Deflection of Beams

Lecture 1 – Introduction & Equation of the Elastic Line

Department of Mechanical, Materials & Manufacturing Engineering

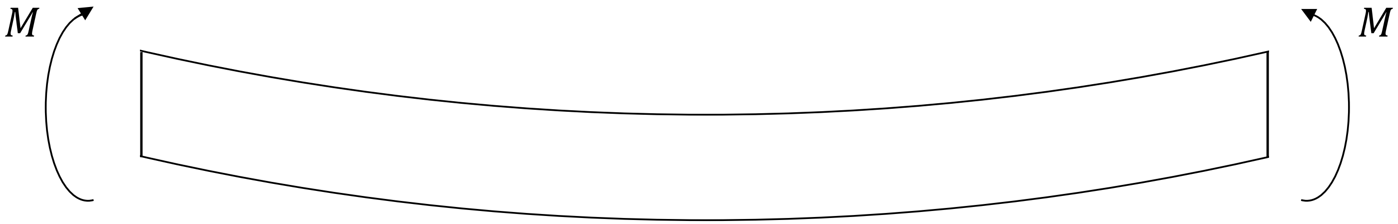
MMME2053 – Mechanics of Solids

University of Nottingham
UK | CHINA | MALAYSIA

Deflection of Beams

Introduction

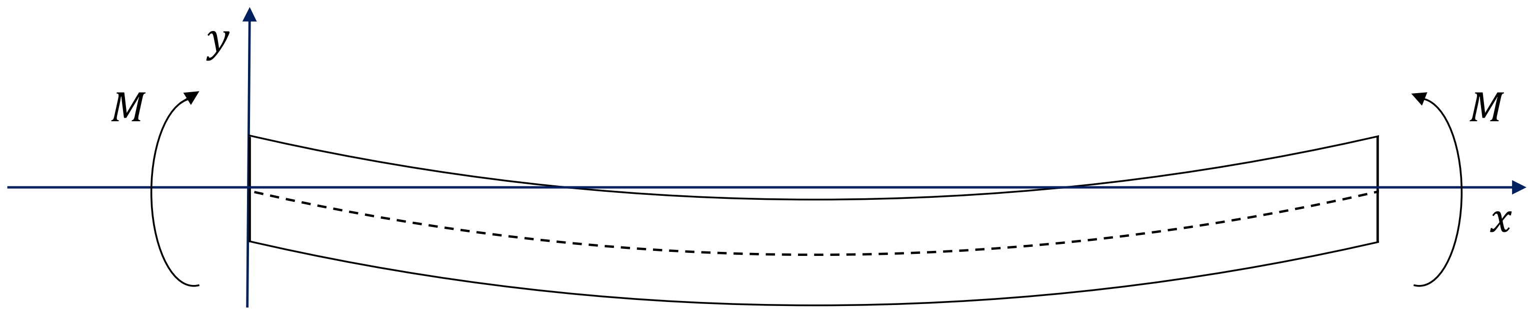
Whereas the design of engineering structures and components is very often dictated by the strength of the materials used and consequently the stresses within the structure, often the limiting factor is the allowable deflection.



This is particularly important for engineering artefacts made from materials of lower stiffness, e.g. aluminium, plastics, composites, etc., but may also be critical for high stiffness structures comprising slender members.

It is therefore important, as part of the design process, to be able to calculate maximum deflections in a structure in addition to the position at which they occur.

Here, following the derivation of the fundamental deflection equation for a beam, a flexible procedure is introduced, called Macaulay's Method, which allows for slopes and deflections to be calculated at any position along a beam span.



In particular, the method allows us to deal with different types of loading, such as point loads, uniformly distributed loads and point bending moments.

Deflection of Beams

Learning Outcomes

1. Know how to derive the differential equation of the elastic line (i.e. deflection curve) of a beam (synthesis);
2. Employ Macaulay's method, also called the method of singularities, to determine bending moment expressions for beams where there are discontinuities in the bending moment distribution arising from discontinuous loading (application);
3. Be able to solve this equation by successive integration in order to yield the slope, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and the deflection, y, of a beam at any position, x, along its span (application);
4. Recognise and use different singularity functions in the bending moment expression, relating to different loading conditions, including point loads, uniformly distributed loads and point bending moments (comprehension);
5. Employ Macaulay's method for statically indeterminate beam problems (application).

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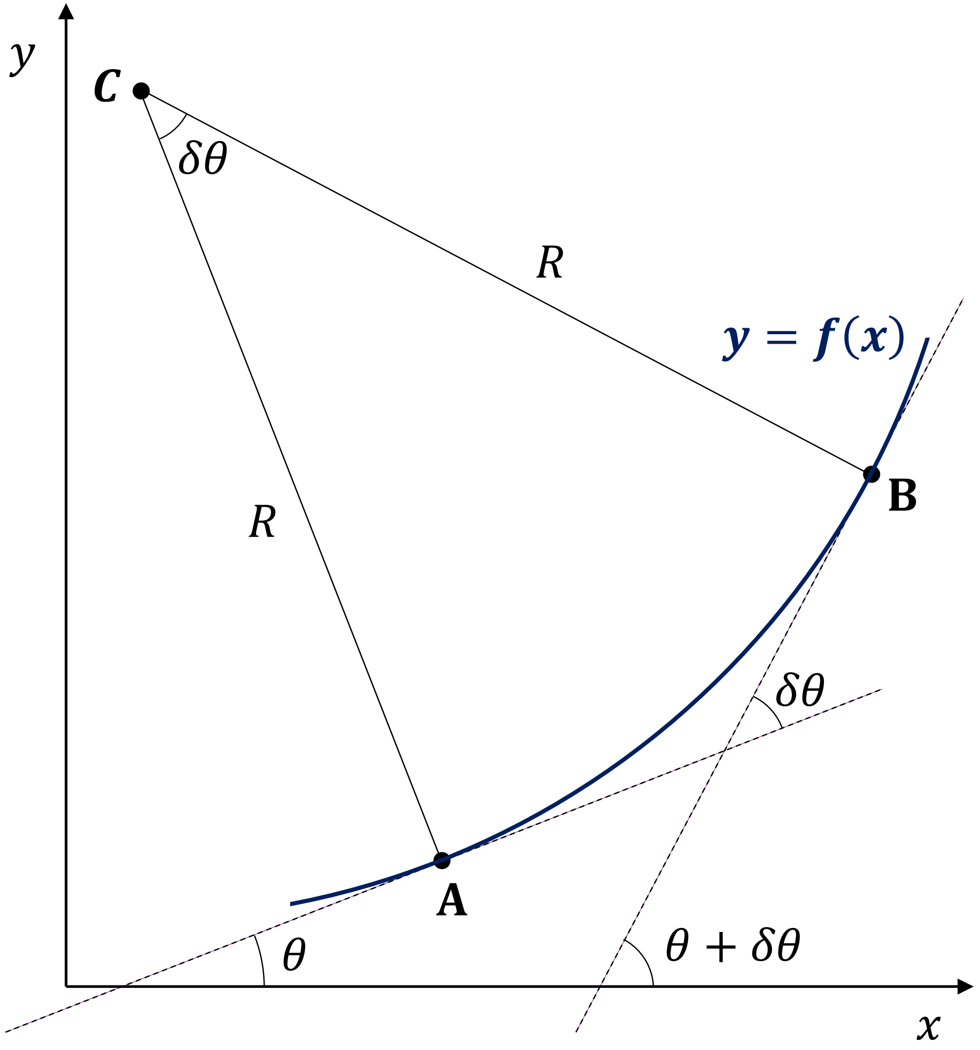
Equation of the Elastic Line

Mathematical Manipulation of a Generic Curve



Gradient at A: $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\mathrm{A}} = \tan\theta$

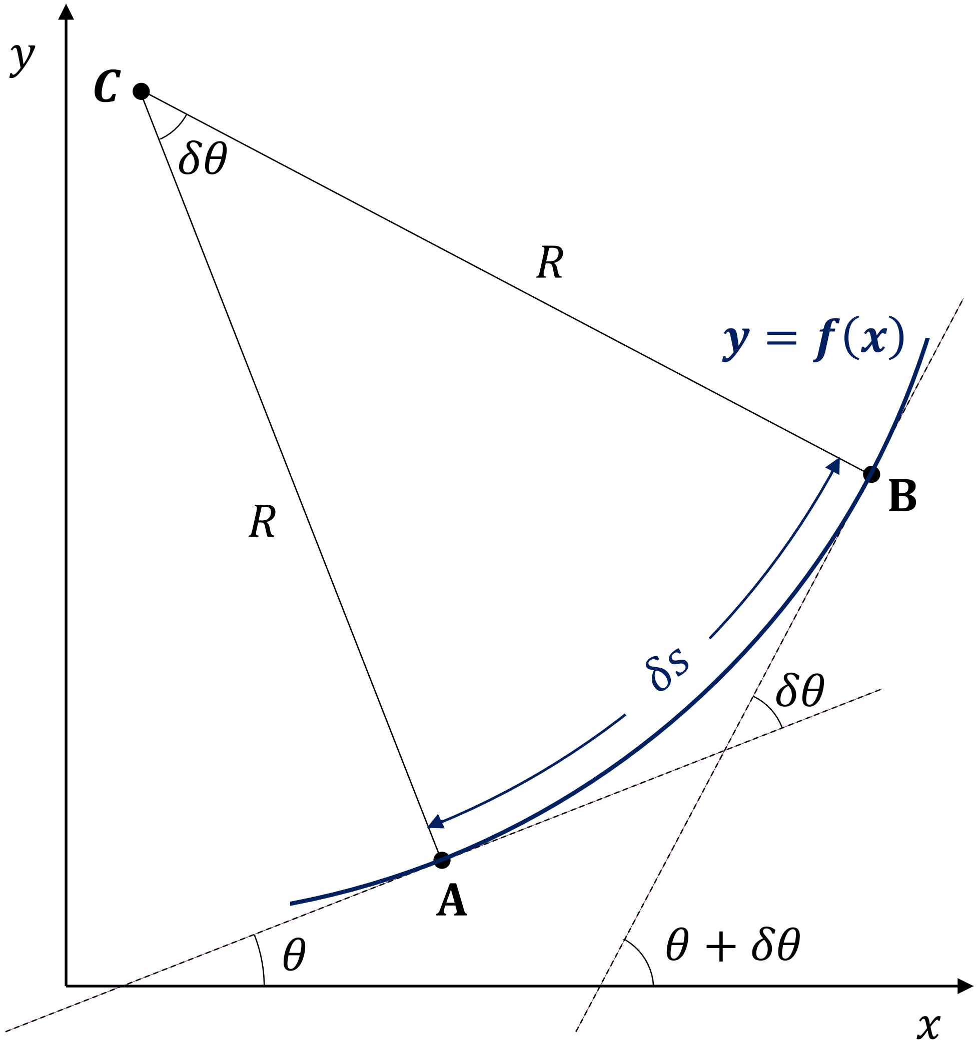
Gradient at B: $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\mathrm{B}} = \tan(\theta + \delta\theta)$

Letting the normal to the curve at points A and B meet at point C, if points A and B are close, the lengths AC and BC are similar. I.e.:

$$\mathrm{A}C \approx \mathrm{B}C\ (= R)$$

Length AB can therefore be thought of as a small arc of a circle of radius, R.

Note that angle $\mathrm{A}C\mathrm{B} = \delta\theta$, since, as the tangent turns through angle $\delta\theta$, so does the normal.



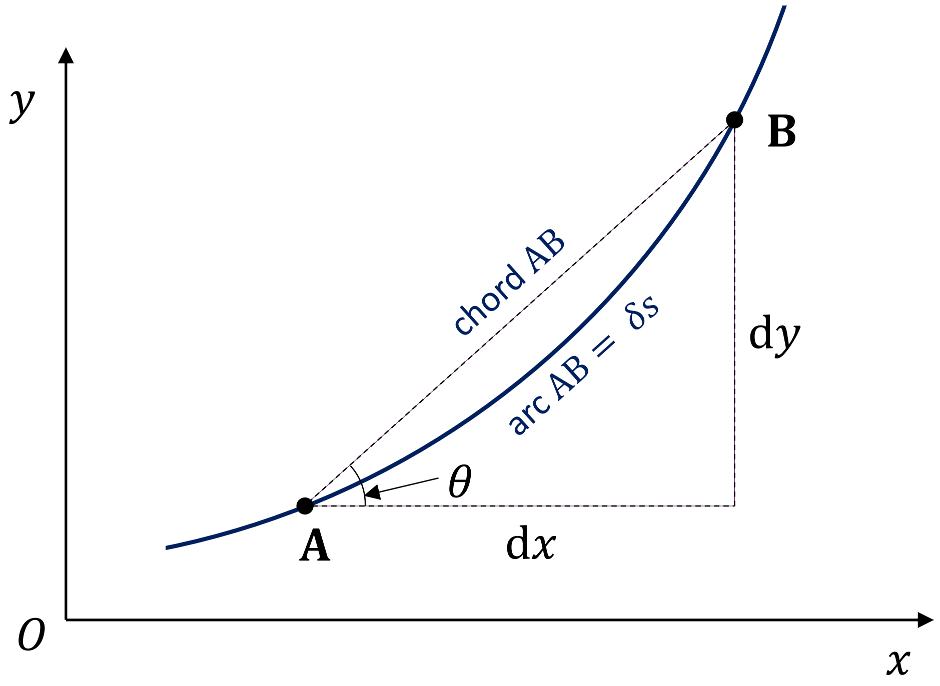
Therefore arc AB $= \delta s = R\delta\theta$

$$\therefore \frac{1}{R} = \frac{\delta\theta}{\delta s}$$

As $\delta s \to 0$ (i.e. as points A and B become closer):

$$\frac{\delta\theta}{\delta s} \to \frac{\mathrm{d}\theta}{\mathrm{d}s}$$

$$\therefore \frac{1}{R} = \frac{\mathrm{d}\theta}{\mathrm{d}s} \qquad (1)$$



Since δs is small, the arc AB $(= \delta s)$ $\approx$ the chord AB

Therefore, when $\delta s \to 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta \qquad (2)$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta \qquad (3)$$

Differentiating equation (2) with respect to s gives: $\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}s}(\tan\theta)$

Multiplying the left−hand side of this equation by $\frac{\mathrm{d}x}{\mathrm{d}x}$, and the right−hand side by $\frac{\mathrm{d}\theta}{\mathrm{d}\theta}$ gives: $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}\theta}(\tan\theta)\frac{\mathrm{d}\theta}{\mathrm{d}s}$

Rearranging this and substituting in equation (3): $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\cos\theta = \sec^2\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}s}$

$$\therefore \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sec^3\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}s} = \left\{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}^{3/2} \qquad (4)$$

where: $\sec^3\theta = (\sec^2\theta)^{3/2} = (1+\tan^2\theta)^{3/2} = \left\{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}^{3/2}$

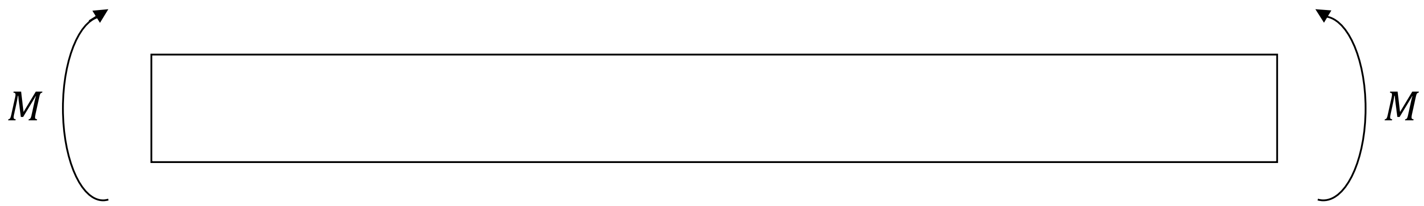
Rearranging equation (4): $\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left\{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}^{3/2}}$

Substituting this into equation (1) gives: $\frac{1}{R} = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left\{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}^{3/2}} \qquad (5)$

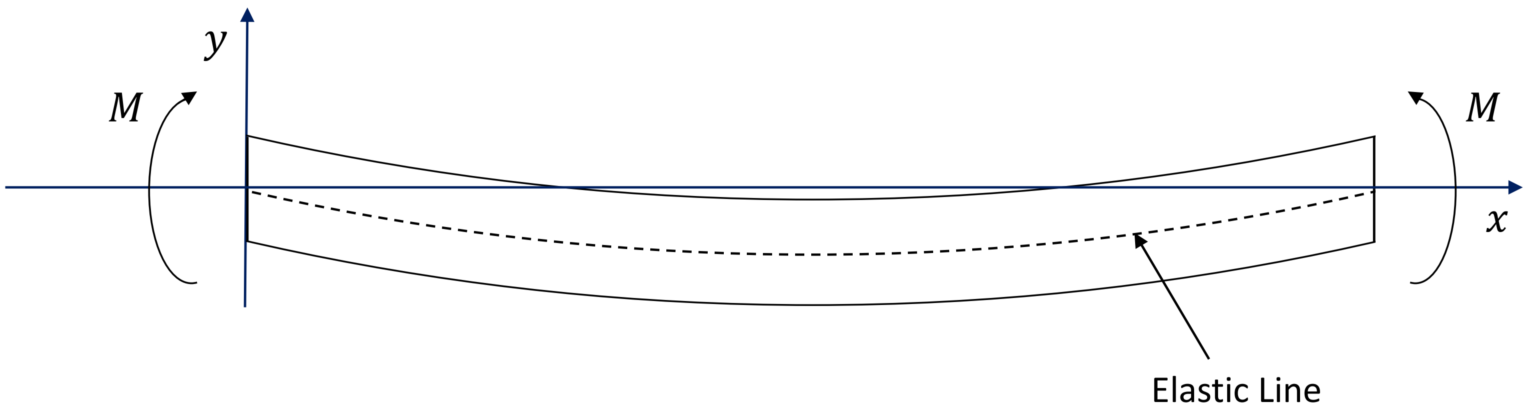
Equation of the Elastic Line

Application to a Beam Under Bending

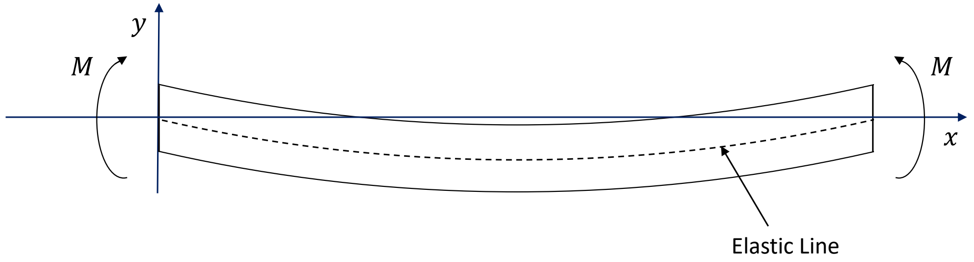
If we take a section of a span of a beam, as shown below, which is under pure bending, i.e. there is a constant bending moment along this section and no shear force.



Under these pure bending conditions, the neutral axis (the axis on which there is zero stress) of the beam, also named the elastic line or deflection curve, is a circular arc with radius of curvature, R, as shown below.



The transverse deflection of the elastic line is given by the co-ordinate, y, of any position along its length, x



The elastic beam bending equation, which can be used to describe the bending moment, M, as a function of the radius of curvature, R, is: $\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$

$$\therefore \frac{1}{R} = \frac{M}{EI}$$

Substituting this into equation (5), which also represents the shape of an arc, gives: $$\frac{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}}{\left\{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}^{3/2}} = \frac{M}{EI} \qquad (7)$$

For small deflections, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is small. Therefore: $\left\{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}^{3/2} \approx 1$

Therefore equation (7) simplifies to: $\boldsymbol{EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = M} \qquad (8)$

This is the is the 2nd order differential equation of the elastic line, relating the deflection, y, to the applied bending moment, M, Young's modulus, E, 2nd moment of area, I, and position along beam span, x.

$$\boldsymbol{EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = M}$$

Successive integration of this equation, with respect to x, will yield the slope, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and the deflection, y, as function of position, x, along the beam:

$$EI\frac{\mathrm{d}y}{\mathrm{d}x} = Mx + \mathrm{A} \qquad\qquad EIy = \frac{Mx^2}{2} + \mathrm{A}x + \mathrm{B}$$

A complication arises where discontinuities in M exist, such as where there are point loads and/or point bending moments or where there is an abrupt change in distributed loading. Various methods have been developed to solve such problems with discontinuities. Here we introduce and develop the procedure called Macaulay's Method, a versatile solution procedure which can handle most discontinuities we are likely to encounter.

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