MTHS2007 Advanced Mathematics and Statistics for Mechanical Engineers

Chapter 6: partial differential equations

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UNITED KINGDOM · CHINA · MALAYSIA

6 Introduction to Partial Differential Equations (PDEs)

PDEs are differential equations in which there is more than one independent variable. They arise in the modelling of a wide range of physical phenomena including electromagnetism, fluid dynamics, elasticity and heat conduction, so their solution underlies much of modern Engineering Science.

In this module we will concentrate on use of an important technique for the solution of linear PDEs (PDEs in which the dependent variable and its derivatives appear in a linear fashion).

6.1 The three most important PDEs

Laplace's equation Comes up in solving problems in electrostatics and fluid motion, and is written here

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0, \qquad \text{later: add bcs}$$

for a 2D problem: find $\varphi(x, y)$ for (x, y) in some region in the plane.

The wave equation describes waves on a string:

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \frac{\partial^2 \varphi}{\partial x^2} \qquad \text{later: add bcs } \textit{and} \text{ ics}$$

Here the spatial dimension is 1, but there are higher-dimensional versions.

The heat/diffusion equation describes diffusion or heat transport

$$\frac{\partial \varphi}{\partial t} = D \frac{\partial^2 \varphi}{\partial x^2} \qquad \text{later: add bcs } \textit{and} \text{ ics}$$

Here again there are higher-dimensional versions.

The principle of superposition

The examples give so far are **linear** and **homogeneous** and so are subject to the **principle of superposition**.

- This means that if φ_1 and φ_2 are any two solutions then

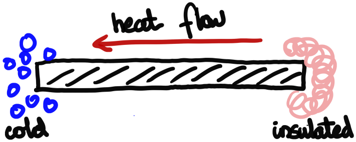
$$\varphi = a\varphi_1 + b\varphi_2.$$

  is also a solution, for any constants a and b.

- We can apply the superposition principle to build solutions of problems subject to homogeneous boundary conditions, such as

$$\varphi|_{x=0} = 0 \qquad \left.\frac{\partial \varphi}{\partial x}\right|_{x=L} = 0.$$

Here is an example of a problem we hope to be able to solve at the end of this section:



Problem Solve the heat equation

$$\frac{\partial \varphi}{\partial t} = D \frac{\partial^2 \varphi}{\partial x^2} \qquad \text{for } 0 < x < L \text{ and } t > 0,$$

subject to the *boundary conditions*

$$\varphi(0, t) = 0 \quad \text{and} \quad \varphi_x(L, t) = 0, \qquad \text{for } t > 0$$

(where $\varphi_x \equiv \partial\varphi/\partial x$) and the *initial condition*

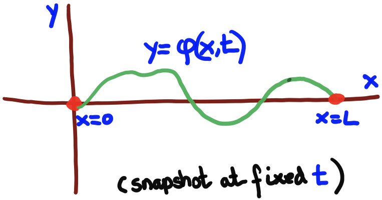
$$\varphi(x, 0) = \sin\left(\frac{\pi x}{2L}\right), \qquad \text{for } 0 < x < L.$$

The Wave Equation

Force (vertical component) on a small segment of string at height $y = \varphi(x,t)$ is

$$\text{force} \propto \frac{\partial^2 \varphi}{\partial x^2}$$

(assuming small displacements).

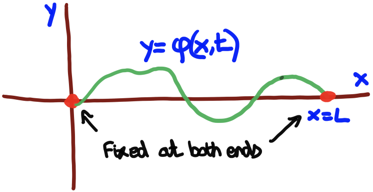


Newton's law then leads to the equation

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \frac{\partial^2 \varphi}{\partial x^2},$$

where

- $\varphi(x,t) =$ (small) lateral displacement of the string at time t and position x.
- c is a constant depending on the density and tension of the string - the **wave speed**



For an explicit solution of the equation we need both

- *Boundary conditions*: telling us how the string is attached at the ends, such as

$$\varphi(0,t) = \varphi(L,t) = 0.$$

- *Initial conditions*: telling is the initial position and velocity for each piece of the string, which amounts to

$$\varphi(x,0) \qquad \text{and} \qquad \varphi_t(x,0)$$

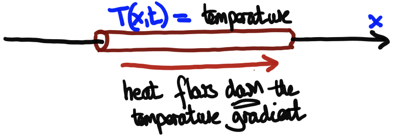
being specified (recall $\varphi_t \equiv \partial\varphi/\partial t$).

The Diffusion Equation

The **diffusion equation** is also known as the **heat equation**.

Fourier's law (or *Fick's Law* for concentration diffusion) says

$$\text{heat flow} \propto -\frac{\partial T}{\partial x}.$$



The rate of change of temperature of a small segment is determined by the net flow of heat into the segment:

$$\Rightarrow \qquad \frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2},$$

where

- $T(x,t)$ = temperature at time t and position x.
- D is a constant depending on crossection and material properties of rod - the **thermal diffusivity** or **diffusion constant**.

For an explicit solution of the equation we need

- *Boundary conditions*: telling us what the the physical set-up of the rod is at each end. For example, if both ends are insulated (no heat flow), then by Fourier's law

  $$T_x(0,t) = T_x(L,t) = 0$$

  (recall $T_x \equiv \partial T/\partial x$). Alternatively, the temperature at one or both ends might be fixed, as in

  $$T(0,t) = T(L,t) = 0.$$

- *Initial conditions* tell us the initial temperature profile of the rod:

  $$T(x,0) = \text{given function.}$$

  Note that the equation is first-order in time, so we don't need $T_t(x,0)$.

Laplace's equation

This comes up in a wide range of applications, such as **fluid mechanics** and **electrostatics** but a convenient motivation for us comes from steady-state heat transfer.

The diffusion equation in two dimensions (for example, for the temperature of a metal plate) is

$$\frac{\partial T}{\partial t} = D\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right).$$

For steady heat flow, $\partial T/\partial t = 0$, so

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,$$

where now we regard $T(x, y)$ as being a function only of the two Cartesian coordinates (x, y).

For an explicit solution of the equation we need boundary conditions only - but along the 1D boundary of the 2D plate. Consider the example of a **rectangular plate**:

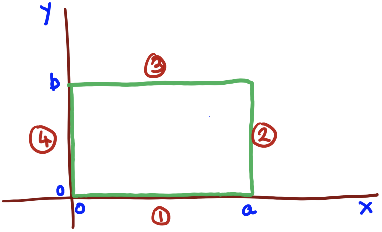
Boundary conditions specifying T along the boundary amounts to imposing

① $T(x,0) = f(x)$ for $0 < x < a$

② $T(a,y) = g(y)$ for $0 < y < b$

③ $T(x,b) = h(x)$ for $0 < x < a$

④ $T(0,y) = p(y)$ for $0 < y < b$,



where f, g, h and p are all functions given to us by the person whose job is to construct the physical model.

6.2 Solution by separation of variables

In the case of a PDE with two independent variables (x, t) or (x, y), we can *separate variables* by looking for solutions of the form

$$\varphi(x, t) = X(x)T(t) \quad \text{or} \quad \varphi(x, y) = X(x)Y(y). \tag{1}$$

To understand why this is useful, consider the PDE

$$\frac{\partial^2 \varphi}{\partial x \partial t} + \varphi \sin x = 0.$$

Substituting in (1) gives

$$X'(x)T'(t) + X(x)T(t)\sin x = 0.$$

Rearrange this so that all terms depending on t are on the LHS, and all terms depending on x on the RHS,

$$\frac{T'(t)}{T(t)} = -\frac{X(x)}{X'(x)}\sin x.$$

The equation

$$\frac{T'(t)}{T(t)} = -\frac{X(x)}{X'(x)} \sin x.$$

must be true for all values of t and x. This means that the LHS and the RHS must be (the same) constant.

Hence

$$\frac{T'(t)}{T(t)} = C \quad \text{and} \quad -\frac{X(x)}{X'(x)} \sin x = C,$$

where the constant C is as yet undetermined - it is called a *separation constant*.

The separation procedure has turned the PDE into two ODEs

$$T' = CT \quad \text{and} \quad X' + \frac{\sin x}{C} X = 0.$$

The general solution of

$$T' = CT \qquad \text{is} \qquad T(t) = Ae^{Ct},$$

where A is an arbitrary constant. The remaining equation is separable:

$$\begin{aligned} X' + \frac{\sin x}{C} X = 0 \quad &\Rightarrow \quad \frac{X'}{X} = -\frac{1}{C}\sin x \\ &\Rightarrow \quad \ln(X) = \int -\frac{1}{C}\sin x \mathrm{d}x \\ &\qquad\qquad\quad = \frac{1}{C}\cos x + \text{const} \\ &\Rightarrow \quad X(x) = Be^{\frac{1}{C}\cos x}. \end{aligned}$$

Then

$$\varphi(x,t) = X(x)T(t) = A'e^{Ct + \frac{1}{C}\cos x},$$

where A, B, $A' = AB$ and C are constants.

6.3 Separable Solution of the Wave equation

Suppose that we want to find a solution of the wave equation

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \frac{\partial^2 \varphi}{\partial x^2} \qquad \text{for } 0 < x < L \tag{2}$$

and subject to the *homogeneous boundary conditions*

$$\varphi(0, t) = 0 \qquad \text{and} \qquad \varphi(L, t) = 0. \tag{3}$$

Let's look for a solution of the form

$$\varphi(x, t) = X(x)T(t). \tag{4}$$

Then we can substitute (4) into (2) to obtain

$$X(x)T''(t) = c^2 X''(x)T(t) \Rightarrow \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}.$$

Observe that $T''/(c^2 T)$ is a function of t only while X''/X is a function of x only. *It follows that they must both be equal to the same constant.*

Denote the separation constant by

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

Note that λ is arbitrary so the choice of sign here is optional - it is chosen like this to make the following calculation neater. Then

$$X''(x) = -\lambda X(x) \quad \Rightarrow \quad X''(x) + \lambda X(x) = 0$$

and

$$T''(t) = -c^2 \lambda T(t) \quad \Rightarrow \quad T''(t) + \lambda' T(t) = 0,$$

where

$$\lambda' = c^2 \lambda.$$

The values of λ must now be chosen so that the boundary conditions can be satisfied and/or the time dependence is physically reasonable.

Using boundary conditions to fix the separation constant

Let's start by finding $X(x)$. The equation

$$X''(x) + \lambda X(x) = 0$$

leads to the auxiliary equation (on substituting $X = e^{mx}$)

$$m^2 + \lambda = 0.$$

There are three possibility here:

(i) **Case** $\lambda > 0$. Here the solutions for X are oscillatory in x. Let $\lambda = k^2 \Rightarrow m = \pm ik$. Then

$$X(x) = A\cos kx + B\sin kx.$$

For appropriate valies of k, we can satisfy the boundary conditions this way - we will return to this case!

Using boundary conditions to fix the separation constant

(ii) **Case** $\lambda = 0$. Here the auxiliary equation has a repeated root $m = 0$ and the general solution for $X(x)$ is

$$X(x) = A + Bx.$$

Recall that the boundary conditions in this example are

$$\text{(a)} \quad \varphi(0,t) = 0 \qquad \text{and} \qquad \text{(b)} \quad \varphi(L,t) = 0.$$

The condition on $x = 0$ gives

$$\varphi(0,t) = X(0)T(t) = 0 \quad \text{(for all } t > 0\text{)} \qquad \Rightarrow X(0) = 0 = A.$$

The condition on $x = L$ then gives

$$X(L) = 0 = BL \quad \Rightarrow B = 0 \quad \Rightarrow X(x) = 0.$$

A solution that vanishes for all x is of no interest - it is a *trivial solution*.

(iii) **Case** $\lambda < 0$. Let $\lambda = -\kappa^2 \Rightarrow m = \pm\kappa$. Then

$$X(x) = Ae^{\kappa x} + Be^{-\kappa x}.$$

Now the boundary conditions give

$$X(0) = 0 = A + B \qquad \Rightarrow B = -A$$

and

$$X(L) = 0 = Ae^{\kappa L} + Be^{-\kappa L} = A(e^{\kappa L} - e^{-\kappa L}) \qquad \Rightarrow A = 0 = B.$$

Here again there is only the *trivial solution* $X(x) = 0$, which is of no interest.

Using boundary conditions to fix the separation constant

Let us return to **Case** (i), $\lambda = k^2 > 0$. Here we found the general solution

$$X(x) = A\cos kx + B\sin kx.$$

The condition on $x = 0$ gives

$$X(0) = 0 = A.$$

So we now know any such solution must be of the form

$$X(x) = B\sin kx.$$

Next, the condition on $x = L$ gives

$$X(L) = 0 = B\sin kL.$$

Now, although $B = 0$ satisfies this condition, the resulting solution is the *trivial* one, $X(x) = 0$, which is of no interest.

Using boundary conditions to fix the separation constant

For a *nontrivial* solution we need

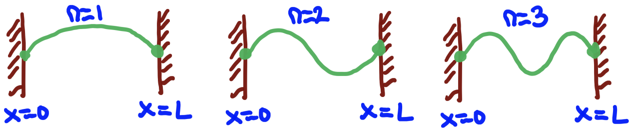
$$\sin kL = 0 \quad \Rightarrow \quad kL = n\pi, \quad \text{where } n = 1, 2, 3, \cdots$$

$$\Rightarrow \quad k = \frac{n\pi}{L} = "k_n".$$

The resulting solutions

$$X_n(x) = B \sin k_n x = B \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, 3, \cdots$$

are examples of *standing waves*:



- We don't count $n = 0$ because $k = 0$ gives us the trivial solution.
- We don't count $n < 0$ because these give the same solutions again, with a minus sign.

Time dependence of standing modes

Given

$$\lambda = k^2 \qquad \rightarrow \qquad \lambda = k_n^2 = \left(\frac{n\pi}{L}\right)^2,$$

we next solve

$$T''(t) + c^2\lambda T(t) = 0 \quad \Rightarrow \quad \begin{aligned} T(t) &= C\cos(ck_n t) + D\sin(ck_n t) \\ &= C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right) \\ &\equiv T_n(t). \end{aligned}$$

The combined solution of the wave equation can be written

$$\varphi_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{L}\right)\left[C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right)\right]$$

(we can drop B since there are already enough arbitrary constants in the brackets).

Matching initial conditions using Fourier series

The solutions

$$\varphi_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right)\right]$$

match the boundary conditions but not necessarily any imposed initial conditions (unless we're very lucky). We now take advantage of the **principle of superposition** to write a more general solution

$$\varphi(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[C_n\cos\left(\frac{n\pi ct}{L}\right) + D_n\sin\left(\frac{n\pi ct}{L}\right)\right],$$

which still satisfies the boundary conditions $\varphi(0,t) = 0 = \varphi(L,t)$. The initial form

$$\varphi(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$

looks like a **Fourier** (sine) series!

Matching initial conditions using Fourier series

The general solution looks like a Fourier (sine) series

$$\varphi(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

with time-dependent Fourier coefficients

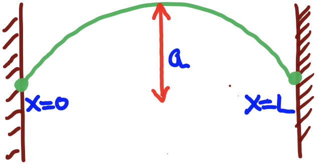
$$B_n(t) = C_n \cos\left(\frac{n\pi ct}{L}\right) + D_n \sin\left(\frac{n\pi ct}{L}\right).$$

We can find the remaining undetermined coefficients by comparing initial conditions

$$\begin{aligned}
\varphi(x,0) &= \sum_{n=1}^{\infty} B_n(0) \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \\
\varphi_t(x,0) &= \sum_{n=1}^{\infty} B_n'(0) \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} D_n \sin\left(\frac{n\pi x}{L}\right).
\end{aligned}$$

Example Solve the wave equation in $0 < x < L$ subject to the initial conditions

$$\varphi(x,0) = \frac{4a}{L^2}x(L-x)$$
$$\varphi_t(x,0) = 0.$$



Solution First impose the initial velocity

$$\varphi_t(x,0) = \sum_{n=1}^{\infty} B_n'(0)\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty}\frac{n\pi c}{L}D_n\sin\left(\frac{n\pi x}{L}\right) = 0$$

to deduce

$$D_n = 0$$

(for all n).

Matching initial conditions using Fourier series

We know therefore that

$$\varphi(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right),$$

where

$$\varphi(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = \frac{4a}{L^2}x(L-x).$$

From the theory of Fourier series *it can be shown* (but not here) that

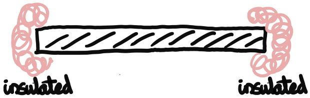
$$\frac{4a}{L^2}x(L-x) = \frac{32a}{\pi^3}\sum_{\text{odd } n}\frac{1}{n^3}\sin\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 < x < L.$$

Therefore

$$\varphi(x,t) = \frac{32a}{\pi^3}\sum_{\text{odd } n}\frac{1}{n^3}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right).$$

Solve the heat/diffusion equation

$$\frac{\partial \varphi}{\partial t} = D\frac{\partial^2 \varphi}{\partial x^2} \qquad \text{for } 0 < x < L \qquad (5)$$



subject to the boundary conditions

$$\varphi_x(0,t) = 0 \quad \text{and} \quad \varphi_x(L,t) = 0. \qquad (6)$$

As usual look for solutions of the form

$$\varphi(x,t) = X(x)T(t). \qquad (7)$$

Substituting (7) into (5) we get

$$X(x)T'(t) = DX''(x)T(t) \Rightarrow \frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)}.$$

As before, $T'/(DT)$ is a function of t only while X''/X is a function of x only and *it follows that they must both be equal to the same constant.*

Denote the separation constant by

$$\frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

The equation for X is just like in the wave equation:

$$X''(x) = -\lambda X(x) \quad \Rightarrow \quad X''(x) + \lambda X(x) = 0,$$

while the time-dependent part is now a first-order ODE:

$$T'(t) = -\lambda DT(t).$$

Solutions are of the form $T(t) = Ce^{-\lambda Dt}$, where C is any constant.

These seem reasonable if $\lambda \geq 0$ but are *exponentially growing* if $\lambda < 0$, which is *unphysical* for the diffusion equation.

Using boundary conditions to fix the separation constant

For $X(x)$, have the same three cases as in the wave equation.

(i) **Case** $\lambda > 0$. Letting $\lambda = k^2$ we get, as in the wave equation,

$$X(x) = A\cos kx + B\sin kx.$$

We will return to this case!

(ii) **Case** $\lambda = 0$. Here the general solution for $X(x)$ is

$$X(x) = A + Bx.$$

It satisfies the boundary conditions (7) if

$$X'(0) = X'(L) = 0 = B \qquad \Rightarrow X(x) = A = \text{constant}.$$

The corresponding solution of $T' = -\lambda DT = 0$ is $T = \text{constant}$. The result is a physically reasonable *steady-state solution*

$$\varphi(x,t) = A,$$

where A is any constant.

Using boundary conditions to fix the separation constant

(iii) **Case** $\lambda < 0$. Let $\lambda = -\kappa^2$. Then

$$X(x) = Ae^{\kappa x} + Be^{-\kappa x}.$$

Recall that the corresponding solution for T is exponentially growing in this case, which is unphysical (for the heat/diffusion equation). This case had better be excluded by the bcs! The boundary conditions give ($\varphi_x(0, t) = 0 \Rightarrow$)

$$X'(0) = 0 = \kappa A - \kappa B \qquad \Rightarrow B = A$$

and ($\varphi_x(L, t) = 0 \Rightarrow$)

$$X'(L) = 0 = \kappa A e^{\kappa L} - \kappa B e^{-\kappa L} = \kappa A(e^{\kappa L} - e^{-\kappa L}) \qquad \Rightarrow A = 0 = B.$$

Here again there is only the *trivial solution* $X(x) = 0$, which is of no interest. The unphysical case is excluded.

Using boundary conditions to fix the separation constant

Let us return to **Case** (i), $\lambda = k^2 > 0$. Here we found the general solution

$$X(x) = A\cos kx + B\sin kx.$$

The condition on $x = 0$ gives ($\varphi_x(0, t) = 0 \Rightarrow$)

$$X'(0) = 0 = kB.$$

So we now know any such solution must be of the form

$$X(x) = A\cos kx.$$

The condition on $x = L$ gives ($\varphi_x(L, t) = 0 \Rightarrow$)

$$X'(L) = 0 = -kA\sin kL.$$

The solution with $A = 0$ is *trivial* and of no interest.

Using boundary conditions to fix the separation constant

For a *nontrivial* solution we need (this next bit is very close to the wave equation example)

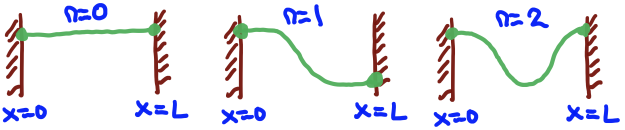
$$\sin kL = 0 \quad \Rightarrow \quad kL = n\pi, \quad \text{where } n = 1, 2, 3, \cdots$$

$$\Rightarrow \quad k = \frac{n\pi}{L} = \text{"}k_n\text{"}.$$

Remark

The case $n = 0$ also works here but it is strictly speaking the solution from **Case** (ii). We add it to the final list of solutions below:

$$X_n(x) = A\cos k_n x = A\cos\left(\frac{n\pi x}{L}\right) \quad n = 0, 1, 2, 3, \cdots.$$



Time dependence of standing modes

Recall that the corresponding solutions of the time-dependent ODE, with

$$\lambda = k^2 \qquad \to \quad \lambda = k_n^2 = \left(\frac{n\pi}{L}\right)^2,$$

are

$$T(t) = C\mathrm{e}^{-\lambda Dt} = \mathrm{e}^{-k_n^2 Dt} = C\mathrm{e}^{-n^2\pi^2 Dt/L^2} \equiv T_n(t).$$

The combined solution of the diffusion equation can be written

$$\varphi_n(x,t) = X_n(x)T_n(t) = C\cos\left(\frac{n\pi x}{L}\right)\mathrm{e}^{-n^2\pi^2 Dt/L^2}.$$

We now take advantage of the **principle of superposition** to write a more general solution

$$\varphi(x,t) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)\mathrm{e}^{-n^2\pi^2 Dt/L^2},$$

which still satisfies the boundary conditions $\varphi_x(0,t) = 0 = \varphi_x(L,t)$.

Matching initial conditions using Fourier series

This more general solution

$$\varphi(x,t) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 Dt/L^2},$$

can be thought of as a Fourier (cosine) series

$$\varphi(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n(t) \cos\left(\frac{n\pi x}{L}\right)$$

with time-dependent Fourier coefficients

$$\begin{aligned} A_0 &= 2C_0 \\ A_n(t) &= C_n e^{-n^2\pi^2 Dt/L^2} \end{aligned}$$

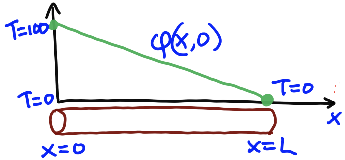
and the constants C_n can be determined by matching to a (single) initial condition as a Fourier (cosine) series:

$$\varphi(x,0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n(0) \cos\left(\frac{n\pi x}{L}\right) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right).$$

Matching initial conditions using Fourier series

Example Solve the heat equation in $0 < x < L$ subject to the initial condition

$$\varphi(x,0) = 100\frac{L-x}{L}$$



and insulating bcs at each end.

Solution It can be shown (but not here) that the given initial condition can be written as a Fourier (cosine) series

$$\varphi(x,0) = 50 + \frac{400}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^2}\cos\left(\frac{n\pi x}{L}\right).$$

By matching this initial condition with the general solution $\varphi(x,t)$ we get

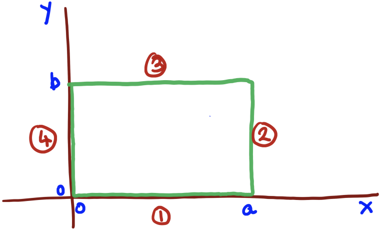
$$\varphi(x,t) = 50 + \frac{400}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^2}\cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 Dt/L^2}.$$

6.4 Separable Solution of the Laplace equation

Solve the heat/diffusion equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad (8)$$

in the rectangle $0 < x < a$ and $0 < y < b$ subject to the boundary conditions



①	$\varphi(x, 0) = 0$	for	$0 < x < a$
②	$\varphi_x(a, y) = 0$	for	$0 < y < b$
③	$\varphi(x, b) = h(x)$	for	$0 < x < a$
④	$\varphi(0, y) = 0$	for	$0 < y < b.$

This time we look for solutions of the form

$$\varphi(x, y) = X(x)Y(y). \qquad (9)$$

Substituting $\varphi(x, y) = X(x)Y(y)$ into (8) we get

$$X''(x)Y(y) + X(x)Y''(y) = 0 \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda.$$

As before, X''/X is a function of x only and -Y''/Y is a function of y only and *it follows that they must both be equal to the same constant*, which we have set equal to $-\lambda$.

The two resulting ODEs can be written as

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ Y''(y) - \lambda Y(y) &= 0. \end{aligned}$$

Note that λ appears with opposite sign in these equations! We have no *a priori* reason to expect one sign of λ to be more physical than the other.

Using boundary conditions to fix the separation constant

(i) **Case** $\lambda > 0$. Letting $\lambda = k^2$ we get

$$\begin{aligned} X(x) &= A\cos kx + B\sin kx \\ Y(y) &= Ce^{ky} + De^{-ky}. \end{aligned}$$

(ii) **Case** $\lambda = 0$. Here the general solution for $X(x)$ and $Y(y)$ are

$$\begin{aligned} X(x) &= A + Bx \\ Y(y) &= C + Dy. \end{aligned}$$

(iii) **Case** $\lambda < 0$. Letting $\lambda = -k^2$ we get

$$\begin{aligned} X(x) &= Ae^{kx} + Be^{-kx} \\ Y(y) &= C\cos ky + D\sin ky. \end{aligned}$$

In cases (ii) and (iii), the bcs $\varphi(0, y) = 0 = \varphi_x(a, y) \Rightarrow X(0) = 0 = X'(a)$ give only the trivial solution $X(x) = 0$.

Using boundary conditions to fix the separation constant

Let us return to **Case** (i), $\lambda = k^2 > 0$. Here we found the general solution

$$X(x) = A\cos kx + B\sin kx.$$

The condition on $x = 0$ gives ($\varphi(0, t) = 0 \Rightarrow$)

$$X(0) = 0 = A.$$

So we now know any such solution must be of the form

$$X(x) = B\sin kx.$$

The condition on $x = a$ gives ($\varphi_x(a, t) = 0 \Rightarrow$)

$$X'(a) = 0 = kB\cos ka.$$

The solution with $B = 0$ is *trivial* and of no interest.

Using boundary conditions to fix the separation constant

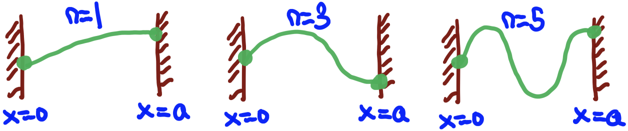
For a *nontrivial* solution we need

$$\cos ka = 0 \quad \Rightarrow \quad ka = \frac{n\pi}{2}, \quad \text{where } n = 1, 3, 5, \cdots$$

$$\Rightarrow \quad k = \frac{n\pi}{2a} = \text{"}\, k_n \text{"}.$$

The list of solutions is then:

$$X_n(x) = B \sin k_n x = B \sin\left(\frac{n\pi x}{2a}\right) \quad n = 1, 3, 5, \cdots .$$



Towards a full solution

The corresponding solution for $Y(y)$ is

$$Y_n(y) = C\mathrm{e}^{k_n y} + D\mathrm{e}^{-k_n y} = C\mathrm{e}^{n\pi y/(2a)} + D\mathrm{e}^{-n\pi y/(2a)}$$

and the combined solution of the Laplace equation can be written

$$\varphi_n(x,y) = X_n(x)Y_n(y) = \sin\left(\frac{n\pi x}{2a}\right)\left(C\mathrm{e}^{n\pi y/(2a)} + D\mathrm{e}^{-n\pi y/(2a)}\right).$$

We use **principle of superposition** to write a more general solution

$$\varphi(x,y) = \sum_{n \text{ odd}} \sin\left(\frac{n\pi x}{2a}\right)\left(C_n\mathrm{e}^{n\pi y/(2a)} + D_n\mathrm{e}^{-n\pi y/(2a)}\right)$$

which still satisfies the boundary conditions $\varphi(0,y) = 0 = \varphi_x(a,y)$. We choose the coefficients C_n and D_n to satisfy the remaining bcs on $y = 0$ and $y = b$.

Towards a full solution

The boundary condition on $x = 0$ gives

$$\varphi(x,0) = 0 = \sum_{n \text{ odd}} \sin\left(\frac{n\pi x}{2a}\right)(C_n + D_n) \quad \Rightarrow D_n = -C_n.$$

Then

$$\begin{aligned}\varphi(x,y) &= \sum_{n \text{ odd}} C_n \sin\left(\frac{n\pi x}{2a}\right)\left(e^{n\pi y/(2a)} - e^{-n\pi y/(2a)}\right) \\ &= \sum_{n \text{ odd}} 2C_n \sin\left(\frac{n\pi x}{2a}\right) \sinh\left(\frac{n\pi y}{2a}\right).\end{aligned}$$

Finally, the remaining coefficients C_n are determined by matching to the boundary condition on $y = b$ (as a function of x):

$$\varphi(x,b) = \sum_{n \text{ odd}} 2C_n \sin\left(\frac{n\pi x}{2a}\right) \sinh\left(\frac{n\pi b}{2a}\right) = h(x).$$