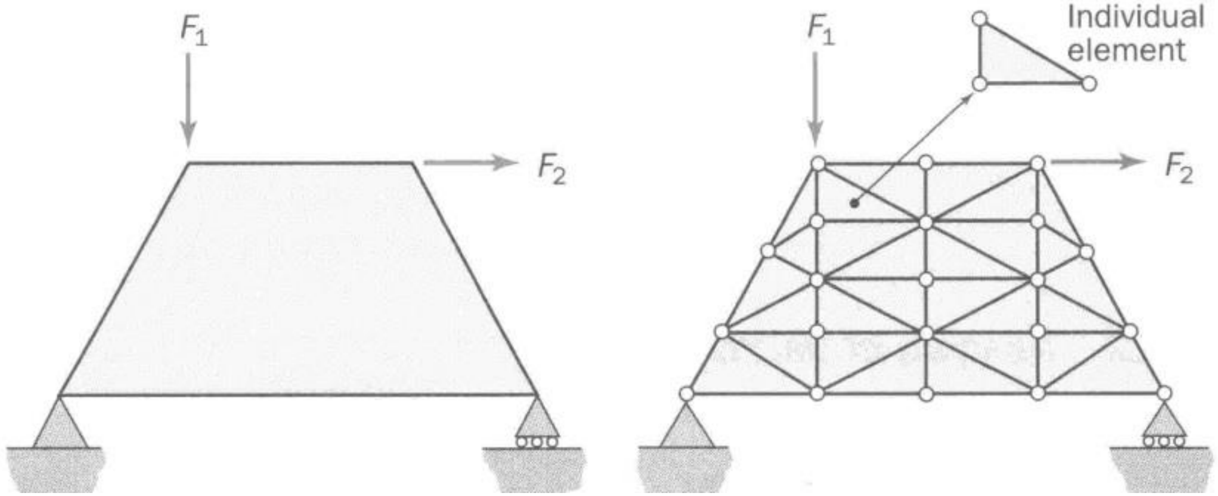
http://www.williamsbikes.com/



http://simmsmachineryinternational.com/

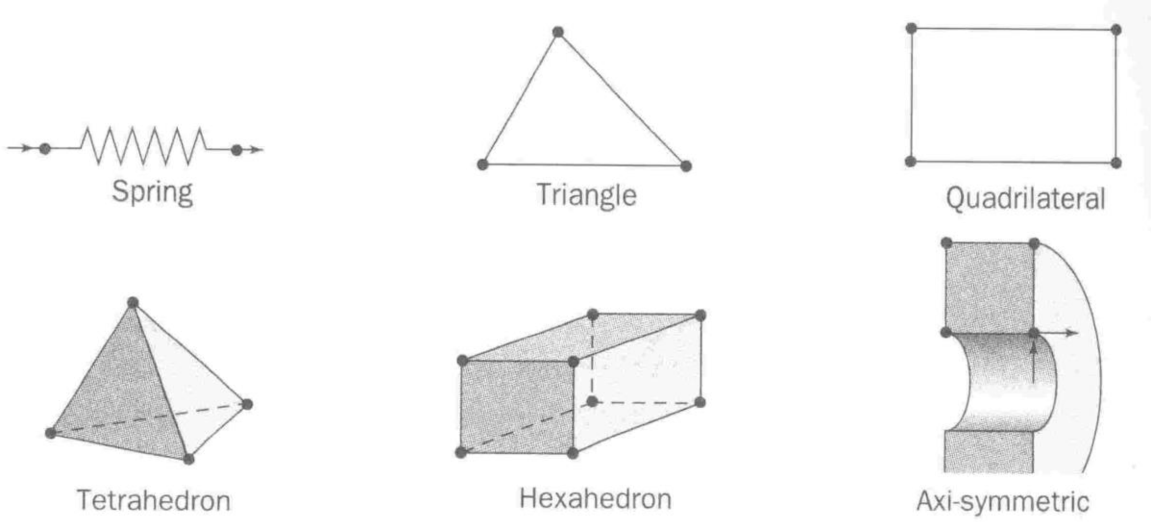
Finite Element Analysis

- **How does FEM work?**
- The basic concept of FEM is to discretise a domain into a number of smaller ‘finite elements’
- These finite elements are appropriately connected on their boundaries at ‘nodes’



- The exact solution of the global function is approximated locally in each element using simplified functions while maintaining equilibrium, deformation compatibility and stress-strain relationships.

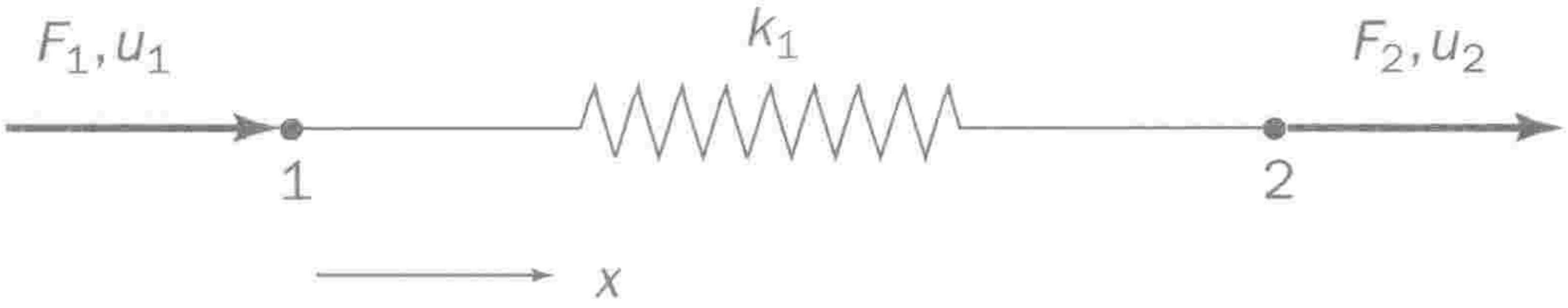
- **How does FEM work?**
- There are many different types of element that can be used dependent on the requirements of the solution, or the type of problem being analysed



- The accuracy of the FE solution depends on the number of elements and while the analysis of each element is relatively simple, the complete analysis for a large number of elements is extremely tedious, hence the need for computers.

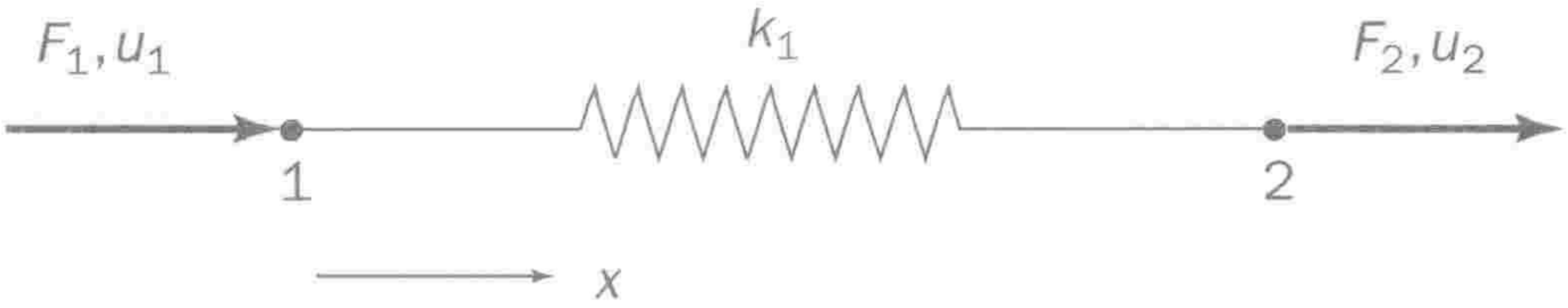
1D Spring Element

- Simplest element that we will use to build up understanding of how the Finite Element Method works



- The points of attachment to other parts or elements are called nodes (1 & 2). The nodal forces and displacements (*F* & *u* respectively) as well as the spring stiffness, k_1, are also shown. For a bar of length *L*, area *A* and elastic modulus *E*, $k_1 = AE/L$

1D Spring Element



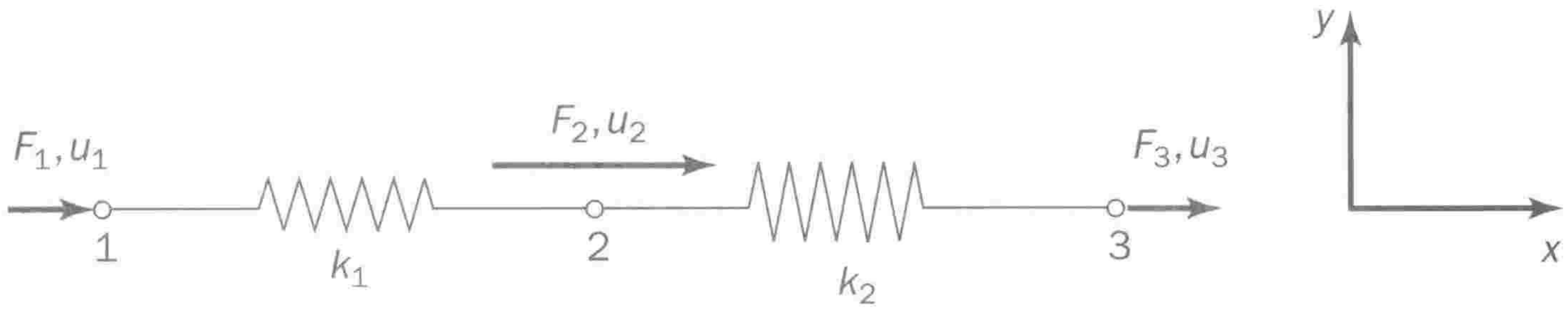
- Note that $[K^e]$ is symmetrical. In general, whether for an element or a complete structure, the stiffness matrix is always symmetrical.

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Assembly of Spring Elements

- To perform analysis of real problems, we need to combine more than one element

- Shown below are two connected spring elements:



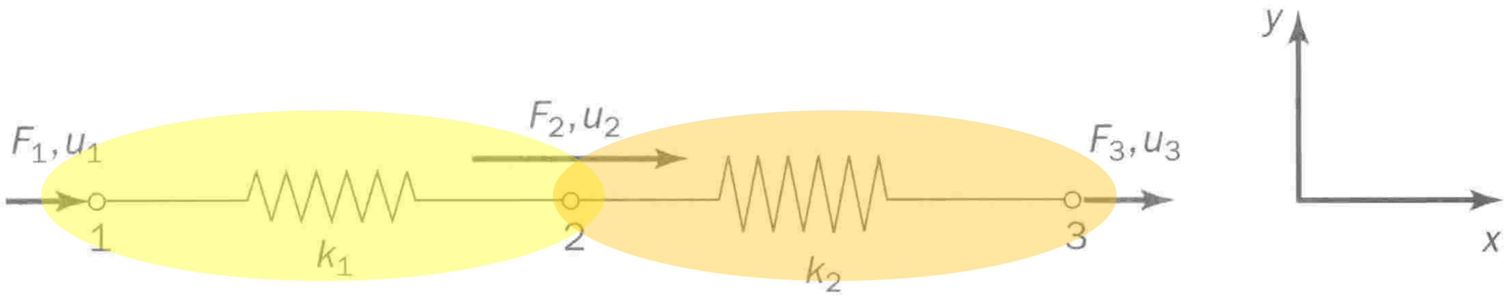
Assembly of Spring Elements

- For element 1:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

- And element 2:

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

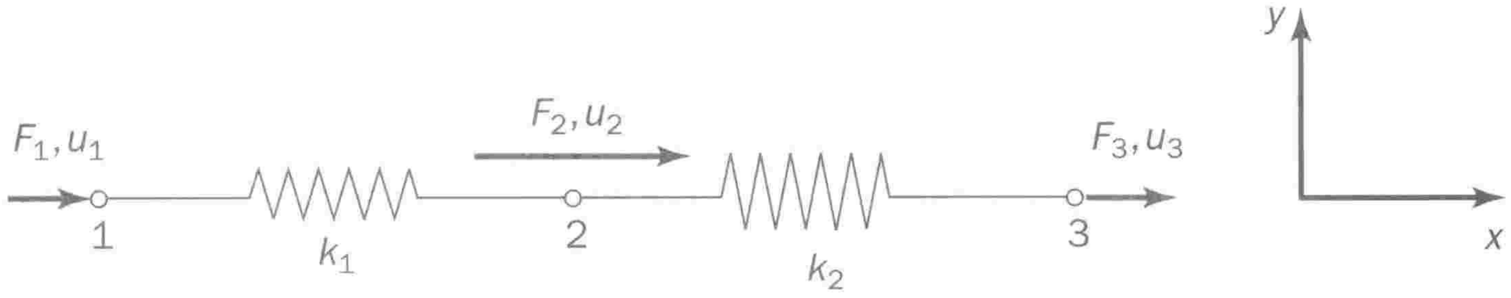


Assembly of Spring Elements

- If we expand the matrices to make them equivalent:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$



Assembly of Spring Elements

- We can then combine the matrices to form one stiffness matrix for the whole system:

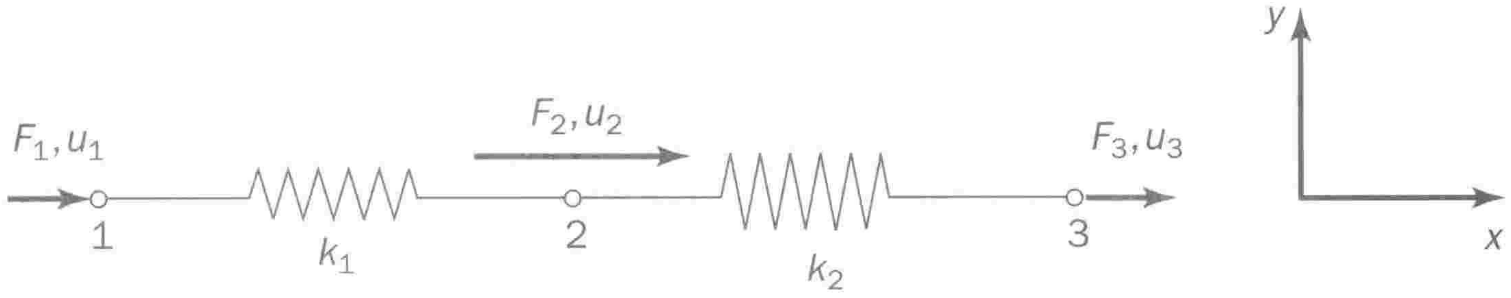
$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

e1 e2

- Or:

$$\{F\} = [K]\{u\}$$

- Where **[*K*]** is the **Global** stiffness matrix



Application of Spring Elements

- Now that we have a system of equations in matrix form that define the relationships between the forces and displacements in our system, we can start to use it to solve problems.

- If we subject the system to a force(s) and sufficient boundary conditions are specified, the remaining forces and displacements can be found.

Application of Spring Elements

1. Determine *AE/L*

2. Construct the element stiffness matrices

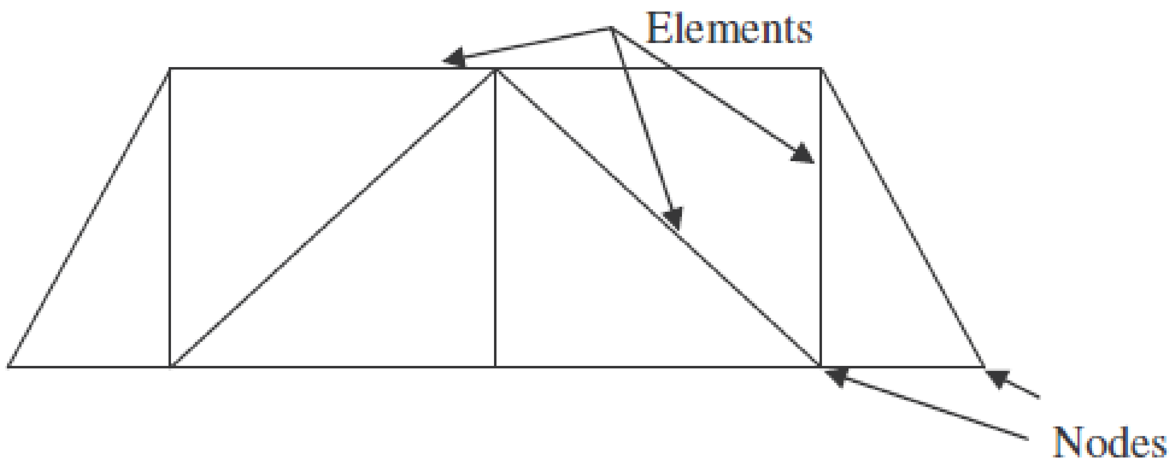
3. Combine to form the global stiffness matrix

4. Apply the boundary conditions

5. Solve for displacements

Truss Elements

- 1D spring elements are a good simple introduction to demonstrate the matrix method and the concept of stiffness matrices, however, they have limited practical use.
- Truss elements are an extension of the spring element where each node has 2 degrees of freedom (DOFs) and they can be connected to form frame structures
- Truss element structure:

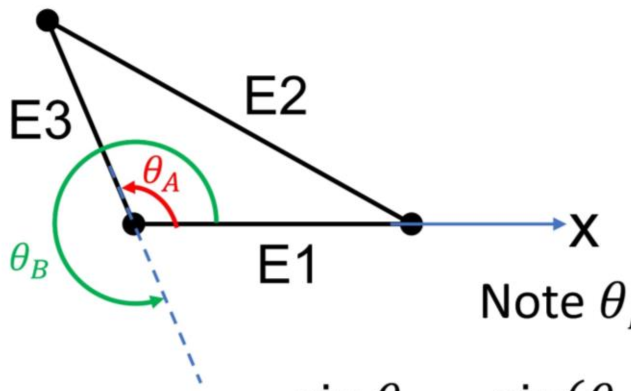


Truss Elements

$$[K^e]_g = \left(\frac{AE}{L}\right)\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

- Stiffness matrix for a truss element in global coordinates
- This element stiffness matrix is also symmetrical
- Dimension of matrix is 4 x 4 as each element has 4 unknown deflections (2 nodes with 2 DOFs)
- Again $c = \cos\theta$ and $s = \sin\theta$

Truss Elements



For element 3 (E3), the angle can be either taken as θ_A or θ_B.

Note $\theta_B = \theta_A + \pi$, $\sin\pi = 0, \cos\pi = -1$

$$\sin\theta_B = \sin(\theta_A + \pi) = \sin\theta_A \cos\pi + \cos\theta_A \sin\pi = -\sin\theta_A$$

$$\cos\theta_B = \cos(\theta_A + \pi) = \cos\theta_A \cos\pi - \sin\theta_A \sin\pi = -\cos\theta_A$$

$$(\sin\theta_B)^2 = (-\sin\theta_A)^2 = (\sin\theta_A)^2$$

$$(\cos\theta_B)^2 = (-\cos\theta_A)^2 = (\cos\theta_A)^2$$

$$\sin\theta_B\cos\theta_B = (-\sin\theta_A)(-\cos\theta_A) = \sin\theta_A\cos\theta_A$$

$$[K^e]_g = \left(\frac{AE}{L}\right)\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

Note $c = \cos\theta$ and $s = \sin\theta$

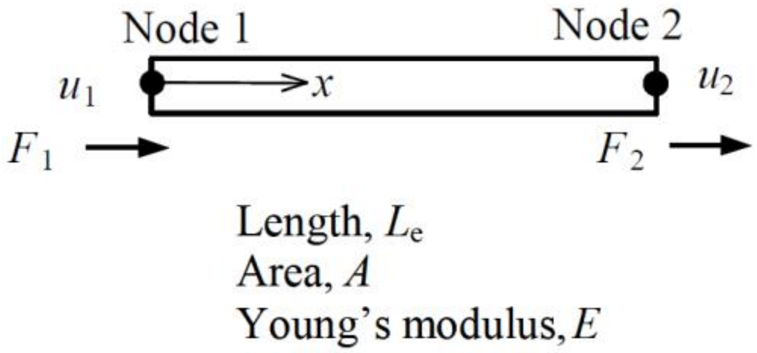
Therefore, the stiffness matrix of E3 in global coordinates are the same if you use either θ_A or θ_B in the calculation.

Convention: use the anticlockwise angle from the x-axis in the direction of the ordered DOFs

Shape Functions

- For our 1D bar, the displacement pattern may be represented as a linear polynomial

$$u = \alpha_1 + \alpha_2 x$$



- α_1 and α_2 are constants which may be determined from the nodal displacements and the geometry of the element

- At node 1, x = 0 so, $u = u_1 = \alpha_1$
- At node 2, x = L so, $u = u_2 = \alpha_1 + \alpha_2 L$

Shape Functions

- Or in matrix form:

$$u(x) = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u(x) = [N_1(x) \quad N_2(x)] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u(x) = [N]\{u\}$$

where [*N*] (or $N_1(x)$ and $N_2(x)$) are the **shape functions** of the element, which specify the variation in displacement within the element, here it is linear

Shape Functions

- The shape functions can also be used to determine the strain in the element

$$\varepsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[N]\{u\} \qquad u(x) = [N]\{u\}$$

$$\varepsilon_x = \frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\varepsilon_x = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\varepsilon_x = [B]\{u\}$$

Shape Functions

- And then the stress (for a uniaxial bar):

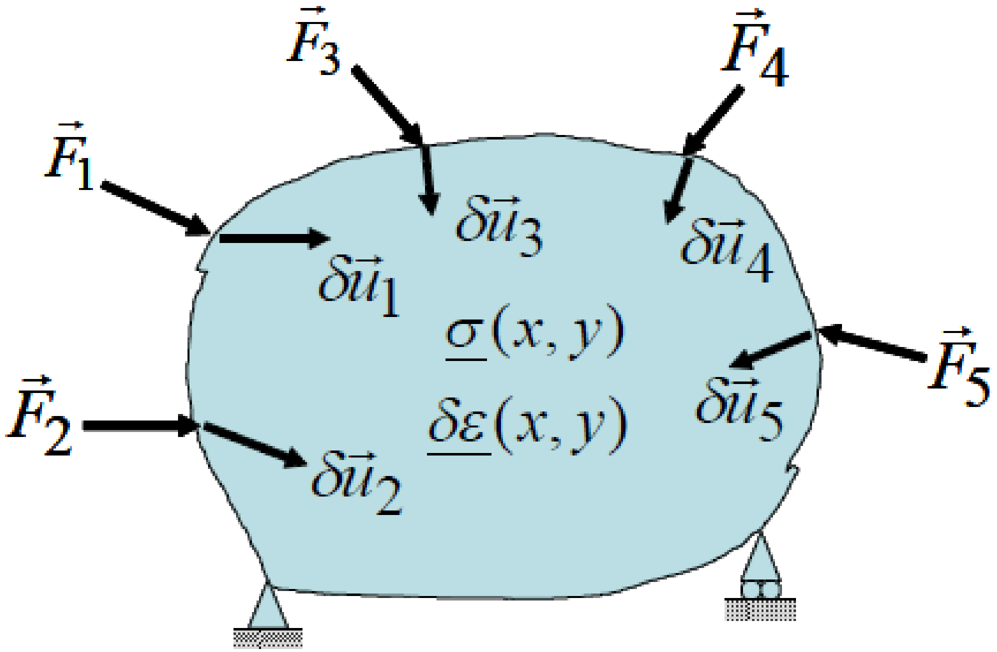
$$\sigma = E\varepsilon = [E][B]\{u\}$$

$$\sigma = E\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

- In this case, [*E*] is just a single value. For more complicated 2D and 3D elements, more elements are required in [*E*] but the principle is the same

The Principle of Virtual Work

- The Principle of Virtual Work states that the work done by a set of forces moving through a set of small, compatible displacements is zero



$$\delta W = \delta W_{int} - \delta W_{ext} = 0$$

The Principle of Virtual Work

- Consider:

$$EA\int_0^{L_e}\{B\}^T\{B\}dx = EA\int_0^{L_e}\begin{Bmatrix}-\dfrac{1}{L_e}\\ \dfrac{1}{L_e}\end{Bmatrix}\begin{Bmatrix}-\dfrac{1}{L_e} & \dfrac{1}{L_e}\end{Bmatrix}dx$$

$$= \frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = [K]$$

- So the stiffness equation becomes

$$[K]\{u\} = \{F\}$$

- We can derive the stiffness matrix for more complex 2D and 3D elements by following the same process

2D Approximations

- Although a lot of real world problems are 3D, we can make some assumptions and use a 2D approach for some problems to reduce the overall size of our model

- Useful assumptions include the **plane stress, plane strain** and **axisymmetric** assumptions

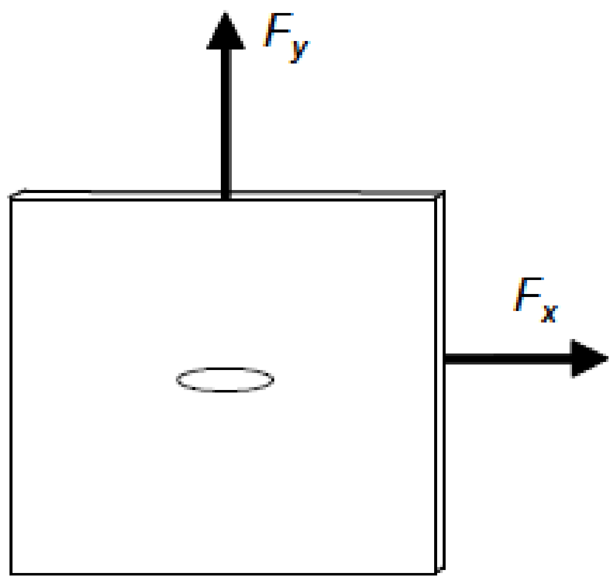
2D Approximations

- **Plane Stress** Approximation

- Consider a thin plate which is only loaded in the in-plane directions

- The normal stress σ_z must be zero on the front and back faces

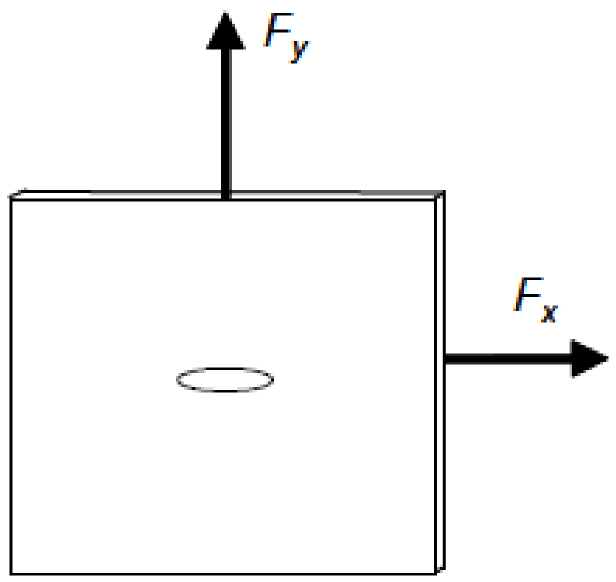
- Because the plate is thin, then we can assume that $\sigma_z \approx 0$ throughout the thickness



2D Approximations

- **Plane Stress** Approximation

- The only non-zero components of stress are σ_x, σ_y and τ_{xy} and we can determine all of the strain components, i.e. ε_x, ε_y, ε_z and γ_{xy}, from these stress components using generalised Hooke's law (for elastic behaviour)



$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \varepsilon_z = \frac{-\nu}{E}(\sigma_x + \sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$ (shear strain)

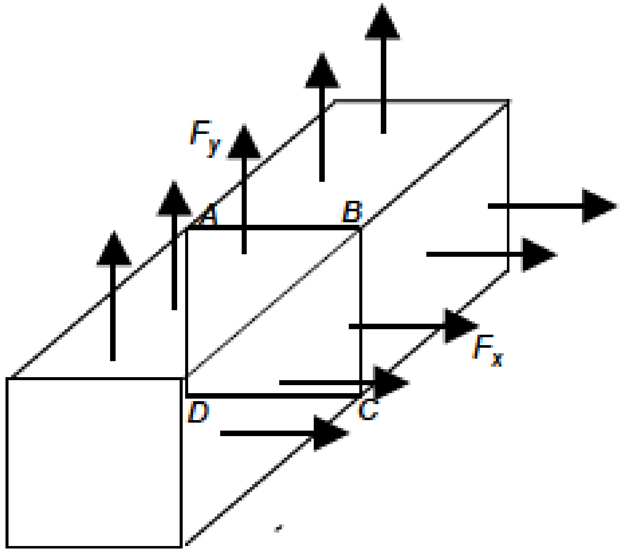
2D Approximations

- **Plane Strain** Approximation

- Consider a very thick plate or long member of regular cross-section, again only loaded in the in-plane directions

- A plane ABCD, remote from the ends experiences negligible strain in the z-direction $\varepsilon_z \approx 0$, so $\sigma_z = \nu(\sigma_x + \sigma_y)$

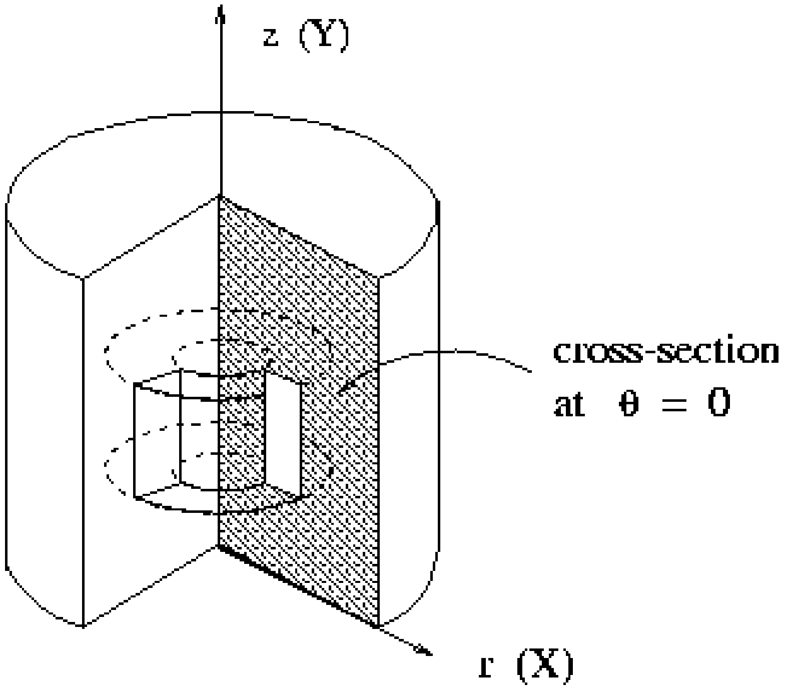
- We can determine the z-direction stresses from the x and y-direction normal stresses



$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right)$$

2D Approximations

- **Axisymmetric** Approximation
- Used to represent cases with geometry and loading that is rotationally symmetric (r, z, θ coordinates)
- Because of symmetry about the z axis, the stresses are independent of the θ coordinate
- All derivatives with respect to θ vanish and the displacement component in the θ direction, the shear strains $\gamma_{r\theta}$ and $\gamma_{\theta z}$ and the shear stresses $\tau_{r\theta}$ and $\tau_{\theta z}$ are all zero

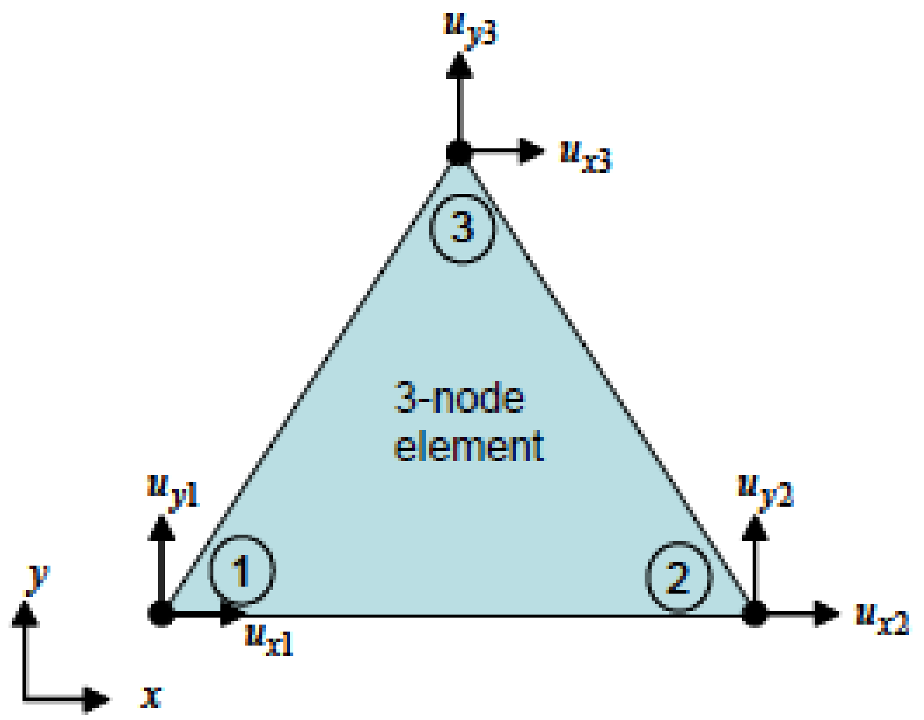


Constant Strain Triangle

- The simplest 2D element
- A 3-noded, linear triangular element, 2 DOF per node
- Linear variation of displacements within the element:

$$u_x(x,y) = C_1 + C_2 x + C_3 y$$

$$u_y(x,y) = C_4 + C_5 x + C_6 y$$



Constant Strain Triangle

- From before:

$$\{C\} = [A]^{-1}\{u\} \qquad \text{(as } \{u\} = [A]\{C\}\text{)}$$

- So:

$$\{\varepsilon\} = [X]\{C\} = [X][A]^{-1}\{u\} = [B]\{u\}$$

(strain-displacement relation)

Constant Strain Triangle

- Substituting for the strains

$$\{\sigma\} = [D][B]\{u\}$$

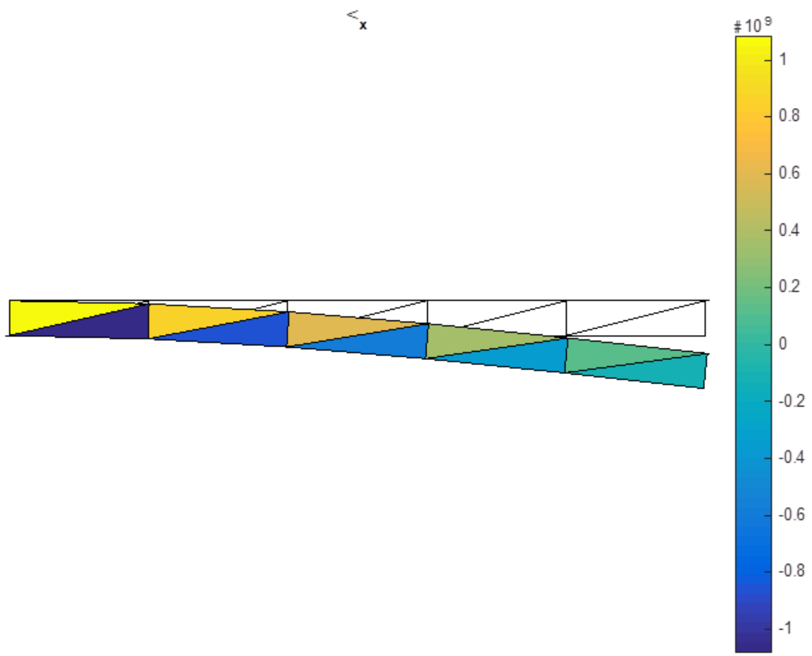
(stress-displacement relation)

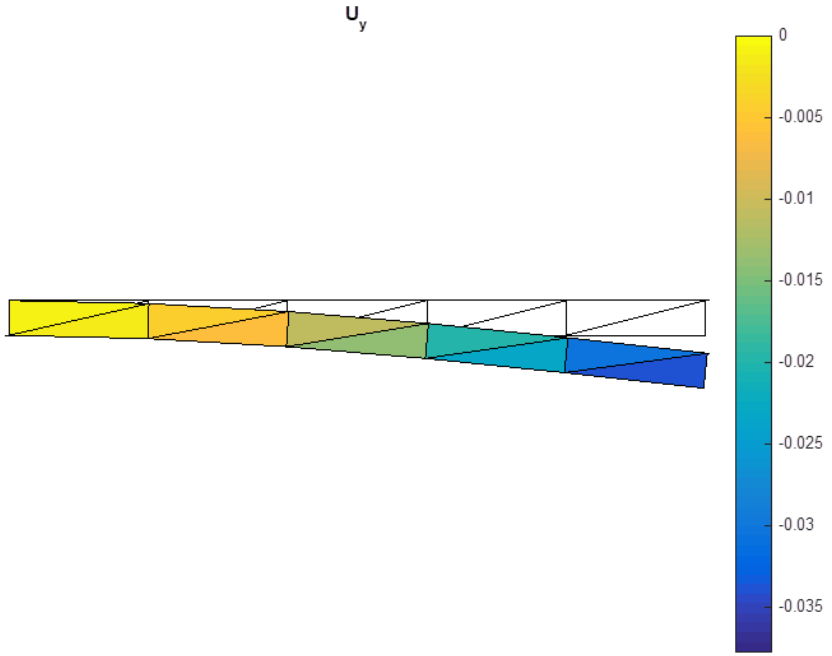
- Which allows us to determine the element stresses from the element displacements

Constant Strain Triangle Example

- A steel cantilever beam of length 0.5m width 0.01m and depth 0.025m, is subjected to a point load at the end of 5kN, determine the maximum stress in the beam and the maximum deflection of the tip of the cantilever. *E* = 200 GPa

N_{el} = 10 (1 through thickness)

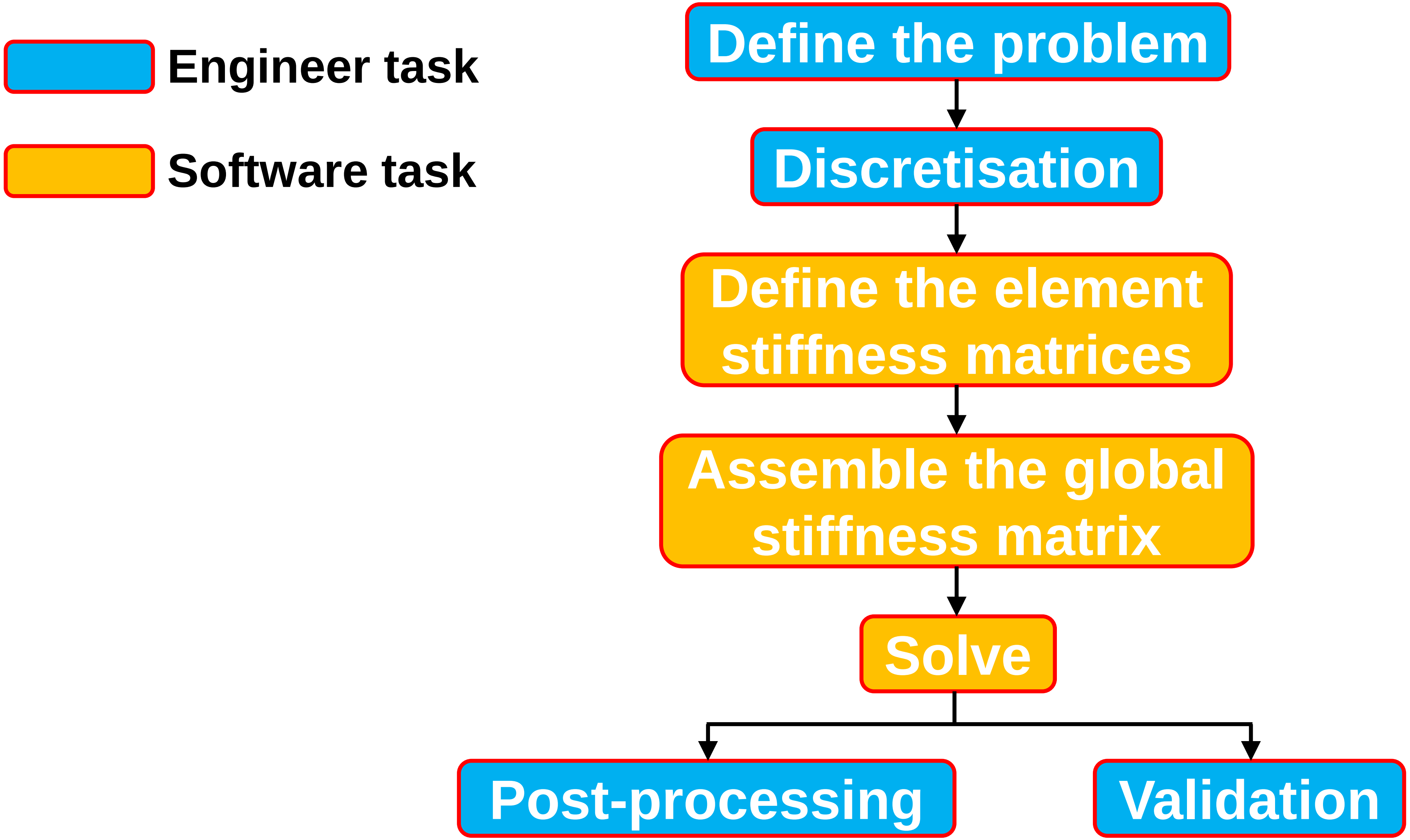




σ_{x_max} = 1082 MPa

u_{y_min} = -0.0378 m

Finite Element Analysis - Workflow



https://www.youtube.com/watch?v=GHjopp47vvQ