MTHS2007 Advanced Mathematics and Statistics for Mechanical Engineers

Chapter 4: applications of Fourier series

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The application of Fourier series to represent a function can be very helpful in solving differential equations by simplifying any given forcing functions (RHSs of ODEs) or replacing boundary/initial conditions (ODEs and PDEs) by using expansions in terms of sine or cosine functions.

If the differential equation is linear, the superposition principle can then be used to find the general solution for an ODE with a periodic RHS, for example.

Example Periodically driven mechanical oscillator

Fourier series are particularly useful if the RHS of an ODE is a periodic function. Consider, for example the case of a periodically forced mechanical oscillator

$$M\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F(t), \tag{1}$$

where $F(t)$ is a periodic function in t, satisfying

$$F(t) = F(t+T).$$

We can denote by

$$\Omega = \frac{2\pi}{T}$$

the corresponding circular frequency. It is convenient to re-write this as

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = f(t) \tag{2}$$

where

$$\gamma = \frac{c}{2M}, \qquad \omega_0 = \sqrt{\frac{k}{M}} \quad \text{and} \quad f(t) = \frac{1}{M}F(t).$$

Since $f(t)$ has period $T = 2\pi/\Omega$ we can write it as a Fourier series,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(n\Omega t\right) + b_n \sin\left(n\Omega t\right)\right).$$

We have previously seen how to solve this problem with a single cosine term on the right.

We can solve the more general problem by finding a corresponding particular integral for each of the RHSs $\cos(n\Omega t)$ and $\sin(n\Omega t)$ and adding them up in the right combination.

This is equivalent to the following procedure:

As we know that $x_\mathrm{p}(t)$ is constructed out of a constant (arising from the constant term in f), and terms containing $\cos(n\Omega t)$ and $\sin(n\Omega t)$ arising from the cosine and sine terms in f respectively, there must be a particular integral that is periodic in t, with period $T = 2\pi/\Omega$.

This means that we can write

$$x_\mathrm{p}(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos(n\Omega t) + B_n \sin(n\Omega t)\right), \qquad (3)$$

at least as long as no term in the Fourier series for $f(t)$ is a solution of the homogeneous equation associated with (2).

This means

$$\frac{\mathrm{d}x_\mathrm{p}}{\mathrm{d}t} = \Omega \sum_{n=1}^{\infty} \left(nB_n \cos(n\Omega t) - nA_n \sin(n\Omega t) \right)$$

and

$$\frac{\mathrm{d}^2 x_\mathrm{p}}{\mathrm{d}t^2} = \Omega^2 \sum_{n=1}^{\infty} \left(-n^2 A_n \cos(n\Omega t) - n^2 B_n \sin(n\Omega t) \right) .$$

We can insert these into the ODE (2) and compare coefficients.

$$\begin{array}{rcl} \omega_0^2 A_0 & = & a_0 , \\ -n^2\Omega^2 A_n + 2n\Omega\gamma B_n + \omega_0^2 A_n & = & a_n \\ -n^2\Omega^2 B_n - 2n\Omega\gamma A_n + \omega_0^2 B_n & = & b_n \end{array} \left.\vphantom{\begin{array}{c} a \\ a \end{array}}\right\} \quad \text{for} \quad n \geq 1.$$

For convenience the last two equations here can be rewritten

$$\begin{aligned} (\omega_0^2 - n^2\Omega^2) A_n + 2n\Omega\gamma B_n &= a_n \\ (\omega_0^2 - n^2\Omega^2) B_n - 2n\Omega\gamma A_n &= b_n \end{aligned}$$

We can solve these equations for A_n and B_n to find

$$A_0 = \frac{a_0}{\omega_0^2}, \tag{4}$$

$$A_n = \frac{(\omega_0^2 - n^2\Omega^2)a_n - 2n\Omega\gamma b_n}{(\omega_0^2 - n^2\Omega^2)^2 + (2n\Omega\gamma)^2}, \tag{5}$$

$$B_n = \frac{(\omega_0^2 - n^2\Omega^2)b_n + 2n\Omega\gamma a_n}{(\omega_0^2 - n^2\Omega^2)^2 + (2n\Omega\gamma)^2}. \tag{6}$$

With these we can then write down the Fourier series representation of the particular integral. Adding this to the complementary function will yield the general solution of the inhomogeneous equation.

Example

A weighted spring with natural frequency ω_0 is forced by the 2π periodic forcing function

$$f(t) = \begin{cases} (t+\pi) & \text{for} \quad -\pi \le t < 0 \\ (\pi - t) & \text{for} \quad 0 \le t < \pi. \end{cases}$$

The vertical displacement y of the weight satisfies the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \omega_0^2 y = f(t). \tag{7}$$

Find y as a function of t.

We start by expressing $f(t)$ as a Fourier series. Since f is an even function, we have $b_n = 0$ for all n, and the coefficients a_n are given by

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - t)\cos(nt)\,\mathrm{d}t,$$

Then

$$a_0 = \pi$$

when $n = 0$ and

$$\begin{aligned} a_n &= \frac{2}{\pi}\left[\frac{\pi}{n}\sin(nt) - \frac{1}{n^2}\cos(nt) - \frac{t}{n}\sin(nt)\right]_0^{\pi} \\ &= \frac{2}{n^2\pi}(1-(-1)^n), \end{aligned}$$

when $n \geq 1$ (after some work involving integration by parts).

Substituting the Fourier series for $f(t)$ into (6) gives

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \omega_0^2 y = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}(1-(-1)^n)\cos(nt).$$

Now write the particular integral as

$$y_{\mathrm{p}}(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos(nt) + B_n \sin(nt)\right),$$

ie

$$\frac{\mathrm{d}^2 y_{\mathrm{p}}}{\mathrm{d}t^2} = \sum_{n=1}^{\infty} \left(-n^2 A_n \cos(nt) - n^2 B_n \sin(nt)\right).$$

Inserting all this into the ODE, we get

$$\frac{\omega^2 A_0}{2} + \sum_{n=1}^{\infty} \left((\omega_0^2 - n^2) A_n \cos(nt) + (\omega_0^2 - n^2) B_n \sin(nt)\right)$$

$$= \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left((-1)^n - 1\right) \cos(nt).$$

We can now determine A_n, B_n by comparing coefficients,

$$\begin{aligned} A_0 &= \frac{\pi}{\omega_0^2}, \\ A_n &= \frac{2(1-(-1)^n)}{\pi n^2(\omega_0^2 - n^2)} \\ B_n &= 0 \end{aligned} \Bigg\} \quad (n \geq 1).$$

Two independent solutions to the homogeneous version of (6) (i.e. with $F = 0$) are $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ and it follows that the general solution to (6) is

$$\begin{aligned} y(t) &= C\cos(\omega_0 t) + D\sin(\omega_0 t) + \frac{\pi}{2\omega_0^2} \\ &\quad + \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(nt)}{n^2(\omega_0^2 - n^2)}. \end{aligned} \tag{8}$$

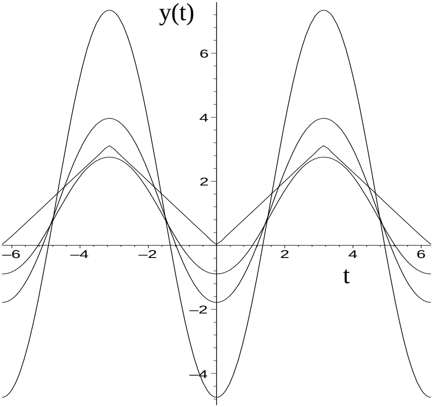

Figure 1: This picture shows y_p for $\omega_0 = 1.1$, $\omega_0 = 1.2$ and $\omega_0 = 1.3$. The closer ω_0 is to one of the values of n, the larger the amplitude of y will be. This effect is called *resonance*.