University of Nottingham
Department of Mechanical Engineering

MM2MS3 - Mechanics of Solids 3
Exercise Sheet 3 - Thick Walled Cylinders

1. Some high pressure hydraulic steel tubing is 4 mm bore, 15 mm outside diameter. If the greatest principal stress is not to exceed one half the yield strength of mild steel, calculate the greatest permissible internal pressure. (assume σ_y = 250 MPa)
[*Ans: 1084 bar*]

Lame's equations:

$$\sigma_r = A - \frac{B}{r^2}$$
$$\sigma_\theta = A + \frac{B}{r^2}$$

for this (pure pressure) case:

$$\sigma_1 = \sigma_{theta} = 125 \text{ MPa}$$

applying BCs:
at $r = R_o$ = 7.5 mm, $\sigma_r = 0$, therefore, $0 = A - \frac{B}{7.5^2}$ or $B = 56.25A$
at $r = R_i$ = 2 mm, $\sigma_r = -p$, therefore, $-p = A - \frac{B}{4} = A - \frac{56.25A}{4}$
which gives $A = \frac{4p}{52.25}$ and $B = \frac{225p}{52.25}$

As the highest hoop stress will occur at the ID ($r = R_i$):

$$125 = \sigma_\theta = \frac{4p}{52.25} + \frac{225p}{52.25 \times 4}$$

or:

$$p = \frac{125}{\left(\frac{4}{52.25} + \frac{225}{52.25 \times 4}\right)}$$

$$\boxed{\boldsymbol{p = 108.4\ \mathrm{MPa}\ (= 1084\ \mathrm{bar})}}$$

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2. A heat exchanger consists of a cylindrical vessel which contains 35 U-shaped tubes 20 mm bore, 30 mm outside diameter. The ends of these tubes are welded into one of the flat end plates. The total length of the tubes within the vessel is 7 m. Calculate the hoop and axial stresses on the inside and outside of the straight parts of these tubes, remote from the bend and the ends, due to pressures of 10 bar in the vessel (i.e. outside the tubes) and 100 bar inside the tubes.
[*Ans: ID: σ_z = 6.2 MPa, σ_θ = 22.4 MPa; OD: σ_z = 6.2 MPa, σ_θ = 13.4 MPa*]

Lame's equations:

$$\sigma_r = A - \frac{B}{r^2}$$
$$\sigma_\theta = A + \frac{B}{r^2}$$

For a cylinder with closed ends (the OD and ID values will be the same):

$$\boxed{\boldsymbol{\sigma_z = \frac{R_i^2 p_i - R_o^2 p_o}{(R_o^2 - R_i^2)} = \frac{10^2 \times 10 - 15^2 \times 1}{15^2 - 10^2} = 6.2\ \mathrm{MPa}}}$$

p_i = 100 bar = 100 x 10^5 Pa = 10 MPa
p_o = 10 bar = 10 x 10^5 Pa = 1 MPa

at r = R_i, $\sigma_r = -p_i = -10 = A - \frac{B}{10^2}$ – (1)
at r = Ro, $\sigma_r = -p_o = -1 = A - \frac{B}{15^2}$ - (2)

subtracting (2) from (1) gives:

$$9 = \frac{B}{10^2} - \frac{B}{15^2}$$

giving

$$B = 1620$$

substitute in for B in (1):

$$A = 16.2 - 10 = 6.2$$

therefore, at the ID:

$$\boxed{\boldsymbol{\sigma_\theta = 6.2 + \frac{1620}{10^2} = 22.4\ \mathrm{MPa}}}$$

and at the ID:

$$\boxed{\boldsymbol{\sigma_\theta = 6.2 + \frac{1620}{15^2} = 13.4\ \mathrm{MPa}}}$$

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3. A thick cylindrical tube, of internal diameter *D* is supported so that there is no longitudinal stress, and is subjected to an internal pressure *p*. If the maximum direct stress is to be limited to a value *Y*, assuming Lame's equations, show that the necessary wall thickness is

$$\left(\left(\frac{Y+p}{Y-p}\right)^{\frac{1}{2}}-1\right)\frac{D}{2}$$

With this wall thickness, show that the increase in outer diameter when the pressure is applied will be

$$\frac{D}{E}\sqrt{Y^2-p^2}$$

Lame's equations:

$$\sigma_r = A - \frac{B}{r^2}$$

$$\sigma_\theta = A + \frac{B}{r^2}$$

$\sigma_\theta$at r = D/2 will be the maximum stress
at r = D/2:
$\sigma_r = -p = A - \frac{4B}{D^2}$ - (1)
also at D/2:
$\sigma_\theta = Y = A + \frac{4B}{D^2}$ - (2)
adding (1) and (2)

$$Y - p = 2A$$

therefore

$$A = \frac{Y-p}{2}$$

subtracting (1) from (2)

$$\frac{8B}{D^2} = Y + p$$

therefore

$$B = \frac{D^2(Y+p)}{8}$$

substituting for A and B into (1):

$$\sigma_r = \frac{Y-p}{2} - \frac{D^2(Y+p)}{8r^2}$$

If we let the thickness of the wall be *T*, at r = *T* + *D*/2 (the outer surface), $\sigma_r = 0$:

$$0 = \frac{Y-p}{2} - \frac{D^2(Y+p)}{8\left(T+\frac{D}{2}\right)^2}$$

rearranging

$$4(Y-p)\left(T+\frac{D}{2}\right)^2 = D^2(Y+p)$$

$$\left(T+\frac{D}{2}\right)^2=\frac{Y+p}{Y-p}\left(\frac{D}{2}\right)^2$$

$$\boxed{\boldsymbol{T=\left(\left(\frac{Y+p}{Y-p}\right)^{\frac{1}{2}}-1\right)\frac{D}{2}}}$$

At radius *T*+*D*/2, $\sigma_r=0$ and $\sigma_z=0$ and

$$\sigma_\theta=\frac{Y-p}{2}+\frac{D^2(Y+p)}{8\left(T+\frac{D}{2}\right)^2}$$

recalling:

$$\varepsilon_\theta=\frac{1}{E}(\sigma_\theta-\nu(\sigma_r+\sigma_z)=\frac{\sigma_\theta}{E}=\frac{u}{r}$$

where *u* is the increase in radius, therefore at the maximum radius (*T*+*D*/2)

$$u=\frac{\sigma_{theta}r}{E}=\frac{Y-p}{2}+\frac{D^2(Y+p)}{8\left(T+\frac{D}{2}\right)^2}\times\frac{\left(\left(\frac{Y+p}{Y-p}\right)^{\frac{1}{2}}\right)\frac{D}{2}}{E}$$

$$=\frac{(Y-p)D}{2E}\left(\frac{Y+p}{Y-p}\right)^{\frac{1}{2}}$$

$$=\frac{D}{2E}(Y^2-p^2)^{\frac{1}{2}}$$

The increase in diameter is 2*u*, therefore

$$\boxed{\boldsymbol{2u=\frac{D}{E}\sqrt{Y^2-p^2}}}$$

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4. A steel shaft 4 cm dia is encased in a bronze sleeve 6 cm outside dia, which is forced into position. Before forcing on the inside diameter is .005 cm smaller than the diameter of the shaft. Find the radial pressure between the shaft and sleeve, the maximum hoop stress in the sleeve, and the change of outside diameter of the sleeve. For steel E= 200 GPa; v= 0.3 and for bronze E= 120 GPa; v= 0.34.
[*Ans: p = 44.64 MPa, σ_θ = 116.1 MPa, ΔD = 0.00357 mm*]

Inner radius of bronze sleeve increases by u_1
Outer radius of steel shaft increases by u_2
$u_1 + u_2 = 0.025\ \text{mm}$ - (a)
On steel shaft surface,

$$\sigma_\theta = \sigma_r = -p$$

At bronze sleeve bore
$\sigma_r = -p = A - \frac{B}{20^2}$ - (1)

At bronze sleeve outer
$\sigma_r = 0 = A - \frac{B}{30^2}$ - (2)

subtracting (2) from (1) gives:

$$p = B\left(\frac{1}{20^2} - \frac{1}{30^2}\right)$$

which means:

$$B = 720p$$

substitute for B in (2) gives:

$$A = \frac{720p}{900}$$

Therefore, for bronze:

$$\sigma_r = \frac{720p}{900} - \frac{720p}{r^2}$$
$$\sigma_\theta = \frac{720p}{900} + \frac{720p}{r^2}$$

for the steel shaft

$$-\frac{u_2}{20} = \frac{1}{200000}\left(-p - 0.3(-p + 0)\right) = -\frac{p(1 - 0.3)}{200000}$$

for the bronze sleeve

$$\frac{u_1}{20} = \frac{1}{120000}\left(\left(\frac{720p}{900} + \frac{720p}{20^2}\right) - 0.34\left(\left(\frac{720p}{900} - \frac{720p}{20^2}\right) + 0\right)\right)$$
$$= \frac{720p}{900 \times 120000}\left(\frac{13}{4} + \frac{5}{4} \times 0.34\right)$$

using (a):

$$\frac{0.025}{20} = \left[\frac{720(13 + 5 \times 0.34)}{900 \times 120000 \times 4} + \frac{(1 - 0.3)}{200000}\right] p$$

therefore:

$$\boxed{\boldsymbol{p = 44.64\ \mathrm{MPa}}}$$

the maximum hoop stress in the bronze sleeve will be at the bore:

$$\boxed{\boldsymbol{\sigma_\theta = \frac{720p}{900} + \frac{720p}{r^2} = 44.64\left(\frac{720}{900} + \frac{720}{20^2}\right) = 116.1\ \mathrm{MPa}}}$$

The change in the outer diameter of the sleeve is $2u_o$ (at the OD, $\sigma_r = \sigma_z = 0$)

$$\frac{u_o}{30} = \frac{1}{120000}\left(\left(\frac{720p}{900} + \frac{720p}{30^2}\right) - 0.34(0 + 0)\right)$$

means

$$u_o = \frac{30 \times 2 \times \left(\frac{720}{900}\right) \times 44.64}{120000} = 0.017856\ \mathrm{mm}$$

therefore, the change in diameter is:

$$\boxed{\boldsymbol{\Delta D = 2u_o = 0.0357\ \mathrm{mm}}}$$

5. A circular saw, 5 mm thick and 900 mm dia has a bore of 100 mm. The steel, of which the saw is made, has a density of 7800 kg m^{-3}, and v= 0.3. Find the maximum speed permitted if the hoop stress is restricted to 240 MPa. What is then the maximum value of the radial stress?
   [*Ans: 4093rpm, σ_r = 94.56 MPa*]

   From the lecture notes:
$$\sigma_r = A - \frac{B}{r^2} - \frac{3+\nu}{8}\rho\omega^2 r^2$$
$$\sigma_\theta = A + \frac{B}{r^2} - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

   at r = 0.05 (ID), $\sigma_r = 0$ therefore:
$$0 = A - \frac{B}{0.05^2} - \frac{3+0.3}{8}7800\omega^2 \times 0.05^2$$
   $0 = A - 400B - 8.04375\omega^2$ - (1)

   at r = 0.45 (OD), $\sigma_r = 0$ therefore:
$$0 = A - \frac{B}{0.45^2} - \frac{3+0.3}{8}7800\omega^2 \times 0.45^2$$
   $0 = A - 4.93827B - 651.544\omega^2$ - (2)

   substituting (2) from (1):
$$0 = 395.0617B - 643.5\omega^2$$
   therefore:
$$B = 1.6289\omega^2$$
   and from (1)
$$A = 659.59\omega^2$$

   therefore:
$$\sigma_\theta = 659.59\omega^2 + \frac{1.6289\omega^2}{r^2} - \frac{1+3\times 0.3}{8}7800\omega^2 r^2$$
$$\sigma_\theta = \omega^2\left(659.59 + \frac{1.6289}{r^2} - 1852.5r^2\right)$$
$$\frac{d\sigma_\theta}{dr} = \omega^2\left(-2\times\frac{1.6289}{r^3} - 2\times 1852.5r\right)$$

   no mathematical stationary point as rearranging gives r^4 = -ve

   Evaluate at ID and OD to determine max value of σ_θ

   At r= 0.05
$$\sigma_\theta = \omega^2\left(659.59 + \frac{1.6289}{0.05^2} - 1852.5\times 0.05^2\right) = 1306.52\omega^2$$

At r= 0.45

$$\sigma_\theta = \omega^2\left(659.59 + \frac{1.6289}{0.45^2} - 1852.5 \times 0.45^2\right) = 292.5\omega^2$$

therefore the maximum value of σ_θ is at the ID

Hoop stress is limited to 240 MPa, therefore at ID

$$240 \times 10^6 = 1306.52\omega^2$$
$$\omega^2 = 183694.088$$
$$\boldsymbol{\omega = 428.6\ \mathrm{rad/s}}$$

$$\boxed{\mathbf{428.6\ rad/s} = \frac{\boldsymbol{\omega}}{\mathbf{2}\boldsymbol{\pi}} \times \mathbf{60} = \mathbf{4092.8\ rpm}}$$

The radial stress can be determined using

$$\sigma_r = \omega^2\left(659.59 - \frac{1.6289}{r^2} - \frac{3+0.3}{8}7800r^2\right)$$

and the maximum can be found where $\frac{d\sigma_r}{dr} = 0$

$$\frac{d\sigma_r}{dr} = \omega^2\left(2 \times \frac{1.6289}{r^3} - 2 \times 3217.5r\right)$$

i.e. when

$$r^4 = \frac{3.2578}{6435} = 5.06 \times 10^{-4}$$
$$r = 0.15\ \mathrm{m}$$

and the value of hoop stress is

$$\boxed{\begin{aligned}\boldsymbol{\sigma_r} &= \mathbf{183694.088} \times \left(\mathbf{659.59} - \frac{\mathbf{1.6289}}{\mathbf{0.15^2}} - \frac{\mathbf{3+0.3}}{\mathbf{8}}\mathbf{7800} \times \mathbf{0.15^2}\right) = \mathbf{94.56 \times 10^6\ Pa} \\ &= \mathbf{94.56\ MPa}\end{aligned}}$$