Laplace Transforms Problem Sheet 4

1. Use the definition (i.e. do not use the table of Laplace transforms) to find the Laplace transform of each of the functions below.

$$2e^{4t}, \qquad 3e^{-2t}, \qquad 5t-3, \qquad 2t^2 - e^{-t}, \qquad 3\cos 5t, \qquad 10\sin 6t.$$

2. Find the Laplace transforms of

(a) $F(t) = \begin{cases} 0 & \text{for} \quad 0 \le t \le 2, \\ 4 & \text{for} \quad t > 2. \end{cases}$ (b) $f(t) = e^{\alpha t}\sinh kt.$

3. Use the table of Laplace transforms to invert the functions given below.

$$\frac{1}{s+1}, \qquad \frac{1}{(s-3)^2}, \qquad \frac{a}{s(s+a)}, \qquad \frac{k^2}{s(s^2+k^2)}, \qquad \frac{6s-4}{s^2-4s+20}, \qquad \frac{e^{-2s}}{s^4}.$$

4. Solve, using Laplace transforms, the differential equation

$$\frac{d^2y}{dt^2} - 2a\frac{dy}{dt} + (a^2+b^2)y = 0, \quad \text{with} \quad y = 0, \quad \frac{dy}{dt} = 1, \quad \text{when} \quad t = 0.$$

5. Solve, using Laplace transforms, the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 1 - H(t-4),$$

with the initial conditions $y(0) = 0, \quad y'(0) = 0.$

6. Using the table of Laplace transforms, show that

(a) $\mathcal{L}\left\{e^{-2t}\sin t\right\} = \dfrac{1}{s^2+4s+5},$

(b) $\mathcal{L}\left\{\dfrac{1}{5}\left[1 - e^{-2t}\cos t - 2e^{-2t}\sin t\right]\right\} = \dfrac{1}{s(s^2+4s+5)},$

(c) $\mathcal{L}\left\{\dfrac{1}{5}\left[1 - e^{-2v}\cos v - 2e^{-2v}\sin v\right] H(v)\right\} = \dfrac{e^{-3s}}{s(s^2+4s+5)},$

where $v = t - 3$.

Use Laplace transform techniques to solve the ordinary differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = H(t-3),$$

with $y(0) = 0$ and $y'(0) = 1$.