Elastic-Plastic Deformations

Lecture 3 – Torsion of Shafts

Department of Mechanical, Materials & Manufacturing Engineering

MMME2053 – Mechanics of Solids

University of Nottingham
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Elastic-Plastic Deformations

Learning Outcomes

1. Know the shapes of uniaxial stress-strain curves and the elastic-perfectly-plastic approximation (knowledge);

2. Know the kinematic and isotropic material behaviour models used to represent cyclic loading behaviour (knowledge);

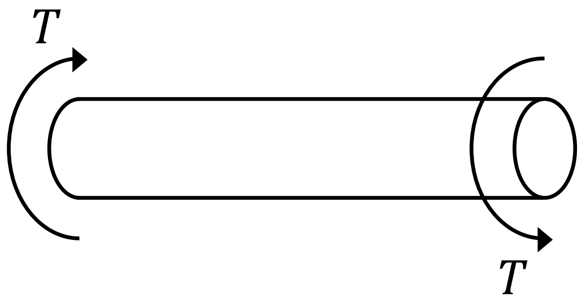
3. Understand elastic-plastic bending of beams (comprehension) and be able to use equilibrium, compatibility and σ-ε behaviour to solve these types of problems for deformation and stress state (application);

4. Understand elastic-plastic torsion of shafts (comprehension) and be able to use equilibrium, compatibility and τ-γ behaviour to solve these types of problems for deformation and stress state (application);

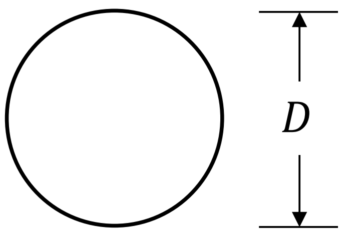
5. Be able to determine residual deformations and residual stresses in beams under bending and shafts under torsion (application).

Elastic-Plastic Torsion of Shafts

Part a of the figure below shows a shaft which is subjected to a torque, T. The circular cross-sectional area of the shaft is shown in part b of the figure.



(a)



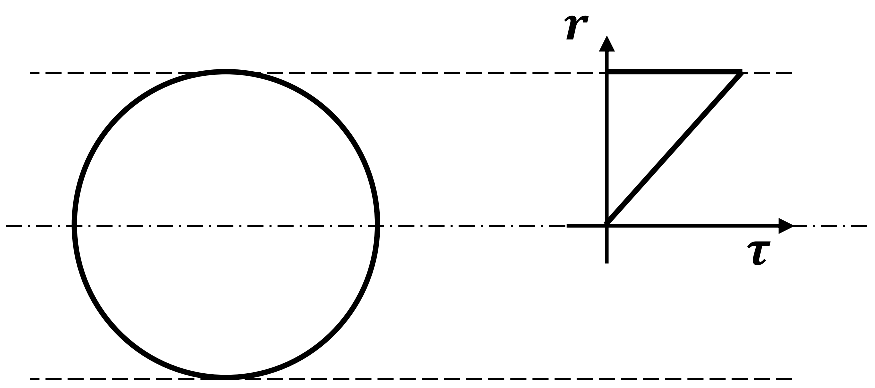
(b)

Stress Distribution

Assuming that the magnitude of the torque is not high enough to cause plasticity (yielding) within the shaft, the elastic shaft torsion equation can be used to describe the shear stress distribution, as a function of r (radius of the shaft), as:

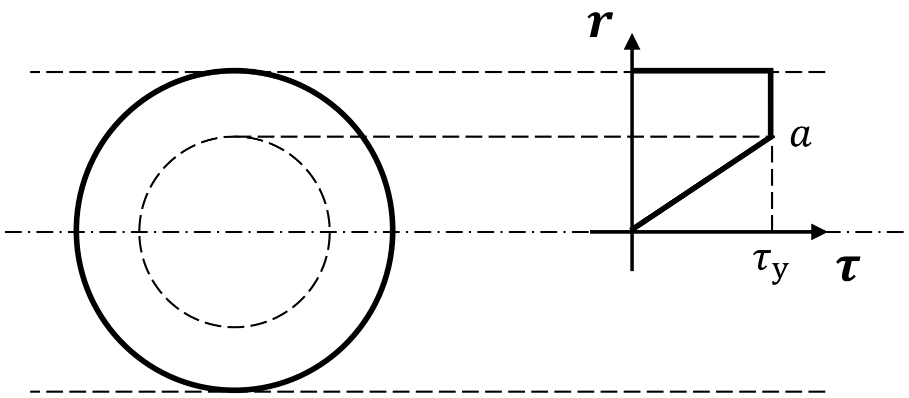
$$\tau = \frac{Tr}{J}$$

In this elastic case then, the shear stress distribution throughout the cross-section is linear, as shown in the figure below.



If the torque is increased to a magnitude which is just high enough to induce plasticity in the shaft, this plasticity will occur at the positions furthest away from the centre of the cross-section, i.e., at the positions of maximum r magnitude (circumference of the cross-section).

As the torque is further increased, the plasticity spreads from the outer edge, to further within the cross-section (towards the centre) as shown in the figure below.



Torque Equilibrium

The applied torque, T, cab be related to the to the position as which yielding occurs, a, as:

$$T = \int_A r\tau dA$$

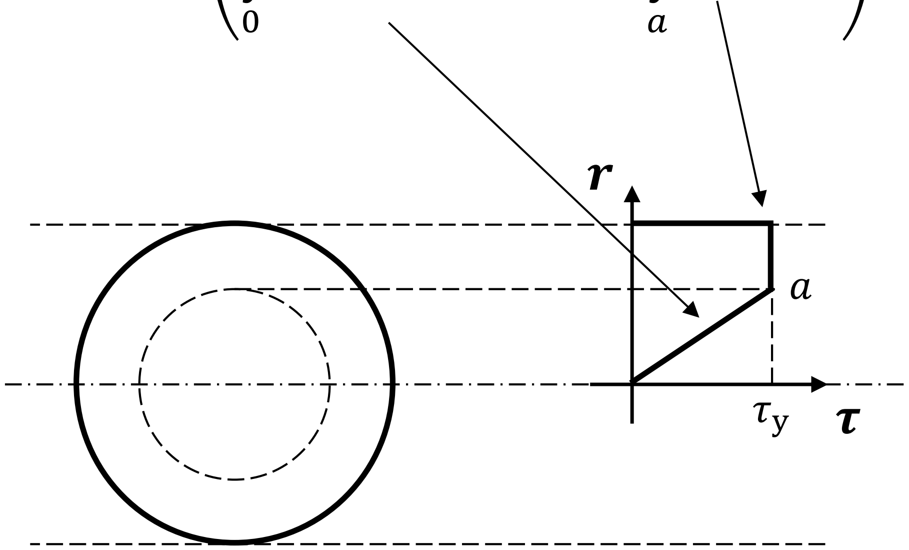
$$= 2\pi \int_0^R \tau r^2 \mathrm{d}r$$

Where R is the outer radius of the cross-section, and:

$$\mathrm{d}A = 2\pi r \mathrm{d}r$$

Substituting the expressions for shear stress for each of the elastic ($0 > r > a$) and plastic ($a > r > R$) regions into this equation gives:

$$T = 2\pi\left(\int_0^a \left(\tau_y \frac{r}{a}\right) r^2 dr + 2\pi \int_a^R \tau_y r^2 dr\right)$$



$$\therefore \mathrm{T} = 2\pi\tau_y\left(\frac{R^3}{3} - \frac{a^3}{12}\right)$$

Compatibility Requirement

In order for the twist of the shaft, θ, due to the applied torque, T, to be calculated, both compatibility and a shear stress-strain relationship are required.

As the region of the cross-section between $0 < r < a$ has only behaved elastically, the elastic shaft torsion equation can be applied. I.e.:

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$$

$$\therefore \frac{r\theta}{L} = \gamma \left(= \frac{\tau}{G}\right)$$

$$\therefore \theta = \frac{\gamma L}{r}$$

As the shaft behaves as one body, the entirety of the shaft (both the elastic and plastic regions) must share this common twist, θ.

Stress-Strain Relationship

Again, as the region of the cross-section between $0 < r < a$ has only behaved elastically, Hooke's law applies here, and so:

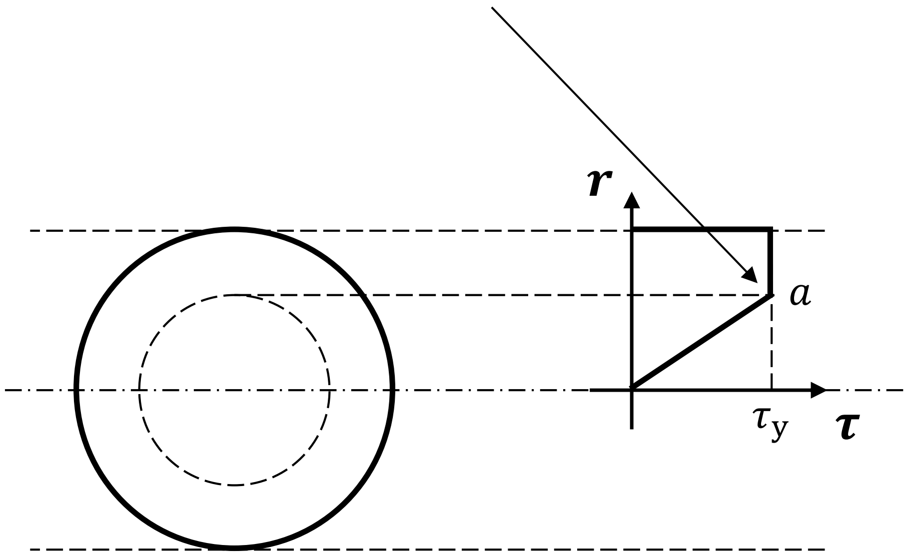
$$\tau = G\gamma$$

Substituting the above equation into the expression for θ on the previous slide:

$$\theta = \frac{\tau L}{Gr}$$

Substituting values for r and τ, from within the elastic region, into this equation, allows for θ to be calculated.

A convenient value of r to use is a, for which the corresponding value of τ is τ_{y}.



Therefore:

$$\theta = \frac{\tau_y L}{Ga}$$

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Unloading

As plasticity has occurred within the shaft during loading, on unloading the shear stress distribution and twist will not return to zero. Rather a residual shear stress distribution and residual twist will remain.

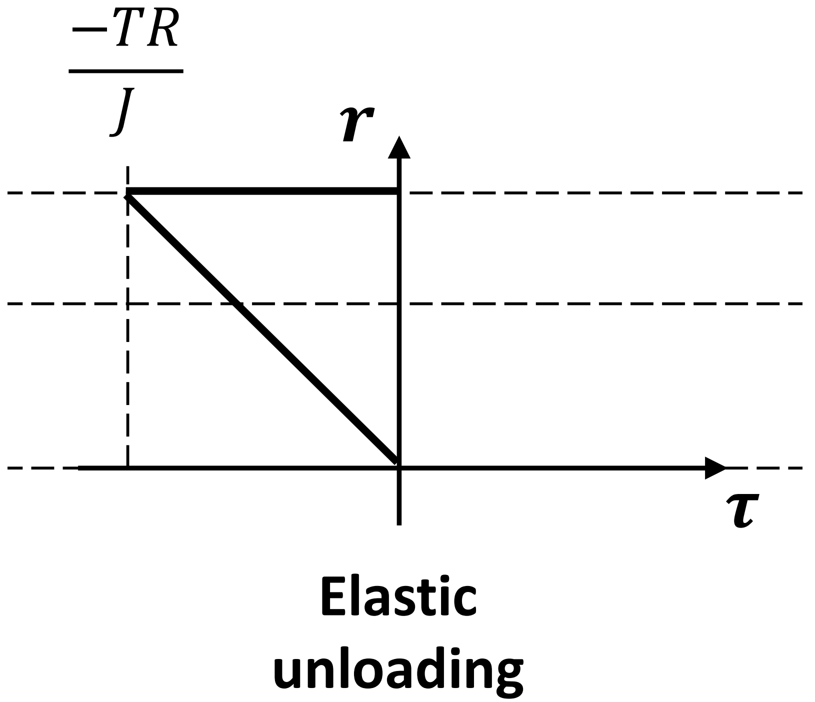
Assuming that the shear stress change which occurs on unloading is purely elastic, then the shear stress change can be calculated from the elastic shaft torsion equation as:

$$\Delta\tau = \frac{\Delta T r}{J}$$

The maximum shear stress change, $\Delta\tau_{\max}$, will therefore occur at $r_{\max}$, and so:

$$\Delta\tau_{\max} = \frac{\Delta T \times r_{\max}}{J} = \frac{-TR}{J}$$

As the unloading behaviour has been assumed to be elastic, the shear stress variation will be linear up to this maximum value.



This residual shear stress distribution will be accompanied by a residual shaft twist, which can be calculated by substituting unloaded beam values for r and τ into the expression derived for θ, which again relate to a position that has only been subjected to elastic behaviour.