Minilecture 1E: Homogeneous equations complex roots

Recall $y = e^{mx}$ is a solution of

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

if

$$am^2 + bm + c = 0$$

Three cases:
- real distinct roots — done
- complex roots — now
- repeated roots — next time

Complex roots

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad b^2 - 4ac < 0$$

Call them

$$m_1 = \alpha + i\beta \qquad m_2 = \alpha - i\beta$$

Solutions

$$y_1(x) = e^{m_1 x} = e^{(\alpha + i\beta)x}$$
$$y_2(x) = e^{m_2 x} = e^{(\alpha - i\beta)x}$$

and general solution:

$$y(x) = A e^{(\alpha + i\beta)x} + B e^{(\alpha - i\beta)x}$$

Euler's formula

$$e^{i\beta x} = \cos\beta x + i\sin\beta x$$

Then general solution is

$$\begin{aligned} y(x) &= A e^{\alpha x} e^{i\beta x} + B e^{\alpha x} e^{-i\beta x} \\ &= e^{\alpha x}\left(A e^{i\beta x} + B e^{-i\beta x}\right) \\ &= e^{\alpha x}\Big(A(\cos\beta x + i\sin\beta x) \\ &\qquad\quad + B(\cos\beta x - i\sin\beta x)\Big) \\ &= e^{\alpha x}\left((A+B)\cos\beta x + i(A-B)\sin\beta x\right) \end{aligned}$$

Define <u>new</u> constants

$$C = A+B \qquad D = i(A-B)$$

Then

$$\boxed{y(x) = e^{\alpha x}\left(C\cos\beta x + D\sin\beta x\right)}$$

Note C, D can be <u>real</u> by choosing A, B appropriately.

Example Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0.$

Auxiliary equation: $m^2 + 2m + 2 = 0$
$= (m+1)^2 + 1$

$\Rightarrow m+1 = \pm i \Rightarrow m = -1 \pm i$

General solution

$$y(x) = Ae^{(-1+i)x} + Be^{(-1-i)x}$$
$$= e^{-x}(Ae^{ix} + Be^{-ix})$$
$$= e^{-x}(A(\cos x + i\sin x) + \cdots)$$
$$\vdots$$
$$= e^{-x}(C\cos x + D\sin x)$$

Example Solve

$$d^2x/dt^2 + \frac{dx}{dt} + \frac{1}{2}x = 0 \qquad (x = e^{mt})$$

Auxiliary equation $0 = m^2 + m + \frac{1}{2} = (m+\frac{1}{2})^2 + \frac{1}{4}$

$\Rightarrow m + \frac{1}{2} = \pm\frac{i}{2} \quad m = -\frac{1}{2} \pm \frac{i}{2}$

General solution:

$$x(t) = A e^{\left(-\frac{1}{2}+\frac{i}{2}\right)t} + B e^{\left(-\frac{1}{2}-\frac{i}{2}\right)t}$$

$$= e^{-\frac{1}{2}t}\left(C\cos \frac{1}{2}t + D\sin \frac{1}{2}t \right)$$