

University of Nottingham  
Department of Mechanical, Materials and Manufacturing  
Engineering  
Dynamics and Control (MMME2046)  
Systems modelling and control

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## LEARNING OUTCOMES

No.	Description	Means to demonstrate achievement
1	Understand the representation of a physical system by a mathematical model	Laboratory activity, exam
2	Predict the response of a first order system to an input	Compulsory question in exam
3	Understand the concept of the root locus in predicting the response of more complex systems	Laboratory activity, Exam
4	Predict whether a given system will be stable	Laboratory activity, Exam

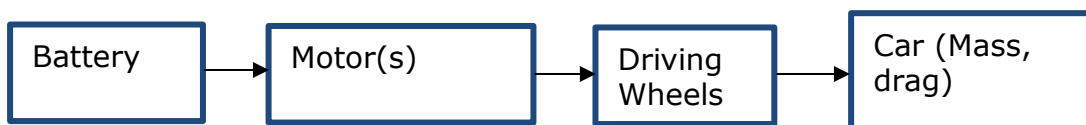
## SYSTEM MODELLING 1

Reading: Nise, Control Systems Engineering, Chapter 1, pp10-20; Dorf and Bishop, Modern Control Systems, Chapter 1, pp23-39; Clifford, Introduction to Mechanical Engineering Part 2, Unit 5: 317-321

### INTRODUCTION

Systems modelling is used to predict the performance of a complex system from the known behaviour of its components – these may be levers, gearboxes, electric motors, hydraulic actuators, even jet engines.

A practical application of systems modelling would be in the design of an in-hub motor for an electric car. Bench testing of the components will give us the data we need, but how will the car perform?

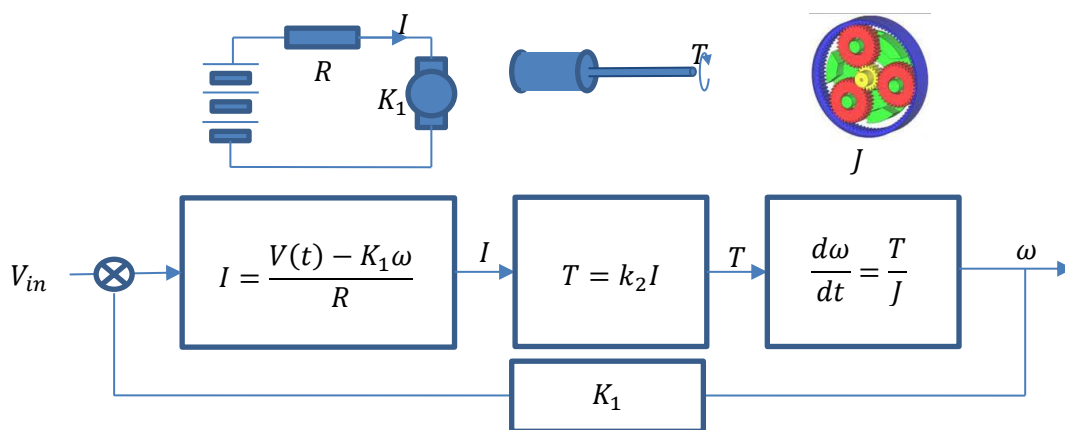


Each of the processes entails a conversion – the battery supplies current to the motor, which converts the electrical energy (voltage × current) into motive

power (torque  $T \times$  angular velocity,  $\omega$ ) which is supplied to the wheels (diameter, coefficient of friction, axle weight) to provide traction force. The traction force is applied to the car, which has a mass, and if the car is moving, has to overcome the aerodynamic drag, which is dependent on the car's velocity.

The conversions are characterised by transfer functions – these are mathematical expressions that relate each input to its corresponding output. Unlike electrical circuits, where we would typically trace a current around the circuit, assuming that the current will be unaffected, the transfer function will change the input into an output every time.

### Example 1: Large Geared Electric Motor



The in-hub motor shown provides both motive force and regenerative braking for an electric vehicle. The battery supplies current to the electric motor which will have a back EMF dependent on the rotation speed of the motor. The armature of the motor converts the current into a torque, which then turns the wheel. The motor and the wheel have a moment of inertia given by  $J$ .

The behaviour of the system – and eventually of the entire car, can therefore be predicted using this systems engineering approach.

In reality, the armature resistance ( $R$ ) is very low, to minimise losses and so if the car is heavy (which it will be, the batteries alone will weigh over 200kg) the stall current when  $\omega = 0$  may burn out the coil very rapidly if  $V$  is not increased gradually, and so the systems engineer will work with the design team to develop control systems to ensure safe and reliable operation.

Another term that is used is control, and usually, a control system is there to reduce the workload of the operator – instead of using a fine adjustment, it is simpler to specify desired position (**input**) and then to have a system that will move the component to its desired position (**output** or **response**).

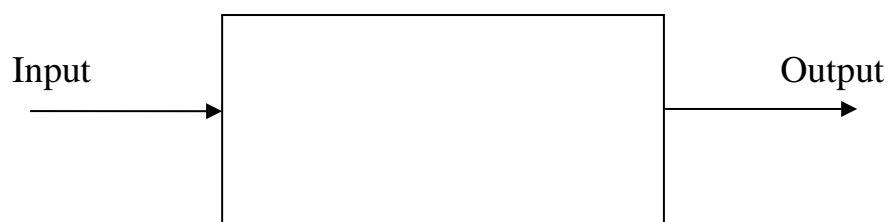
Some common examples of control systems are as follows:

- i) A thermostatically controlled domestic central heating system that keeps the house at a constant temperature.
- ii) An engine or turbine governor system that limits changes in speed as the load changes.
- iii) The system which causes the table of a numerically controlled milling machine to move through a desired path in response to command signals generated by computer.
- iv) The systems that cause the control surfaces on a large aircraft to move in response to movements of the joystick by the pilot (generation 1 was hydraulic control, generation 2 is fly-by-wire).

Clearly, the complexity of control systems can vary enormously. Some systems, such as examples i) and ii) above, are classed as regulators as their purpose is to maintain constant conditions in spite of changes in external conditions. Other systems, such as iii) and iv) above, are known as servo-mechanisms. Their purpose is to cause the system to move to any desired condition within its operating range in response to changes in demand.

There are two classifications of control system:

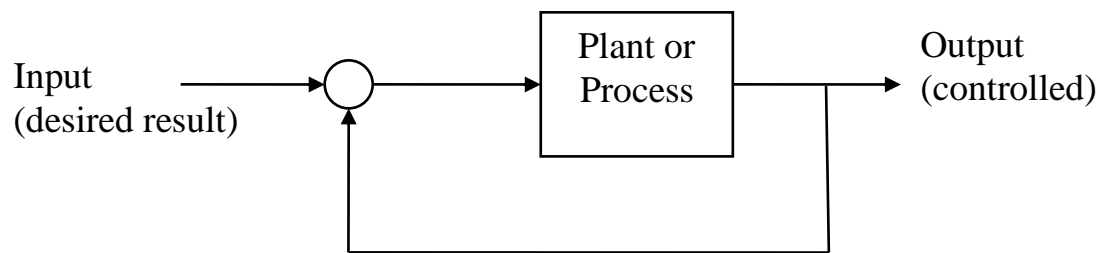
### i) Open-Loop Systems



This is the simplest and crudest form of system. On the basis of knowledge or experience about how the plant or process will behave a guess is made of what input is needed to give the desired output. Large variations can occur in the controlled output because changes in external circumstances are not taken into account. A simple example of an open loop system is a domestic central heating system without a room thermostat.

### ii) Closed-Loop System

In a closed-loop system the output is measured and compared with the desired value. Consequently the system continually attempts to reduce the error between the desired input and the output.



The closed-loop system is therefore "error actuated". An essential feature of any closed-loop system is the use of **feedback**, which allows the output to be compared to the input.

This module is concerned with the study of the behaviour of automatic control systems. We will focus most of our attention on relatively simple closed-loop systems that have a single input and a single controlled output. However, it may be noted that in reality many systems are more complex and may have several inputs and outputs.

The main aim of the module is to introduce you to the fundamental concepts of systems engineering. In particular, you will be introduced to the following:

- i) Simple control systems involving feedback
- ii) Modelling simple components of control systems
- iii) Representation of dynamic systems using block diagrams
- iv) Manipulation and reduction of block diagrams
- v) Transient and steady-state performance of control systems
- vi) Stability analysis of control systems

## SYSTEMS MODELLING 2

### REPRESENTATION OF CONTROL SYSTEMS

Reading: Nise, Control Systems Engineering, Chapter 2, pp49-52; Dorf and Bishop, Modern Control Systems, Chapter 2, pp87-106 (also covers block diagram manipulation, to be covered later).

Clifford, Introduction to Mechanical Engineering Part 2, Unit 5: 321-325

The first step in analysing the behaviour of any system is to clearly define what constitutes the system, the inputs to the system, and the outputs. It must then be represented in a convenient pictorial and mathematical way.

The **block diagram** is the most common pictorial means of depicting control systems. For analysis purposes the function of the components of the system must be described mathematically: this is termed the **transfer function**, which is defined as:

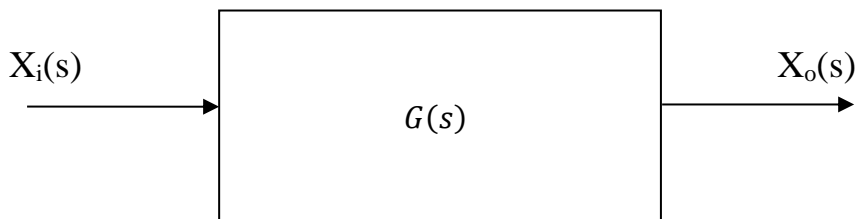
*The transfer function of a linear system is the ratio of the Laplace transform of the output ( $X_o(s)$ ) to the Laplace transform of the input ( $X_i(s)$ ) when the initial conditions are zero.*

The **transfer function**  $G(s)$  is therefore:

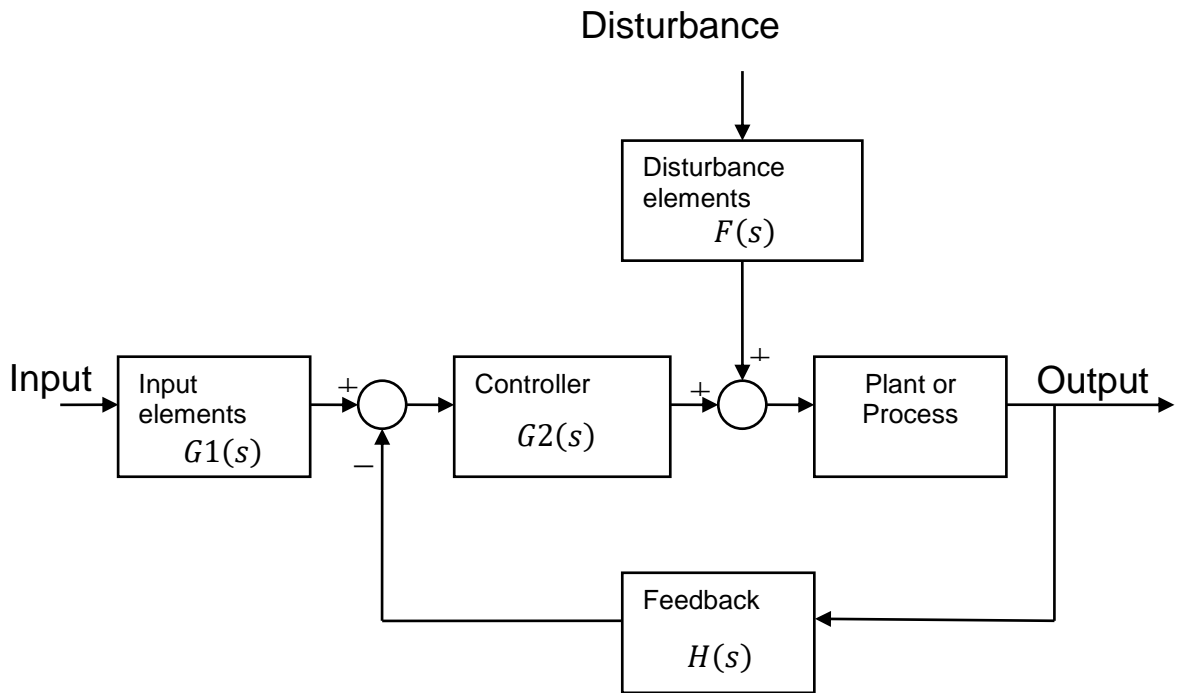
$$G(s) = \frac{X_o(s)}{X_i(s)} = \frac{P(s)}{Q(s)}$$

where  $X_o(s)$  and  $X_i(s)$  are the Laplace transforms of the output and input respectively,  $Q(s)$  is known as the **characteristic function**, and  $Q(s)=0$  is the **characteristic equation**.

The block diagram for an element is drawn as follows:



A typical system will have a block diagram of the following form:



Each box contains the transfer function of the element. A picture of the overall system can be built up from knowledge of its component parts and how they fit together.

Note that because of the properties of Laplace transforms, a separate transfer function for the disturbance, and its influence on the output, can be derived and the final output function will be the sum of these inputs multiplied by their respective transfer functions.

# CONTROL 3

## LAPLACE TRANSFORMS

Nise, Control Systems Engineering, Chapter 2, pp38-48;

Dorf and Bishop, Modern Control Systems, Chapter 2, pp80-89.

Clifford, Introduction to Mechanical Engineering Part 2, Unit 5: 326-333.

The Laplace transform technique is a useful tool for the solution of differential equations and is widely used in control engineering, where it provides a convenient means of describing the **transfer function** of system components.

The Laplace transform of a function  $f(t)$  is written as  $F(s)$  and is defined as:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt$$

where  $s = \sigma + j\omega$  is a complex variable and  $f(t) = 0$  for  $t < 0$ .

When solving a differential equation using Laplace transforms the following step-by-step procedure should be followed:

- STEP 1** Transform the equation from the time-domain to the Laplace domain using the tabulated transforms (taking account of the initial conditions).
- STEP 2** Solve the resulting equations (in the s-domain) by simple algebraic manipulation.
- STEP 3** Take partial fractions and use tabulated transforms to get the solution in the time domain.

In many cases in control theory much useful information can be obtained without having to perform step 3.

Some useful theorems relating to Laplace transforms are given below:

i) Addition and Subtraction.

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

ii) Multiplication by a constant.

$$\mathcal{L}[kf_1(t)] = kF_1(s)$$

iii) Final Value Theorem (see steady state errors later in module)

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

This theorem is only valid if the final value is finite and constant.

iv) Shifting Theorem

$$\text{If } \mathcal{L}[f(t)] = F(s) \text{ then } \mathcal{L}[f(t - \tau)] = e^{-s\tau}F(s)$$

A table of useful Laplace transform pairs is given on page 10.

### Examples of the use of Laplace Transforms

#### Example 1)

Determine the Laplace transform of  $f(t)$  if

$$f(t) = \frac{d^2x}{dt^2}$$

and  $x = 2$ ,  $\frac{dx}{dt} = 1$  at  $t = 0$ .

Solution:

Using entry 2 of the attached table of Laplace transforms, the Laplace transform of  $f(t)$  is:

$$s^n F(s) - s^{n-1}f(0) - \dots - f^{n-1}(0)$$

$$F(s) = s^2X(s) - sx(0) - \dot{x}(0)$$

Substituting the initial conditions into this equation gives:

$$F(s) = s^2X(s) - 2s - 1$$

If the initial conditions are each zero (i.e.  $x(0) = 0$  and  $\dot{x}(0) = 0$ ), it should be noted that the above reduces to:

$$F(s) = s^2X(s)$$

#### Example 2)

Use Laplace transforms to determine the solution to the following second order differential equation:



$$\frac{d^2x}{dt^2} + \omega_n^2 x = \cos(pt)$$

where the initial conditions are taken to be zero.

Solution:

To determine the solution in this case it is necessary to follow the steps indicated on page 6.

STEP 1: Taking Laplace transforms (with zero initial conditions) gives:

$$s^2 X(s) + \omega_n^2 X(s) = \frac{s}{s^2 + p^2}$$

(Entries 2 and 11 in the Table of Laplace transforms are used to obtain the above).

STEP 2: Rearranging gives:

$$X(s) = \frac{s}{(s^2 + \omega_n^2)(s^2 + p^2)}$$

STEP 3: Converting back to the time-domain (using entry 13 in the attached table of Laplace transforms) gives:

$$x(t) = \frac{1}{\omega_n^2 - p^2} [\cos(pt) - \cos(\omega_n t)]$$

Note: the inverse Laplace transform could have been obtained by taking partial fractions and then using the tabulated Laplace transforms.

**Example 3)**

i) Determine the transfer function of the system whose equation of motion is given by:

$$\ddot{x}_0 + ax_0 = ax_i$$

where  $x_0$  is the (time-varying) output and  $x_i$  is the (time-varying) input to the system.

ii) Use the transfer function obtained in i) to determine the time-varying output of the above system subjected to a unit step input.

Solution:

i) Taking Laplace transforms (with zero initial conditions) gives:

$$sX_0(s) + aX_0(s) = aX_i(s)$$

where  $X_0(s)$  is the Laplace transform of the output and  $X_i(s)$  is the Laplace transform of the input.

Re-arranging this equation and using the definition of the transfer function (see handout Control 2), the transfer function  $G(s)$  is given by:

$$G(s) = \frac{X_0(s)}{X_i(s)} = \frac{a}{s + a}$$

ii) The Laplace transform of the output can be deduced simply by multiplying the transfer function by the Laplace transform of the input. Thus, if the input  $x_i$  is a unit step, then (from entry 5 in the table of Laplace transforms):

$$X_i(s) = \frac{1}{s}$$

and from the transfer function the Laplace transform of the output is:

$$X_0(s) = \frac{1}{s} \left( \frac{a}{s + a} \right)$$

The time-domain output of the system is determined by taking the inverse Laplace transform of the above expression. Taking inverse Laplace transforms (using entry 8 in the table of Laplace transforms) gives:

$$x_0(t) = 1 - e^{-at}$$

This is the time-varying output of the system subjected to a unit step input.

### Table of Laplace Transforms

	$f(t)$	$F(s)$
1	$\frac{df(t)}{dt}$	$sF(s) - f(0)$
2	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) \dots - f^{(n-1)}(0)$
3	$\int f(t)dt$	$\frac{1}{s}F(s)$
4	Unit impulse $\delta(t)$	1
5	Unit step 1	$\frac{1}{s}$
6	Unit ramp $t$	$\frac{1}{s^2}$
7	$e^{-at}$	$\frac{1}{s+a}$
8	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
9	$t - \frac{1}{a}(1 - e^{-at})$	$\frac{a}{s^2(s+a)}$
10	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
11	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
12	$e^{-at} \sin(\omega t)$	$\frac{\omega}{(s+a)^2 + \omega^2}$
13	$e^{-at} \cos(\omega t)$	$\frac{s+a}{(s+a)^2 + \omega^2}$
14	$\frac{1}{(\omega^2 - p^2)} \left[ \sin(pt) - \frac{p}{\omega} \sin(\omega t) \right]$	$\frac{p}{(s^2 + p^2)(s^2 + \omega^2)}$
15	$\frac{1}{(\omega^2 - p^2)} [\cos(pt) - \cos(\omega t)]$	$\frac{s}{(s^2 + p^2)(s^2 + \omega^2)}$
16	$\frac{\omega}{\sqrt{1-\gamma^2}} e^{-\gamma\omega t} \sin(\omega t \sqrt{1-\gamma^2})$	$\frac{\omega^2}{s^2 + 2\gamma\omega s + \omega^2}$
17	$1 - \frac{e^{-\gamma\omega t}}{\sqrt{1-\gamma^2}} \sin(\omega t \sqrt{1-\gamma^2} + \varphi)$	$\frac{\omega^2}{s(s^2 + 2\gamma\omega s + \omega^2)}$
18	$t - \frac{2\gamma}{\omega} - \frac{e^{-\gamma\omega t}}{\omega\sqrt{1-\gamma^2}} \sin(\omega t \sqrt{1-\gamma^2} + \varphi)$	$\frac{\omega^2}{s^2(s^2 + 2\gamma\omega s + \omega^2)}$
	Where $\cos \varphi = \gamma$	

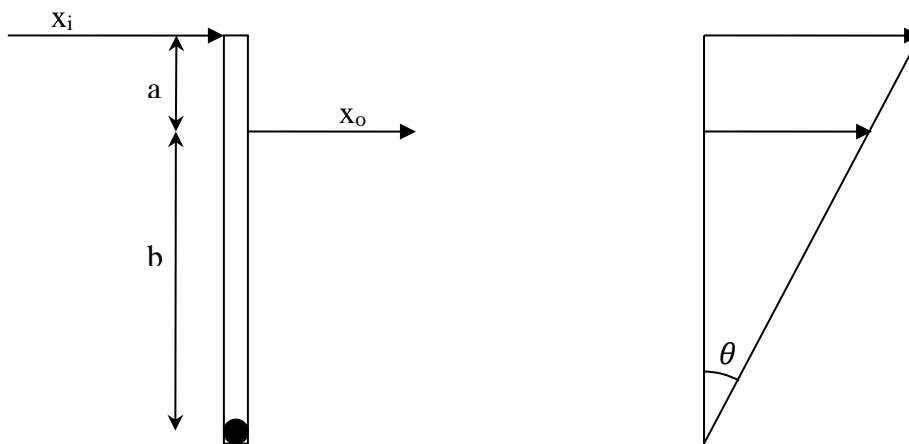
## CONTROL 4

Reading: Nise, Control Systems Engineering, Chapter 2, pp68-87; Dorf and Bishop, Modern Control Systems, Chapter 2, pp72-74, Table 2.5 pp. 98-101.

### MODELLING OF SIMPLE COMPONENTS

In order to describe the behaviour of a complete control system it is necessary to describe in detail what happens in each component of the system. This handout considers the modelling of some simple components which appear in control systems, and derives the appropriate transfer functions and block diagrams.

#### a) Simple Lever System



Determine the **transfer function** and **block diagram** for this rigid lever system.

The two triangles on the right are similar – the angles will be the same, and so the ratios of the sides will be identical:

$$\frac{x_i}{(a + b)} = \frac{x_o}{b}$$

and the relationship between the output and input is:

$$\frac{x_o}{x_i} = \frac{b}{(a + b)}$$

Taking Laplace transforms and assuming zero initial conditions gives:

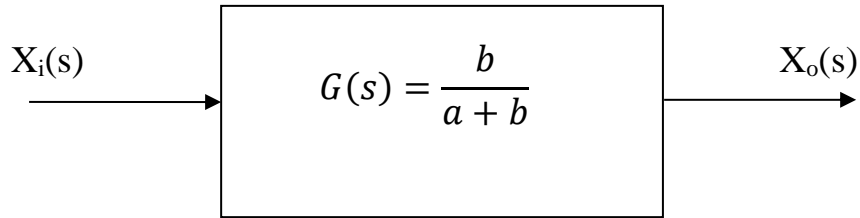
$$\frac{X_o(s)}{X_i(s)} = \frac{b}{(a + b)}$$

Note:  $s$  does not appear on the right hand side of this equation since  $a$  and  $b$  are independent of time.

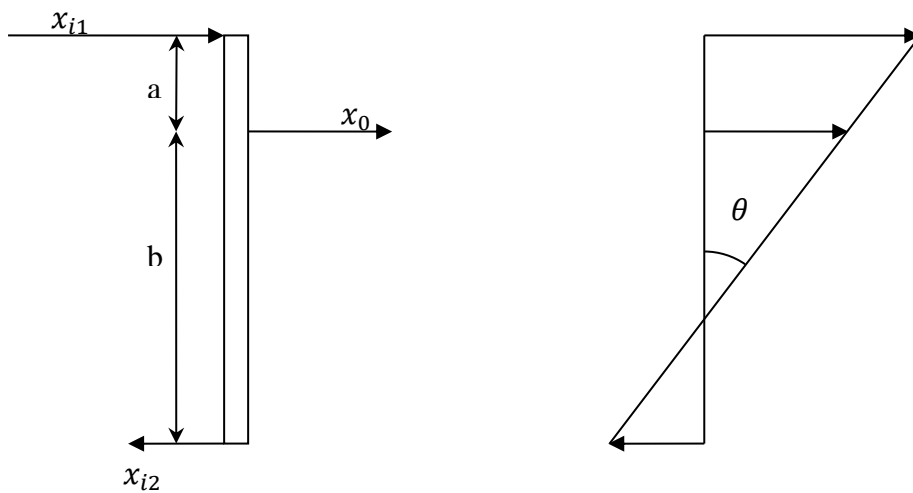
Thus the **transfer function**  $G(s)$  is given by:

$$G(s) = \frac{b}{(a + b)}$$

The **block diagram** for the simple lever system is shown below:



**b) More Complex Lever System**



Determine the **transfer function** and the **block diagram** for the above system.

Conceptually, this system can be treated as two separate transfer functions:

1. Assuming that  $x_{i1}$  is zero, for the triangles to be similar:

$$\frac{x_0}{a} = \frac{-x_{i2}}{a + b}$$

$$x_0 = \frac{-x_{i2} \times a}{a + b}$$

2. Assuming that  $x_{i2}$  is zero:

$$\frac{x_0}{b} = \frac{x_{i1}}{a + b}$$

$$x_0 = \frac{x_{i1} \times b}{a + b}$$

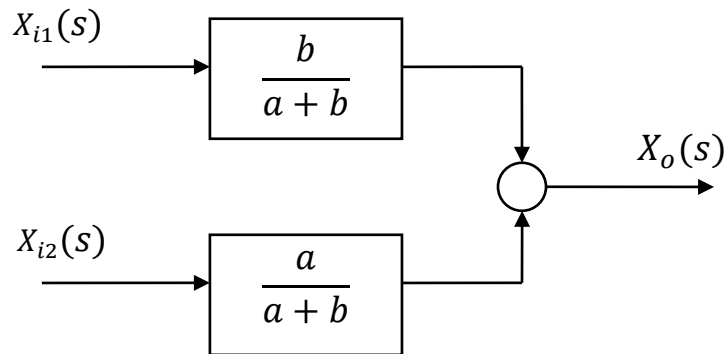
Adding together the two transfer functions to give the general case gives:

$$x_0 = \frac{b}{a+b}x_{i1} - \frac{a}{a+b}x_{i2}$$

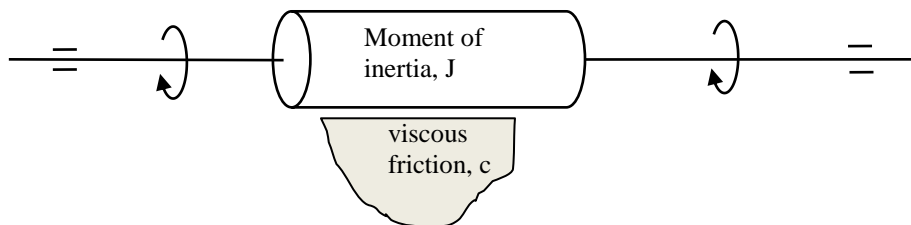
Assuming zero initial conditions and taking Laplace transforms gives:

$$X_0(s) = \frac{b}{a+b}x_{i1}(s) - \frac{a}{a+b}x_{i2}(s)$$

As can be seen, the output is a combination of the two inputs – there is no single transfer function for the input. The **block diagram** for this system is shown below:



### c) Rotor with Viscous Drag



Determine the **transfer function** and **block diagram** for the above system when the input is the drive torque  $l(t)$  and the output is the angular displacement,  $\theta$ .

The equation of motion of the system is given by:

$$J\ddot{\theta} = l - c\dot{\theta}$$

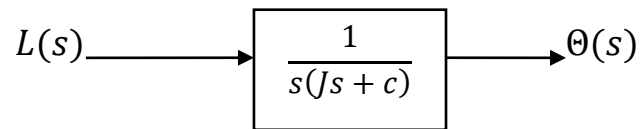
Assuming zero initial conditions and taking Laplace transforms gives:

$$s^2J\Theta(s) = L(s) - sc\Theta(s)$$

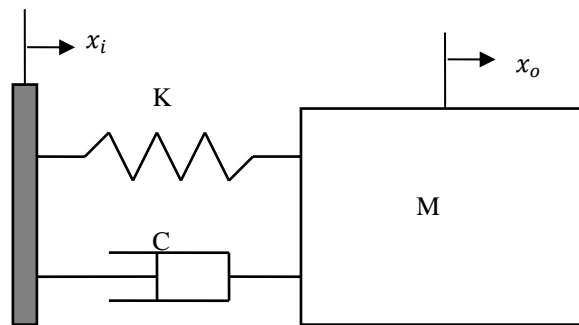
Rearranging, the **transfer function** is given by:

$$G(s) = \frac{\Theta(s)}{L(s)} = \frac{1}{s(Js + c)}$$

The **block diagram** is:



d) **Spring-Mass-Damper System**



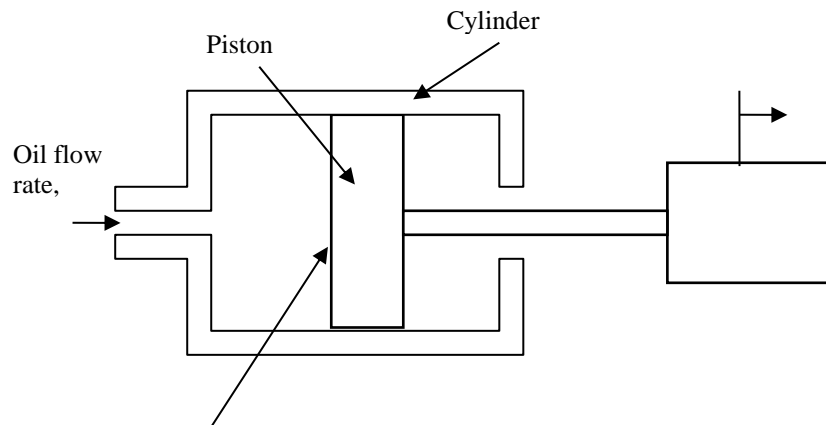
**Exercise:** Noting that the input to the above system is a displacement, show that the **transfer function** for the system is given by:

$$G(s) = \frac{X_o(s)}{X_i(s)} = \frac{Cs + K}{Ms^2 + Cs + K} = \frac{2\gamma\omega_n s + \omega_n^2}{s^2 + 2\gamma\omega_n s + \omega_n^2}$$

where  $\omega_n$  is the natural frequency and  $\gamma$  is the damping ratio given by:

$$\omega_n^2 = \frac{K}{M} \quad \text{and} \quad \gamma = \frac{C}{2\sqrt{KM}}$$

### e) Hydraulic Ram



Determine the **transfer function** between the input  $q(t)$  and the output  $x(t)$ .

Assumptions:

- i) Neglect any leakage past the piston
- ii) Neglect the compressibility of the oil

To obtain the transfer function the continuity equation for the oil flow is used:

$$q(t) = q_{piston} = A \frac{dx}{dt}$$

Taking Laplace transforms with zero initial conditions and rearranging, the transfer function is:

$$G(s) = \frac{X(s)}{Q(s)} = \frac{1}{As}$$

Note: In this simplified case the load mass  $M$  does not appear in the transfer function and the ram acts as an "**integrator**":

$$q(t) = A \frac{dx}{dt} \quad x(t) = \frac{1}{A} \int q(t) dt$$



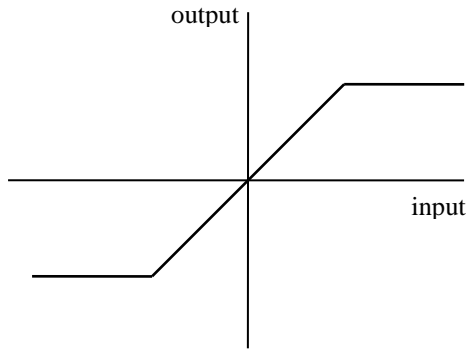
# CONTROL 5

Reading: Nise, Control Systems Engineering, Chapter 2, pp97-102; Dorf and Bishop, Modern Control Systems, Chapter 2, pp77-80.

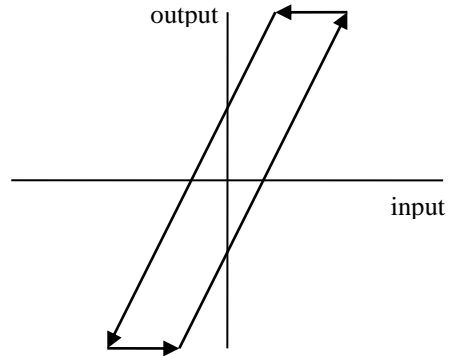
## NON-LINEARITY AND LINEARISATION

Reading:

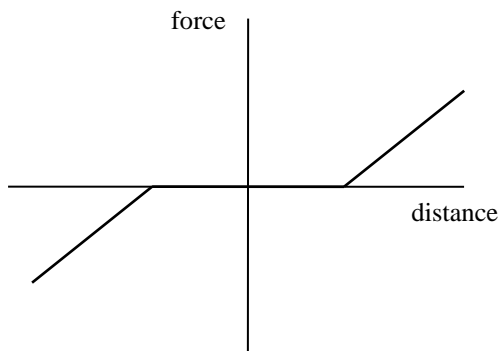
In handout Control 4, the components considered each had a linear input-output relationship. In practice, many control system components have a **non-linear** input-output relationship. Some common examples are shown below:



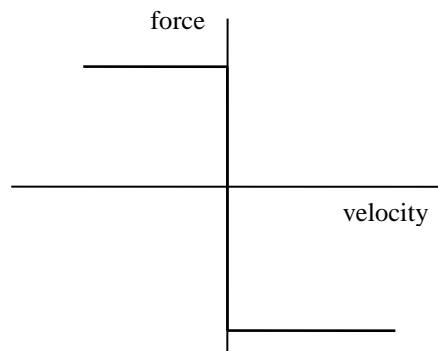
Saturation (Valves and Amplifiers)



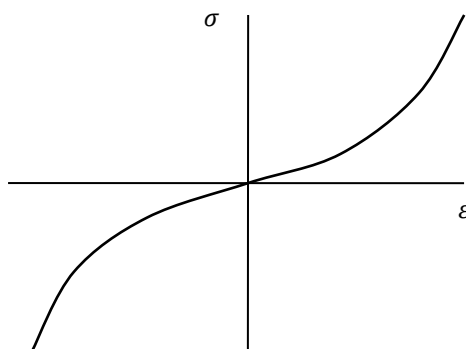
Backlash in Gears



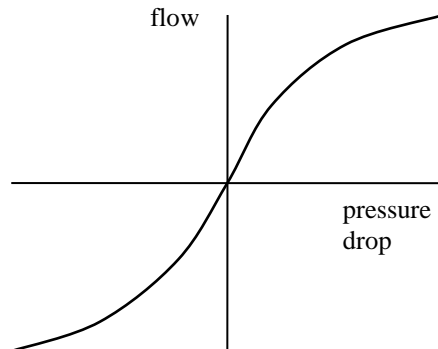
Clearance Effects



Coulomb Friction



Material Non-Linearity

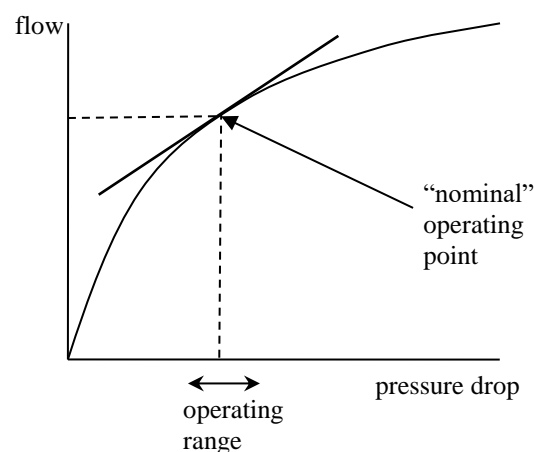


Flow through an Orifice

The analysis of control systems using Laplace transforms and transfer functions is *restricted* to cases where all the components can be assumed to have linear input-output relationships. The presence of a non-linearity means that other approaches have to be used.

In the case of gross non-linearities such as saturation and backlash it is necessary to use techniques such as the Describing Function method or phase-plane analysis (which are both beyond the scope of this module) or to model the system numerically on a computer.

However, in cases such as flow through an orifice, where the non-linearity is "smooth", it is possible to use the process of **linearisation**. The basic idea of linearisation is to replace the non-linear (curved) input-output relationship by a linear (straight line) approximation which is valid over a limited operating range about some nominal operating point. This allows the relatively simple linear analysis techniques to be used, at least in the initial stages of the design process. To illustrate the process of linearisation, consider a component with a non-linear input-output relationship as shown below.



If the component operates about a nominal datum or operation point  $O$ , and if the operation is such that the movement about  $O$  is small (as may be the case in a system designed to maintain a constant condition), departure from the linear approximation will be small. Replacing the curve by the tangent at the datum  $O$  would allow linear analysis to be used.

Further details about non-linearity and linearisation can be found in the following books:

R.J. Richards, "Solving Problems in Control", Longman Scientific and Technical, pages 16, 40.

J. Golten and A. Verwer, "Control System Design and Simulation", McGraw-Hill, Chapter 10.

W. Bolton, "Control Engineering", 2<sup>nd</sup> Edition, Longman, page 70.

# CONTROL 6

Reading: Nise, Control Systems Engineering, Chapter 5, pp250-261; Dorf and Bishop, Modern Control Systems, Chapter 2, pp101-106  
Clifford, Introduction to Mechanical Engineering Part 2, Unit 5: pp348-351.

## BLOCK DIAGRAM MANIPULATION

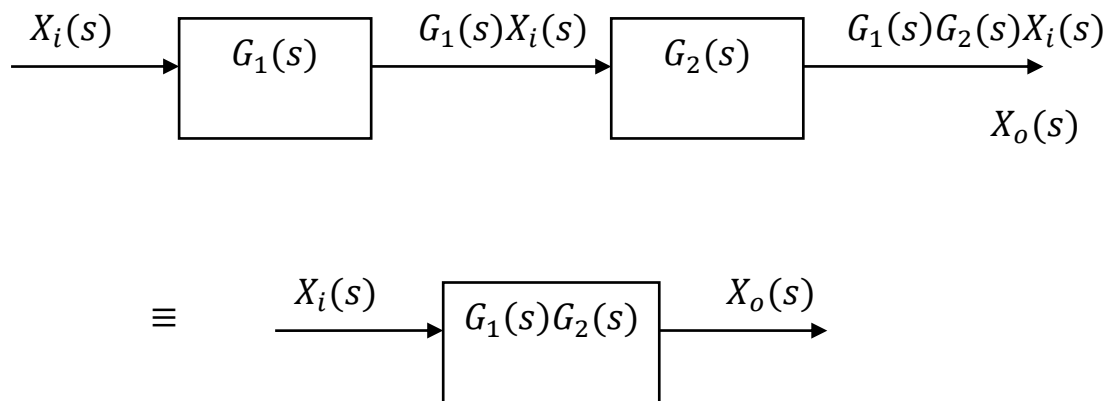
So far, we have considered the way in which a control system is represented (see handouts Control 2 and 3) and the detailed modelling of some simple components of a system (see handouts Control 4 and 5). The next stage in the modelling of a complicated system is to assemble the simple components in turn to represent the complete system. (Examples of this process will be given in handouts Control 8 and 9 for two different types of position control system.)

We will assume here that it is possible to assemble the block diagram for the complete control system from some simple components. However, given that the complete system may appear to be complicated, we will consider here the systematic procedure of **block diagram algebra** which can assist in the simplification of relatively complex block diagrams.

The rules of block diagram algebra are illustrated by the following diagrams.

### a) Elements in Series

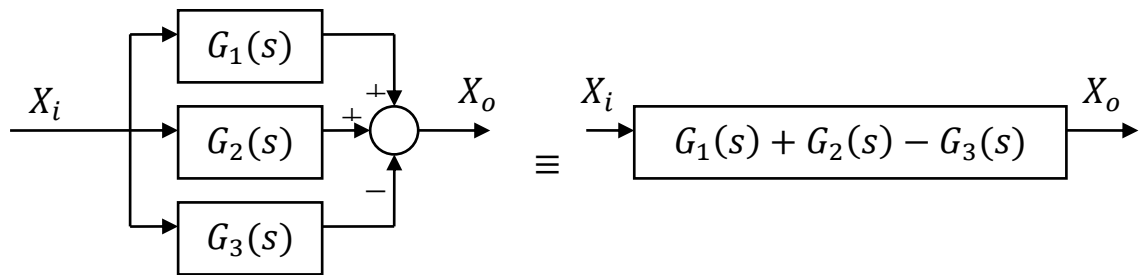
Consider two elements in series.



The overall transfer function of the two elements is the product of the individual component transfer functions. This applies to any number of elements in series. This rule only applies if the elements concerned are **non-interacting** i.e. if the behaviour of one element is not influenced by that of an element which comes after it. If there is interaction between the elements, the interacting units must be considered as a single unit.

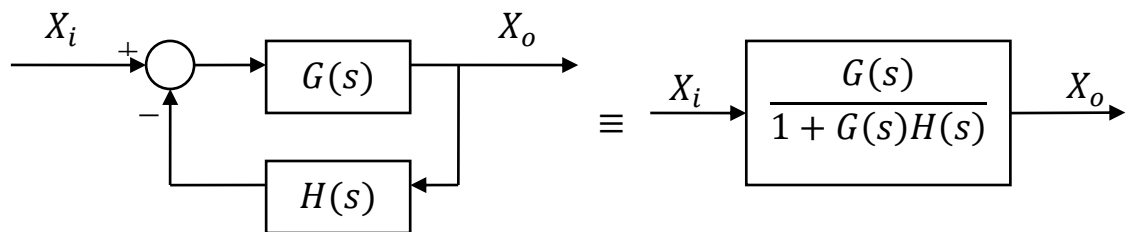
**b) Elements in Parallel**

Consider three elements in parallel.



**c) Closed-Loop transfer function**

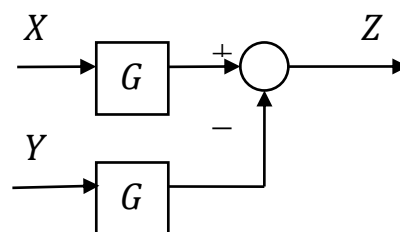
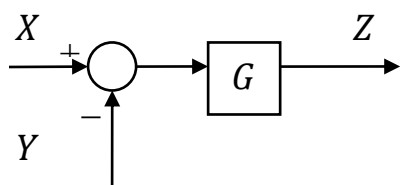
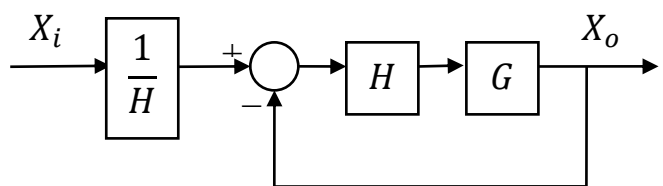
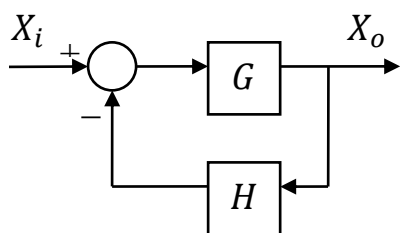
This type of system is considered extensively throughout this course.



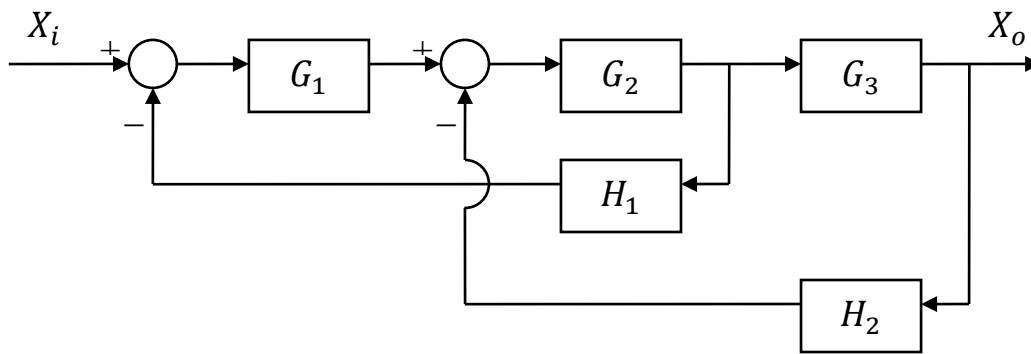
**d) Other Examples**

Block Diagram

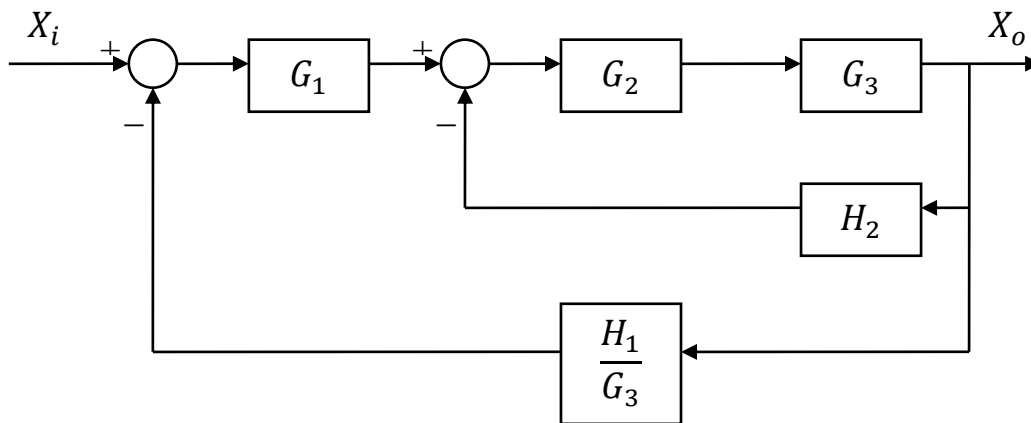
Equivalent Block Diagram



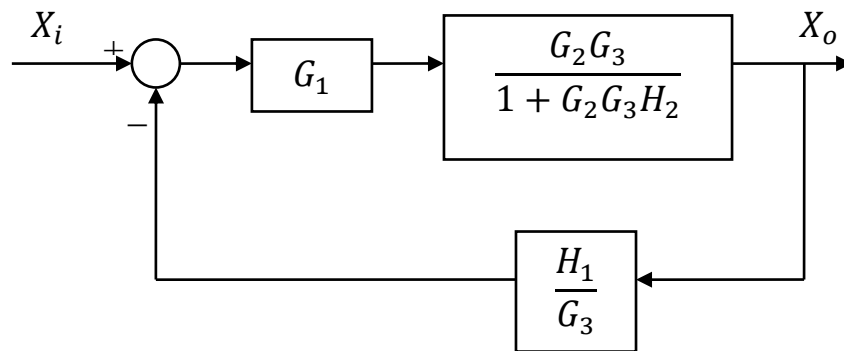
**Example:** Reduce the block diagram to find the overall transfer function.



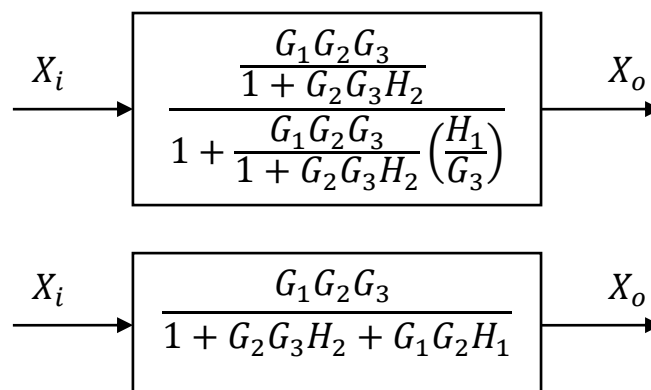
i) Rearrange to avoid interlinking loops.



ii) Eliminate the inner loop.

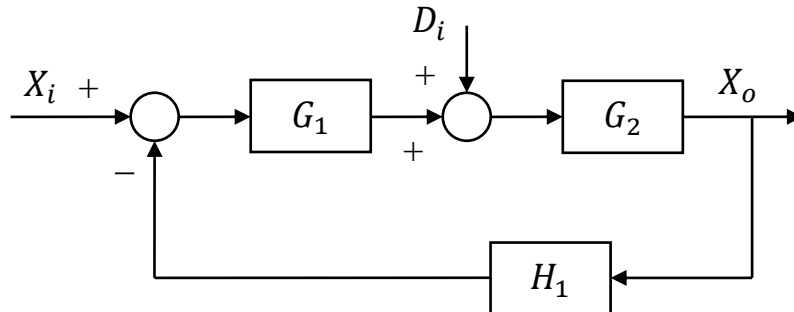


iii) Reduce to a single block (and simplify expression).

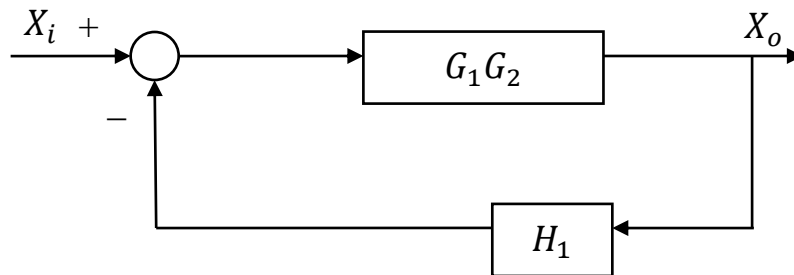


## Disturbance signals/multiple inputs

Sometimes, you will see a block diagram with more than one input, as can be seen below:

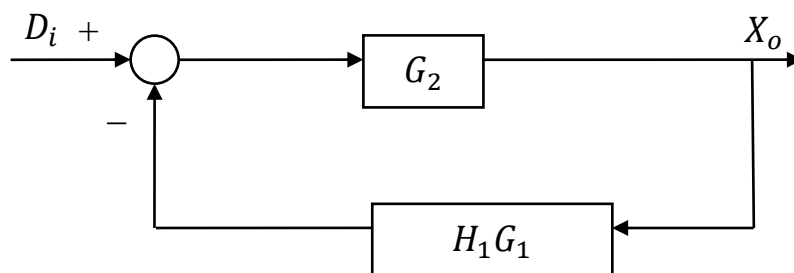


The block diagram and transfer function for  $X_i$  to  $X_o$  is given by:



$$\frac{X_o}{X_i} = \frac{G_1 G_2}{1 + H_1 G_1 G_2}$$

Similarly, for  $D_i$  to  $X_o$



$$\frac{X_o}{D_i} = \frac{G_2}{1 + H_1 G_1 G_2}$$

Using the principle of superposition, the output would be given by:

$$X_o = \left( \frac{G_1 G_2}{1 + H_1 G_1 G_2} \right) X_i + \left( \frac{G_2}{1 + H_1 G_1 G_2} \right) D_i$$

Note that there is no common transfer function for  $X_i$  and  $D_i$ .

## CONTROL 7

Reading:

Nise, Control Systems Engineering, Chapter 4, pp174-182; Dorf and Bishop, Modern Control Systems, Chapter 4, pp269-281; Clifford, Introduction to Mechanical Engineering Part 2, Unit 5: pp348-351.

### INTRODUCTION TO TRANSIENT AND STEADY STATE RESPONSE

When studying control systems it is useful to consider the response of the system to certain '**standard**' inputs. The response of different systems to the same standard input then forms a useful basis for comparing the performance of different systems.

We will consider three such inputs:

- i) Step input
- ii) Ramp input (linear change with time)
- iii) Harmonic input (beyond this module – seen next year)

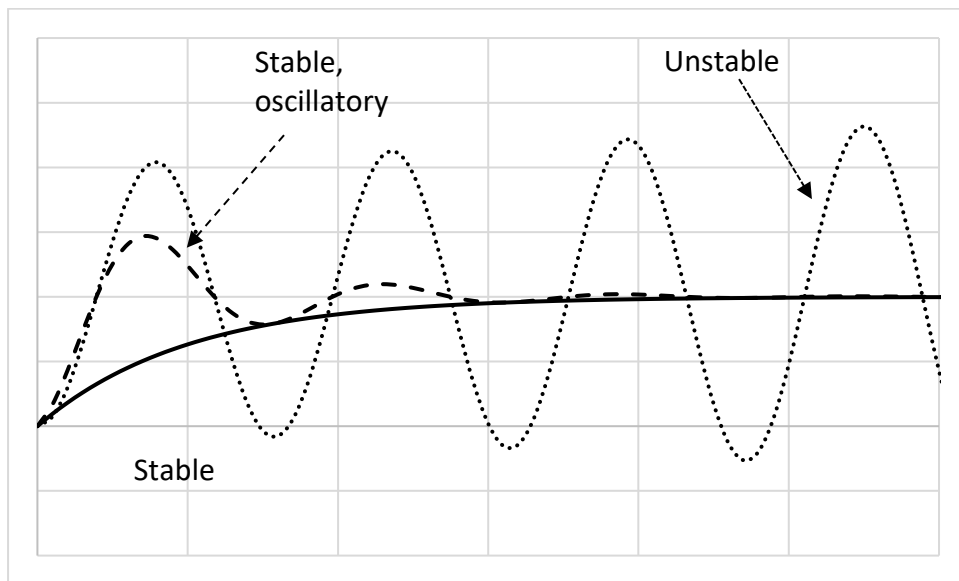
These inputs are useful because:

- a) They are fairly easy to apply in practice, both theoretically and experimentally.
- b) They approximate to operating conditions commonly experienced in control systems.

Other forms of input are possible (eg Impulsive and Random), but we will not consider these in this course.

The purpose of a closed loop system is to ensure that the overall system output follows the system prescribed input as closely as possible. In an ideal system the output would at all times correspond exactly to the input, but this cannot be achieved in a real system. In practice, therefore, the system is judged by a number of criteria, the three most important of which are considered below.

Consider the response of a typical system to a step input.



**i) Is the System stable?**

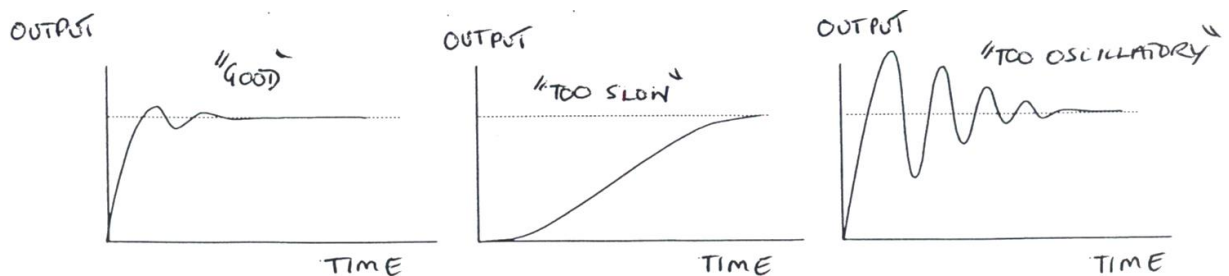
After a disturbance has been applied to the system, the output should settle down to a steady value.

**ii) How Accurate is the System in Steady State?**

The steady state error should be small.

**iii) How Quickly does the system reach steady state?**

The system should reach steady state as quickly as possible without excessive overshoot or oscillation.

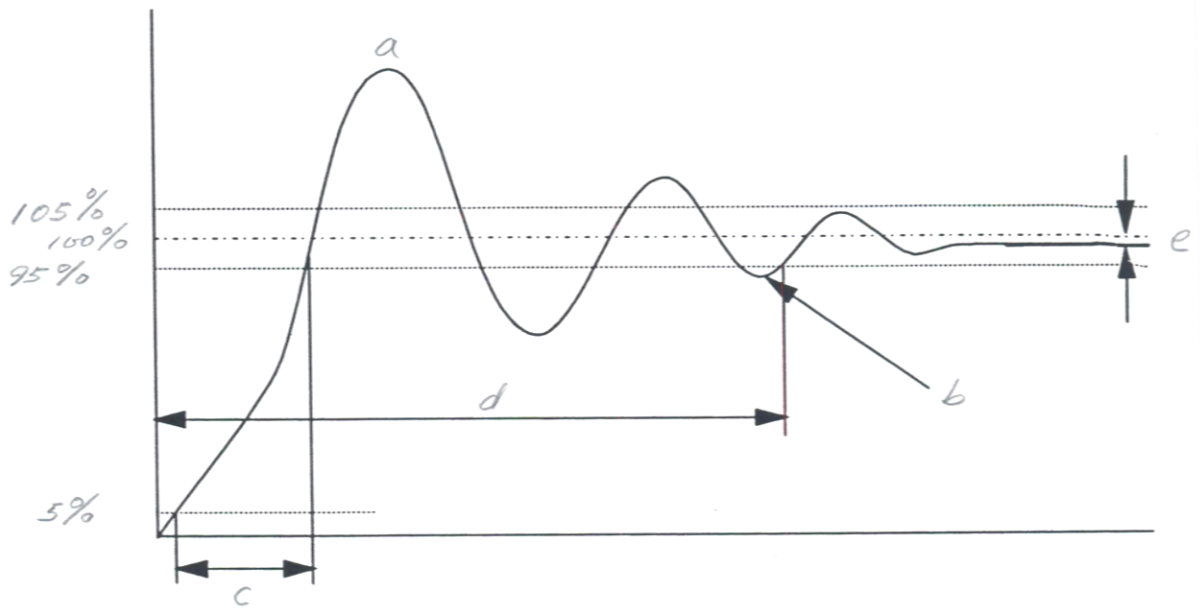


We will consider the question of **stability** in part later in this module, and in detail within the Control and Instrumentation module next year. Aspects of ii) and iii) above will be considered in the following handouts.



### Practical Measures of Transient Response

As will be seen later in the course, for systems of 1st and 2nd order it is possible to characterise the transient nature of the response in terms of the time constant (1st order system) and the natural frequency and damping ratio (2nd order system). However, for systems of 3rd and higher order it is necessary to describe the **step response** of a system in terms of the parameters defined in the diagram below.



- a) Maximum Overshoot as a percentage of step size.
- b) Number of Oscillations before system settles to within a fixed percentage (5% say) of its steady state value.
- c) Rise Time: The time taken for output to rise from 5% to 95% of step size.
- d) Settling Time: The time taken for output to reach and remain within  $\pm 5\%$  of steady state value.
- e) Steady State Error

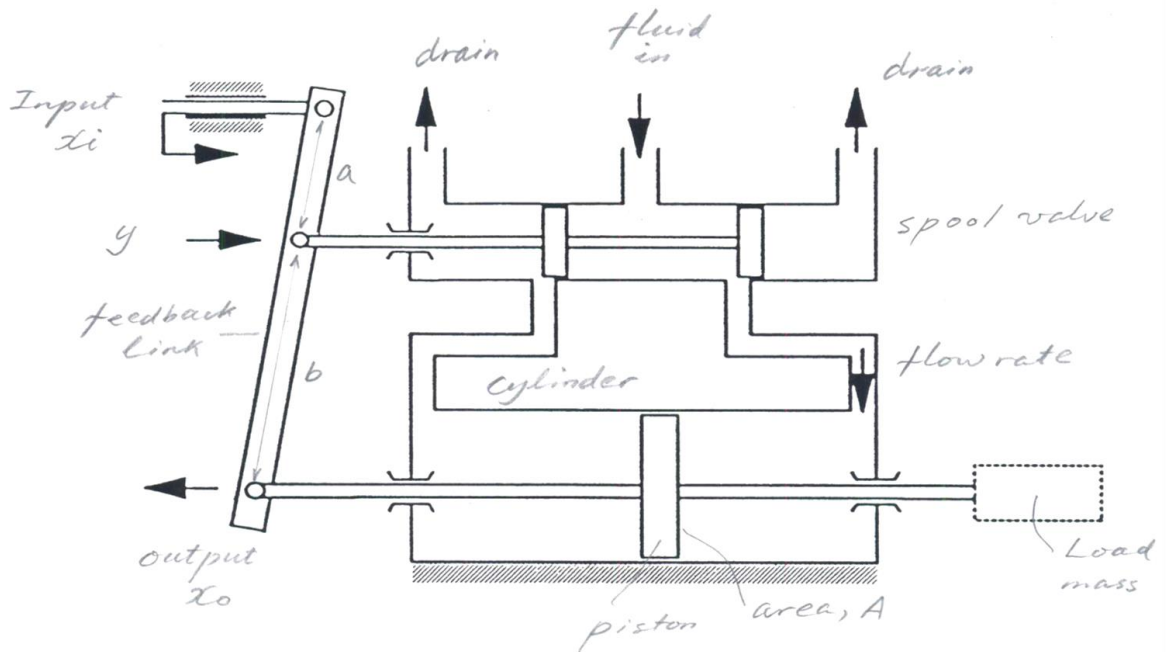
We will next consider the behaviour of two relatively simple control systems (1st and 2nd order systems) in some detail.

## CONTROL 8

Reading: Nise, Control Systems Engineering, Chapter 4, pp176-182.

### TRANSIENT RESPONSE - FIRST ORDER SYSTEMS

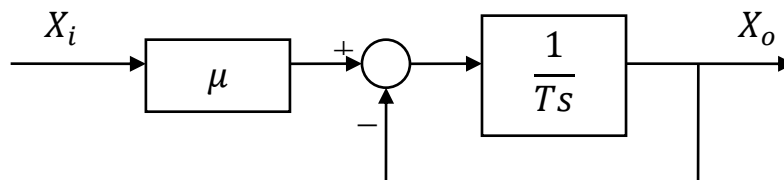
Example: Hydraulic Position Control System



Show that the **transfer function** for the above system may be written as:

$$G(s) = \frac{X_o(s)}{X_i(s)} = \frac{\mu}{1 + Ts}$$

and the **block diagram** can be drawn as follows:



Assume that the rate of flow  $q$  through the **Spool Valve** is given by:

$$q = K y \quad (1)$$

where the mass of the load has been neglected.

Taking Laplace transforms the transfer function for the spool valve is given by:

$$\frac{Q(s)}{Y(s)} = K$$

The velocity of the **Ram Piston** is governed by:

$$A \frac{d x_o}{dt} = q$$

where  $A$  is the cross-sectional area of the piston, and it has been assumed that  
i) there is no leakage and ii) compressibility effects have been ignored.

Taking Laplace transforms the transfer function for the ram piston is given by:

$$\frac{X_o(s)}{Q(s)} = \frac{1}{As} \quad (2)$$

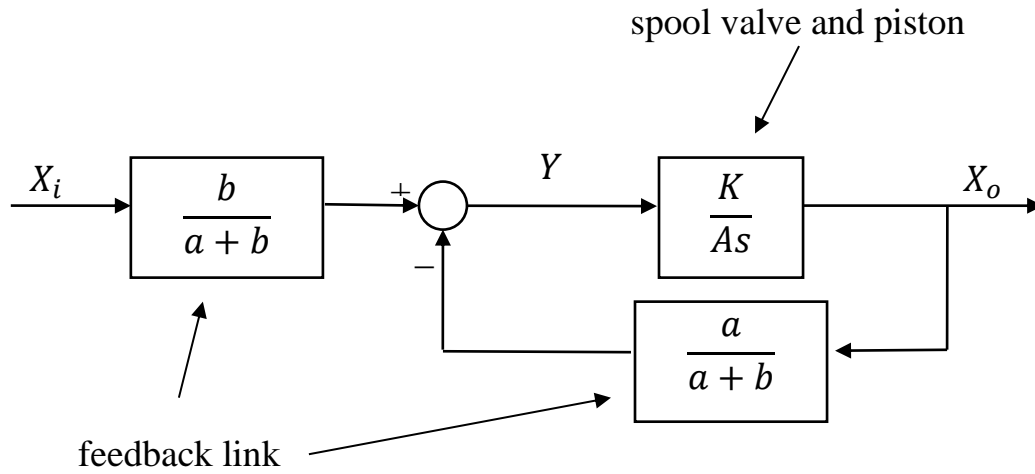
An expression for the input-output relation for a rigid lever system was derived in handout Control 4 (example (b)). Using this result the input-output relationship for the **Feedback Link** is:

$$y = \frac{b}{(a+b)} x_i - \frac{a}{(a+b)} x_o$$

Taking Laplace transforms the transfer function for the feedback link is given by:

$$Y(s) = \frac{b}{(a+b)} X_i(s) - \frac{a}{(a+b)} X_o(s) \quad (3)$$

Using equation, (1), (2) and (3), the block diagram for the **complete system** can be drawn as follows:



From the block diagram it can be seen that:

$$X_o(s) = \left[ X_i(s) \frac{b}{(a+b)} - X_o(s) \frac{a}{(a+b)} \right] \frac{K}{As}$$

Rearranging gives:

$$\left[ 1 + \frac{A(a+b)s}{Ka} \right] X_o(s) = \frac{b}{a} X_i(s)$$

$$[1 + Ts] X_o(s) = \mu X_i(s)$$

i.e.

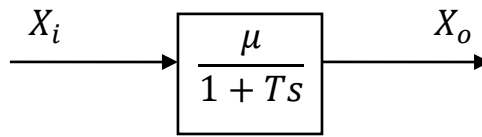
where  $T$  and  $\mu$  are the **time constant** and **steady-state gain** respectively, and:

$$T = \frac{A(a+b)}{Ka} \quad ; \quad \mu = \frac{b}{a}$$

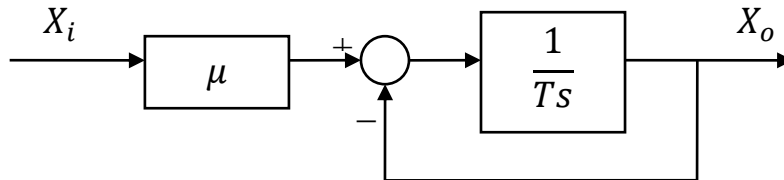
The transfer function for the **complete system** can be written as:

$$\boxed{\frac{X_o(s)}{X_i(s)} = \frac{\mu}{1 + Ts} \quad (4)}$$

This system is **first order** since the highest power of  $s$  in the denominator is one. The corresponding block for a first order system is:

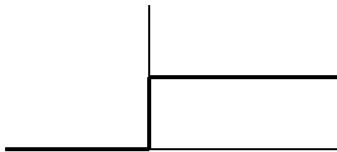


**Exercise:** Show that the block diagram for a system governed by equation (4) can be drawn as follows:



Next, let us consider the response of this system to the following standard inputs: i) a **step input** and ii) a **ramp-input**.

i) **Step-input**



$$t \leq 0 \quad x_i(t) = 0$$

$$t > 0 \quad x_i(t) = \bar{x}_i$$

From the table of Laplace transforms for a step input:

$$X_i(s) = \frac{\bar{X}_i}{s} \quad (5)$$

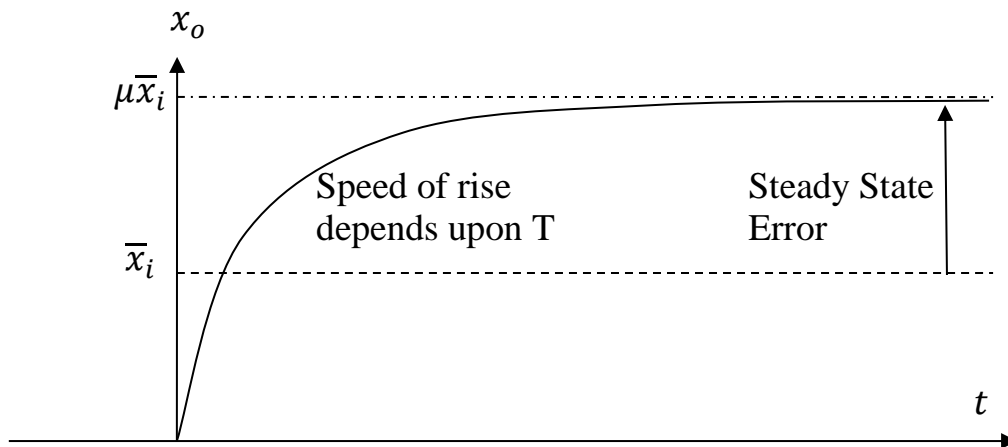
Using equations (4) and (5), the output of the system (to a step input) is given by:

$$X_o(s) = \frac{\mu \bar{X}_i}{s(1 + Ts)} \quad (6)$$

Assuming zero initial conditions and taking the inverse Laplace transform, the output in the time-domain can be shown to be:

$$x_o(t) = \mu \bar{x}_i (1 - e^{-(t/T)}) \quad (7)$$

i.e. the time-history of the response is:



It may be seen that there is an exponential rise to the final position where the speed of the rise is dependent upon  $T$  (ie a small value of  $T$  gives a quick rise, while a large value of  $T$  gives a slow rise).

After a "large" time interval ( $t \rightarrow \infty$ ) the **steady state response**  $x_{ss}$  is given by:

$$x_{ss} = \mu \bar{X}_i \quad (8)$$

The **final value theorem** provides an alternative means of determining the steady state response, in which case there is no need to calculate the inverse Laplace transform of equation (6). From the handout on Laplace transforms (Control 3), the **final value theorem** gives:

$$x_{ss} = \lim_{t \rightarrow \infty} x_o(t) = \lim_{s \rightarrow 0} s X_o(s) \quad (9)$$

For a step input  $X_o(s)$  is given by equation (5), and using equation (9) gives:

$$x_{ss} = \lim_{s \rightarrow 0} s \frac{\mu \bar{X}_i}{s(I+Ts)} = \mu \bar{X}_i \quad (10)$$

This agrees with equation (8) (i.e. the result calculated directly from the response time history).

The above method of calculating the steady state response can also be used to calculate the **steady state error**  $e_{ss}$ , where the error here is defined to be (see block diagram):

$$E(s) = X_i(s) - X_o(s)$$

Substituting for  $X_o(s)$  from equation (6) gives:

$$E(s) = X_i(s) - \frac{\mu X_i(s)}{(1+Ts)} = \frac{(1+Ts - \mu)}{(1+Ts)} X_i(s) \quad (11)$$

The steady state error  $e_{ss}$  is calculated using the **final value theorem** and equation (11) as follows:

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} s \frac{(1+Ts - \mu)}{(1+Ts)} X_i(s) \quad (12)$$

For a step input  $X_i(s)$  is given by equation (5), thus

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} s \frac{(1+Ts - \mu)}{(1+Ts)} \frac{\bar{X}_i}{s} \\ &= \lim_{s \rightarrow 0} \frac{(1+Ts - \mu)}{(1+Ts)} \bar{X}_i = (1 - \mu) \bar{x}_i \end{aligned} \quad (13)$$

i.e. this agrees with the response curve sketched above – remembering that the error is measured *from*  $x_i$  to  $x_o$ , or error =  $(x_i - x_o)$ .

In this example, it is clear that the operator would like to put a small input displacement  $x_i$  to get a displacement which is  $\mu$  times as large at the load mass ( $\mu$  is set by the dimensions of the feedback link). If the operator already knows this, it would perhaps be sensible to look at a new error term,  $e_{new} = \mu x_i - x_o$ . By following through the final value theorem again with this error term it can be shown that the  $e_{new\_ss} = 0$ . This can be confirmed by looking at the response curve which settles at the ‘new’ desired position of  $\mu x_i$ .

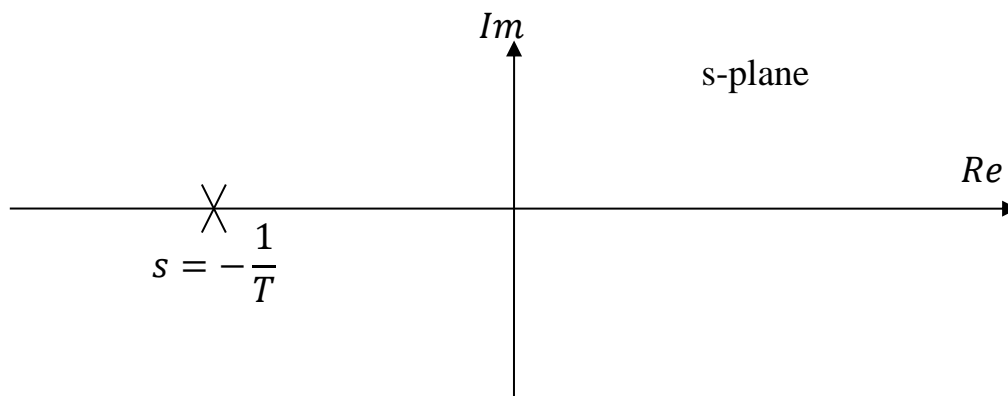
Before considering the ramp-input response of the system it is worthwhile considering which factors in the transfer function govern the transient response of the system.

It may be noted that the transient part of the response (i.e. the  $e^{-t/T}$  term in equation (7)) comes from the  $(s+1/T)$  factor in the denominator of the transfer function. From the definition given in handout Control 2, the **characteristic equation** (i.e.  $P(s)=0$ ) is given by:

$$s + \frac{1}{T} = 0$$

which has one real root at  $s=-1/T$ .

Thus it is the negative value of this root which determines the form of the transient response ( $e^{-t/T}$ ). In general, the roots of the characteristic equation are complex ( $s=a+jb$ ) and the values of the roots can be plotted on the complex s-plane. Further, the position of the roots of the characteristic equation on the s-plane characterise the nature of the transient response. In this particular case the root is real ( $s=-1/T$ ) and the root can be plotted as follows:

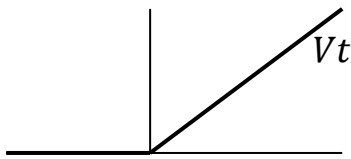


Thus it is the denominator (bottom line) of the transfer function which determines the nature of the transient response.

It follows that when the exponential term is decaying and getting smaller, i.e.  $(e^{-\frac{t}{T}})$ , the corresponding root is in the left hand (negative) side of the s-plane. If the root is in the right hand (positive) side of the s-plane, it means the exponential term is growing, i.e.  $(e^{\frac{t}{T}})$ , and the response would be getting larger – or further away from the desired position. This is a key element of **stability** which we will return to later.



ii) Ramp-input



$$t \leq 0 \quad x_i(t) = 0$$

$$t > 0 \quad x_i(t) = Vt$$

From the table of Laplace transforms:

$$X_i(s) = \frac{V}{s^2} \quad (14)$$

Using equations (4) and (14), the output of the system (to a ramp input) is given by:

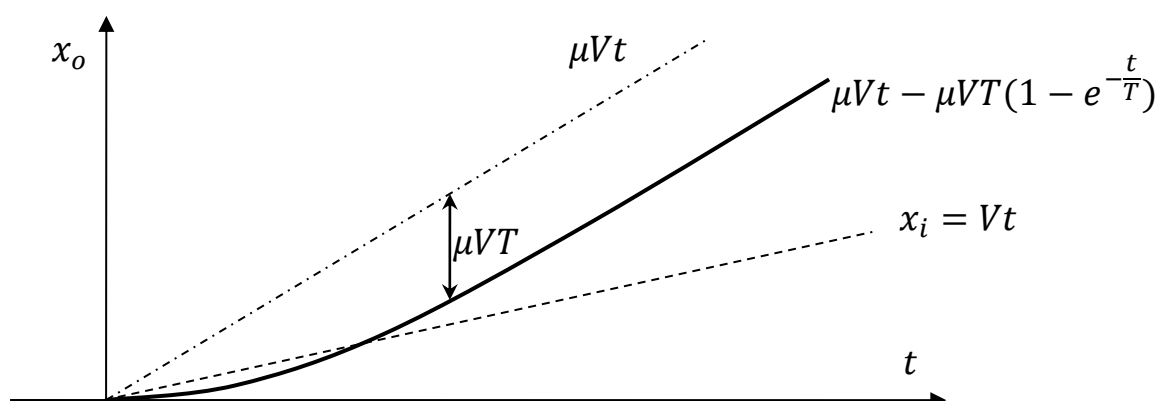
$$X_o(s) = \frac{\mu V}{s^2(1+Ts)} \quad (15)$$

Assuming zero initial conditions and taking the inverse Laplace transform, the output in the time-domain can be shown to be:

$$x_o(t) = \mu Vt - \mu VT \left(1 - e^{-\frac{t}{T}}\right)$$

$$x_o(t) = \mu Vt - \mu VT(1 - e^{-\frac{t}{T}}) \quad (16)$$

So the response in the time domain is:



After a "large" time interval ( $t > 4T$ ) such that  $e^{-t/T} \rightarrow 0$ , the **steady state response**  $x_{ss}$  given by:

$$x_{ss}(t) \approx \mu Vt - \mu VT$$

i.e. the steady-state response lags behind the input by a constant amount. This constant difference is known as the “velocity lag” of the system – the system will always be  $\mu VT$  behind the desired position.

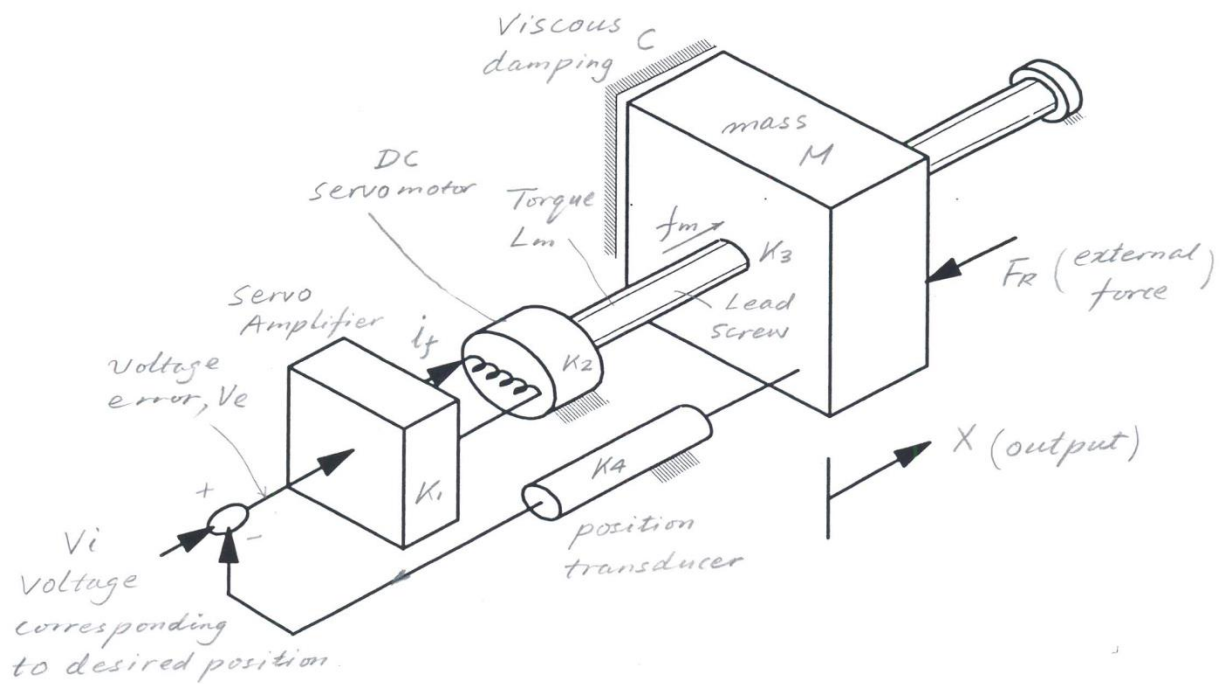
## CONTROL 9

Reading: Nise, Control Systems Engineering, Chapter 4, pp182-196; Dorf and Bishop, Modern Control Systems, Chapter 3, pp269-280

**Reading:**

### TRANSIENT RESPONSE - SECOND ORDER SYSTEMS

Example: Electro-mechanical Position Control System



Show that the **transfer functions** for the above system may be written as:

$$\frac{X(s)}{X_i(s)} = \frac{\omega_n^2}{s^2 + 2\gamma\omega_n s + \omega_n^2}$$

$$\frac{X(s)}{F_R(s)} = \frac{-1}{M(s^2 + 2\gamma\omega_n s + \omega_n^2)}$$

- i) The **Position Transducer** develops an output voltage  $V_x=K_4x$ . The error voltage  $V_e$  is then given by:

$$V_e = V_i - K_4 x$$

- ii) The **Servo Amplifier** develops a current  $i_f$  given by:

$$i_f = K_1 V_e = K_1 (V_i - K_4 x)$$

- iii) The **DC Servomotor** develops a torque  $l_m$  which is proportional to the field current  $i_f$ . ie:

$$l_m = K_2 i_f = K_1 K_2 (V_i - K_4 x)$$

- iii) The torque accelerates the **Lead Screw** and accelerates the mass  $M$ . Assuming that the moment of inertia of the lead screw and any loss of efficiency (ie friction) is negligible, the lead screw exerts a force  $f_m$  on the load mass given by:

$$f_m = K_3 l_m = K_1 K_2 K_3 (V_i - K_4 x)$$

where  $K_3=2\pi/\text{pitch}$  of leadscrew.

Assuming zero initial conditions and taking Laplace transforms gives:

$$F_m(s) = K_1 K_2 K_3 (V_i(s) - K_4 X(s)) \quad (1)$$

- iv) Assuming viscous damping the equation of motion for the **Load Mass M** is:

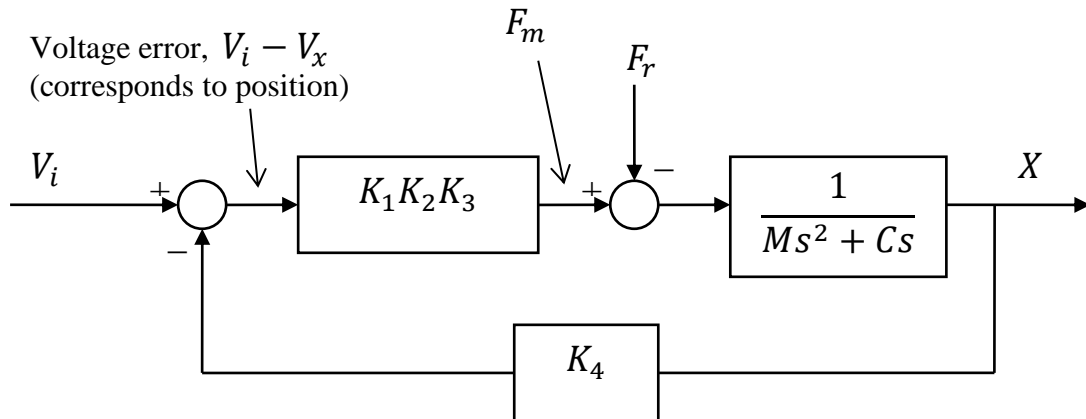
$$M\ddot{x} + C\dot{x} = F_m - F_R$$

Assuming zero initial conditions and taking Laplace transforms gives:

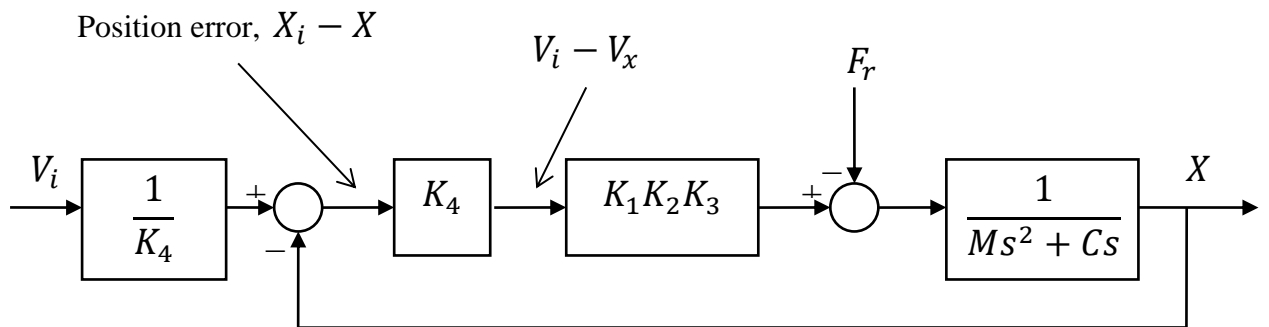
$$[M s^2 + Cs] X(s) = F_m(s) - F_R(s)$$

$$X(s) = \frac{F_m(s) - F_R(s)}{M s^2 + Cs} \quad (2)$$

Using equations (1) and (2), the block diagram for the system can be drawn as follows:



In this case it is more useful to focus on the actual position error, so the block diagram can be rearranged using the techniques given in *Control 6* to ensure that the summation block creates the position error:



Letting  $K=K_1K_2K_3K_4$ , from the block diagram we can find a relationship between the desired position ( $x_i$ ) and the load position ( $x$ ):

$$X(s) = \left( [X_i(s) - X(s)]K - F_R(s) \right) \frac{1}{Ms^2 + Cs}$$

Rearranging gives:

$$[Ms^2 + Cs + K]X(s) = K X_i(s) - F_R(s) \quad (3)$$

Equation (3) is usually rewritten in the 'standard' form using natural frequency and damping ratio as follows:

$$[s^2 + 2\gamma \omega_n s + \omega_n^2]X(s) = \omega_n^2 X_i(s) - \frac{F_R}{M}$$

where  $\omega_n$  and  $\gamma$  are the natural frequency and damping ratio, and

$$\frac{C}{M} = 2\gamma \omega_n \text{ and } \omega_n^2 = \frac{K}{M}$$

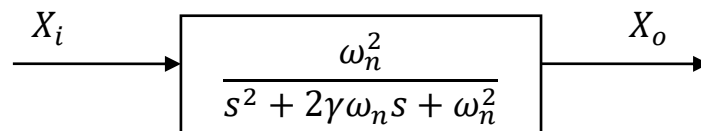
Thus the system has the following behaviour:

$$X(s) = \frac{\omega_n^2 X_i(s)}{s^2 + 2\gamma \omega_n s + \omega_n^2} - \frac{F_R(s)}{M(s^2 + 2\gamma \omega_n s + \omega_n^2)}$$

It can be useful to include the external (or disturbance) force as above. Often it is removed in order to understand the response of the output ( $X(s)$ ) to an input ( $X_i(s)$ ). Subsequently it may be studied to find the effect of the disturbance force on the overall system response. If  $F_R$  is set to zero the system transfer function is:

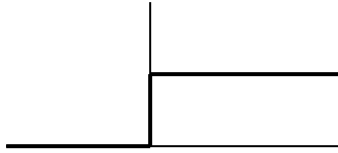
$$\frac{X(s)}{X_i(s)} = \frac{\omega_n^2}{s^2 + 2\gamma \omega_n s + \omega_n^2} \quad (4)$$

The system is **second order** since the highest power of  $s$  in the denominator is two. The corresponding block for this transfer function would be:



Consider now the response of this system to: i) a **step input** and ii) a **ramp-input**.

i) **Step-input**



$$t \leq 0 \quad V_i(t) = 0$$

$$t > 0 \quad V_i(t) = \bar{V}_i$$

From the table of Laplace transforms:

$$X_i(s) = \frac{\bar{V}_i}{K_4 s} = \frac{\bar{X}_i}{s} \quad (5)$$

Using equations (4) and (5), the output of the system is given by:

$$X_o(s) = \frac{\omega_n^2 \bar{X}_i}{s(s^2 + 2\gamma \omega_n s + \omega_n^2)} = \frac{\omega_n^2 \bar{X}_i}{s(s - p_1)(s - p_2)} \quad (6)$$

where  $p_1$  and  $p_2$  are roots of the characteristic equation  $s^2 + 2\gamma \omega_n s + \omega_n^2 = 0$  which are:

$$p_1 = -\gamma \omega_n + \omega_n \sqrt{\gamma^2 - 1} \quad ; \quad p_2 = -\gamma \omega_n - \omega_n \sqrt{\gamma^2 - 1}$$

Assuming a *unit step input* ( $\bar{X}_i = 1$ ) and using partial fractions gives:

$$X_o(s) = \frac{B}{s} + \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2}$$

where (for  $\gamma \neq 1$ )

$$B = 1 \quad ; \quad A_1 = -\frac{1}{2} - \frac{\gamma}{2\sqrt{\gamma^2 - 1}} \quad ; \quad A_2 = -\frac{1}{2} + \frac{\gamma}{2\sqrt{\gamma^2 - 1}}$$

Assuming zero initial conditions and taking the inverse Laplace transform, the output in the time-domain is:

$$x_0(t) = B + A_1 e^{p_1 t} + A_2 e^{p_2 t} \quad (7)$$

This solution, which is valid for  $\gamma \neq 1$ , gives rise to two distinct types of transient response which depend on the value of  $\gamma$ . The response will change characteristic shape as a result. These, together with the  $\gamma = 1$  case, are discussed below by considering the poles of the characteristic equation for the system ( $p_1$  and  $p_2$ ):

i)  $\gamma > 1$        $p_1$  and  $p_2$  are **real and unequal**. For this situation the response is overdamped (non-oscillatory). Typically the system behaves similarly to a first order response corresponding to the slower of the two poles.

ii)  $\gamma < 1$        $p_1$  and  $p_2$  are **complex conjugates** as are  $A_1$  and  $A_2$ . For this situation the response is underdamped (oscillatory). For this situation  $p_1$  and  $p_2$  can be re-expressed as:

$$p_1 = -\gamma \omega_n - j \omega_n \sqrt{1 - \gamma^2}$$

$$p_2 = -\gamma \omega_n + j \omega_n \sqrt{1 - \gamma^2}$$

and the response can be re-written as:

$$x_0(t) = \bar{X}_i \left[ 1 - \frac{e^{-\gamma \omega_n t}}{\sqrt{1 - \gamma^2}} \sin(\omega_n t \sqrt{1 - \gamma^2} + \phi) \right]$$

where

$$\phi = \tan^{-1} \frac{\sqrt{1 - \gamma^2}}{\gamma}$$

The maximum overshoot in this case occurs at

$$t = \frac{\pi}{\omega_n \sqrt{1 - \gamma^2}}$$

and has value:

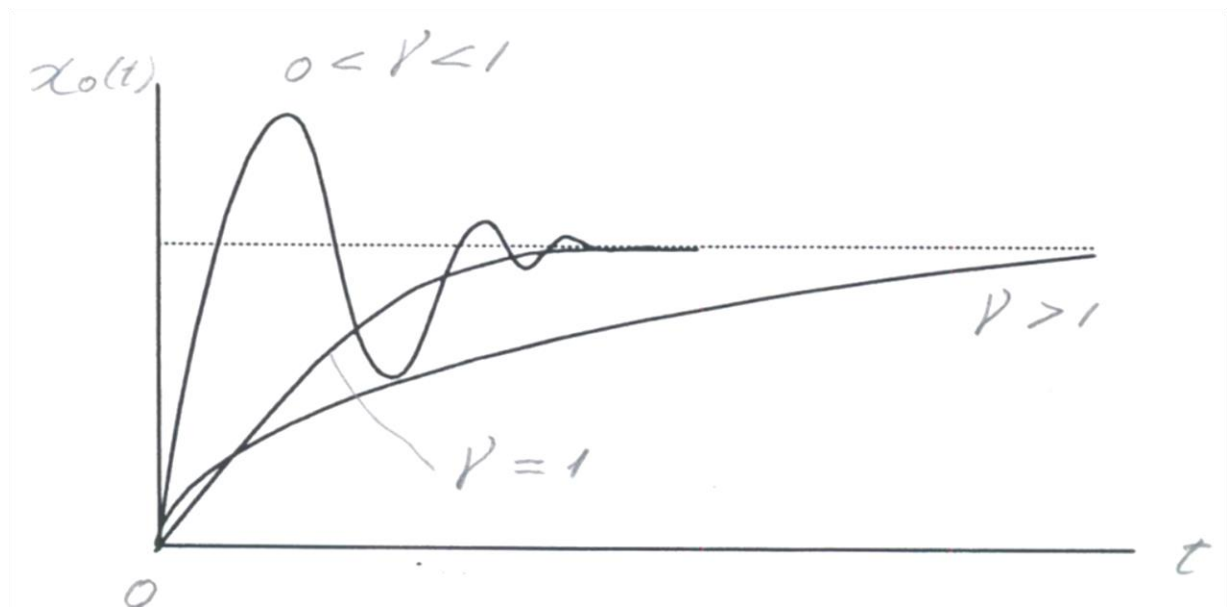
$$x_{\max} = \bar{X}_i \left\{ 1 + e^{-\frac{\gamma \pi}{\sqrt{1 - \gamma^2}}} \right\}$$



- iii)  $\gamma = 1$   $p_1$  and  $p_2$  are **real and equal** ( $=-\omega_n$ ). Here the response is said to be critically damped and can be shown to be given by:

$$x_0(t) = \bar{X}_i \left\{ 1 - (1 + \omega_n t) e^{-\omega_n t} \right\}$$

The transient response of the above three cases are summarised below:

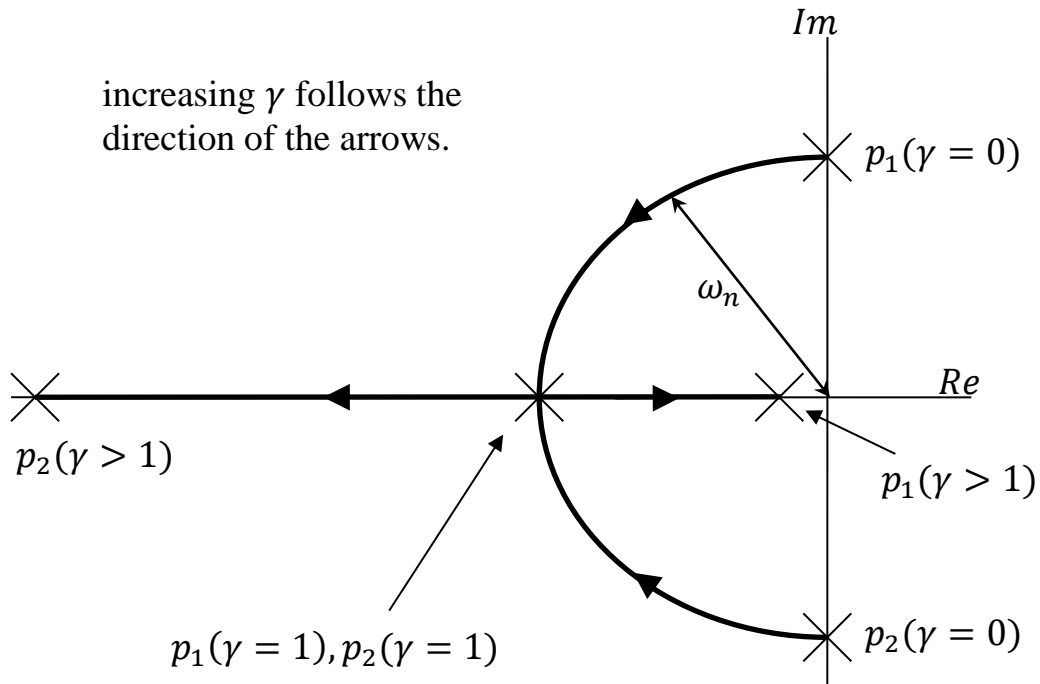


So, the transient response is governed by the roots of the **characteristic equation**, i.e.

$$p_1 = -\gamma \omega_n + \omega_n \sqrt{\gamma^2 - 1} \quad ; \quad p_2 = -\gamma \omega_n - \omega_n \sqrt{\gamma^2 - 1}$$

where the position of the roots on the s-plane characterises the nature of the transient response.

Consider now what happens to the position of the roots of the characteristic equation on the s-plane when  $\gamma$  is varied from 0 to  $\infty$  when  $\omega_n$  is held fixed. The results for this situation are shown below.

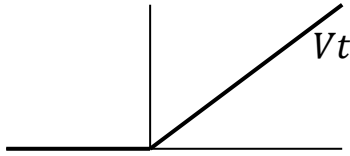


The roots trace out loci on the complex  $s$ -plane as  $\gamma$  increases from 0 and follows the direction of the arrows.

Notes:

- i. Like the first order case, having the poles in the left hand (negative) side of the  $s$ -plane implies the exponential terms are decaying and getting smaller. Therefore, for the system to be stable the poles must be in the left hand side of the  $s$ -plane.
- ii. The further to the left, the faster the exponential decaying term, so for example for  $\gamma > 1$ , the response due to pole  $p_2$  will be a faster decay than the response for pole  $p_1$ . The overall response will look very much like a first order system with a pole at  $p_1$  - the effect of  $p_2$  will decay relatively quickly by comparison.
- iii. A pair of complex poles ( $0 \leq \gamma < 1$ ) gives rise to an oscillatory response. The speed of decay and size of the overshoots will depend on the damping ratio  $\gamma$ .

ii) Response to a Ramp input



$$t \leq 0 \quad V_i(t) = 0$$

$$t > 0 \quad V_i(t) = \Omega t$$

From the table of Laplace transforms:

$$V_i(s) = \frac{\Omega}{s^2}$$

and from the block diagram

$$X_i(s) = \frac{V_i(s)}{K_4} = \frac{\Omega}{s^2 K_4} = \frac{\Omega_X}{s^2} \quad (8)$$

Using equations (4) and (8), the output of the system is given by:

$$X_o(s) = \frac{\omega_n^2 \Omega_X}{s^2 (s^2 + 2\gamma \omega_n s + \omega_n^2)} \quad (9)$$

Assuming zero initial conditions and taking the inverse Laplace transform, the output in the time-domain is:

$$x_o(t) = \Omega_X \left[ t - \frac{2\gamma}{\omega_n} + A_1 e^{p_1 t} + A_2 e^{p_2 t} \right] \quad (10)$$

For large values of  $t$  the transient components of the response (ie the terms involving  $A_1$  and  $A_2$ ) are negligible, giving rise to a “velocity lag”.

From the block diagram for the system considered, the error (i.e. the difference between the desired response (input) and the output)  $E(s)$  is given (for  $F_R=0$ )

$$E(s) = X_i(s) - X(s) = \left( \frac{M s^2 + Cs}{M s^2 + Cs + K} \right) X_i(s) \quad (11)$$

by:

For a ramp input  $X_i(s)$  is given by equation (8). Thus  $E(s)$  is given by:

$$E(s) = \left( \frac{M s^2 + Cs}{M s^2 + Cs + K} \right) \frac{\Omega_x}{s^2} \quad (12)$$

Using the **final value theorem** the steady state error is given by:

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} e(t) = \lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} \left( \frac{Ms + C}{M s^2 + Cs + K} \right) \Omega_x \\ &= \frac{C}{K} \Omega_x = \frac{2\gamma}{\omega_n} \Omega_x \end{aligned} \quad (13)$$

Thus the steady state error is minimised by choosing  $\gamma$  to be small, but this gives a highly oscillatory transient response. A compromise is therefore needed between how accurate the system in steady state is and how quickly the system reaches steady state.

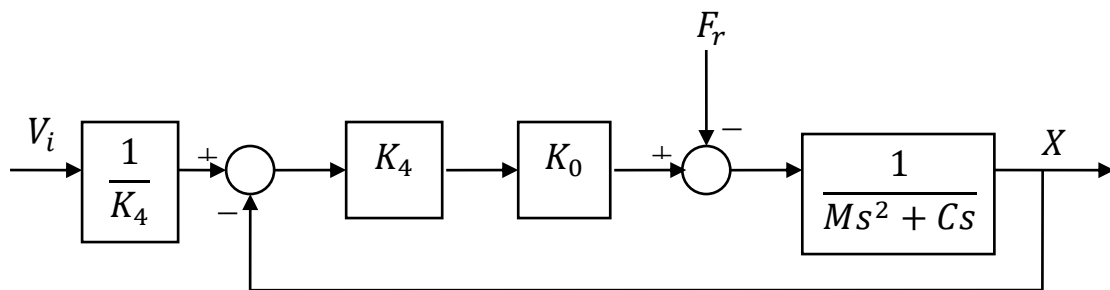
## CONTROL 10

Reading: Nise, Control Systems Engineering, Chapter 9, pp527-533; Dorf and Bishop, Modern Control Systems, Chapter 4, pp275-289

### IMPROVING TRANSIENT AND STEADY-STATE PERFORMANCE

It was shown in handout Control 9 that there is sometimes a compromise between how accurate the system in steady state is and how quickly the system reaches steady state. For this reason, a number of ways of modifying the control of the system to improve transient and steady state performance have been developed. Some of the most commonly used ways of modifying proportional control systems are presented here.

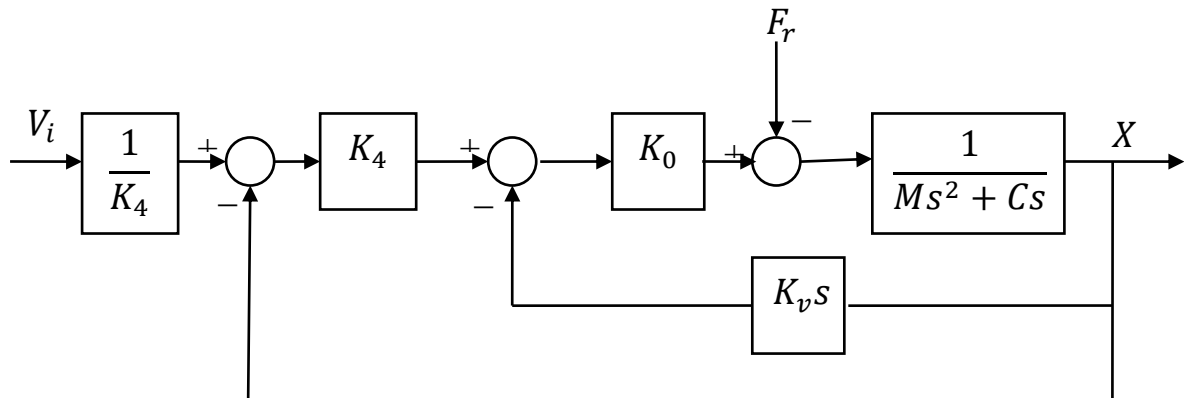
Their influence is examined using the electro-mechanical position control system described in Control 9 (second order system) as an example. Note that it is clear from the block diagram that this is an example of proportional control – the control signal is a constant gain times the position error:



(for simplicity the gain of the servo amplifier, the servo, and the lead screw have been collected into one constant gain, i.e.  $K_0 = K_1K_2K_3$ )

#### a) Velocity Feedback

In addition to feedback of the output position the rate of change of output is fed back. For a position control system this implies feeding back velocity. This can be achieved by using a tachogenerator on the output of the system or on the servo-motor shaft which is added into the voltage supplying the servo amplifier. The block diagram then becomes:



The governing equation follows as:

$$\left[ Ms^2 + (C + K_0 K_v)s + K_0 K_4 \right] X(s) = K_0 K_4 X_i(s) - F_R(s)$$

The velocity feedback has the same effect as viscous damping. Its advantage is that it does not absorb power as viscous damping does, so reasonable damping ratios can be used in large systems without the penalty of high power consumption.

However, the steady-state error to a ramp input remains and can be shown to be:

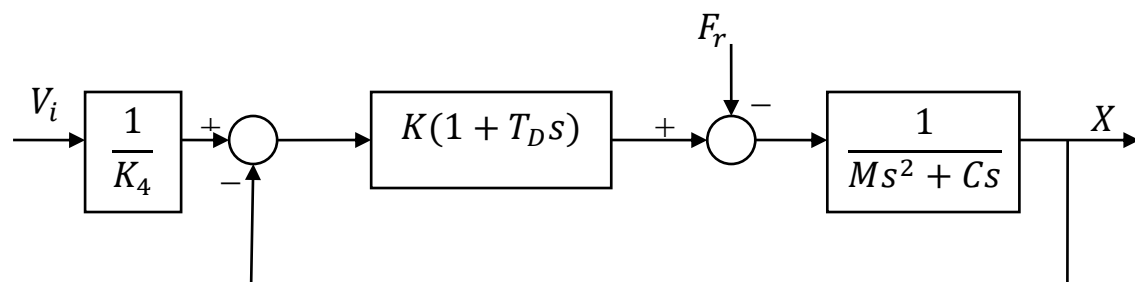
$$e_{ss} = \Omega_X \frac{[C + K_0 K_v]}{K_0 K_4}$$

Comparing this to the steady state error for the unchanged system (eq 13 in Control 9) shows that the addition of velocity feedback has increased the steady state error (assuming a positive value for  $K_v$  in order to add more damping effect).

### b) Proportional and Derivative Control (P+D)

The proportional error is modified by adding to it a quantity proportional to the first derivative of error wrt time (i.e. the rate of change of error). The differentiation can be carried out electronically.

The block diagram for the system is



The governing equation follows as:

$$\left[ Ms^2 + (C + KT_D)s + K \right] X(s) = K[1 + T_Ds] X_i(s) - F_R(s)$$

Notes:

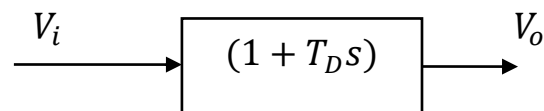
- i) The system damping has been increased without penalty of high power consumption.
- ii) The  $KT_DsX_i(s)$  term indicates that the output controller is modified when the input is changing rapidly. It can therefore be thought of as *anticipating a large overshoot and improves transient response*.
- iii) The steady-state error to a ramp input in this case can be shown to be:

$$e_{ss} = \Omega_x \frac{C}{K}$$

which is independent of  $T_D$ . So the transient will be changed without affecting the steady state error (it is now the same as it was for the unchanged system).

- vi) Derivative action tends to amplify 'noise' in the system. This can be explained as follows:

Consider the following system:



such that:

$$v_o(t) = v_i(t) + T_D \frac{dv_i(t)}{dt}$$

where  $T_D$  is the derivative time.

If the input signal  $v_i$  is such that:

$$v_i(t) = V + v_n \sin \omega t$$

then the output signal will be:

$$V_o(t) = V + v_n \sin \omega t + T_D \omega v_n \cos \omega t$$

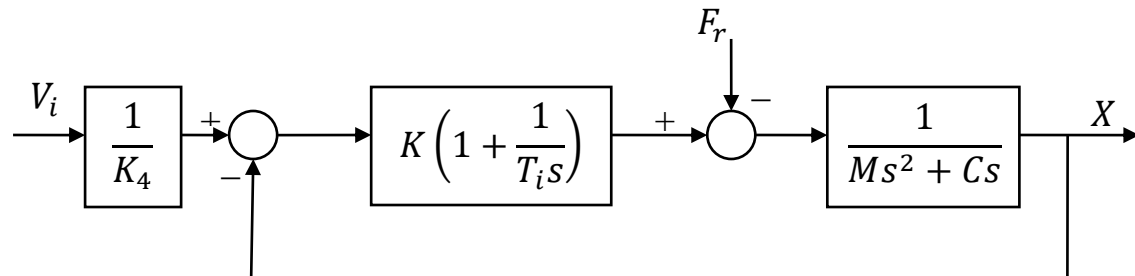
Now if  $\omega > 1/T_D$  then the noise in the signal will amplified significantly.

The solution to this problem is to use filters to reduce high frequency noise in a system. This is particularly important in (digital) computer controlled systems.

### c) Proportional and Integral Control (P+I)

The error signal is modified by adding to it a quantity proportional to the integral of the error wrt time. This can also be carried out electronically.

The block diagram for the system is



where the integration process is represented by  $1/T_i s$ .

The governing equation is now:

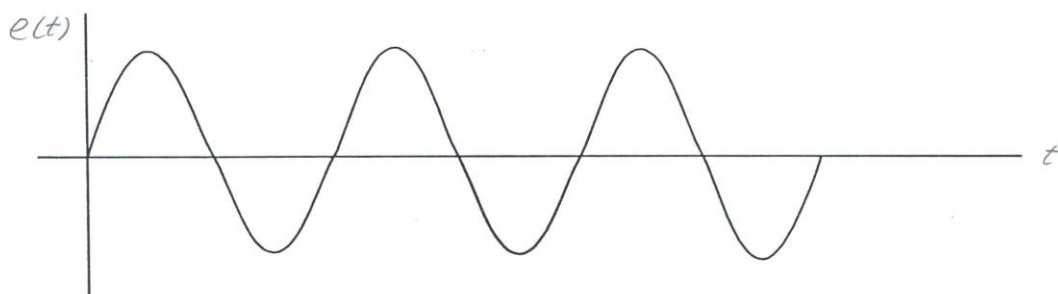
$$\left[ M s^3 + C s^2 + K s + \frac{K}{T_i} \right] X(s) = \left[ K s + \frac{K}{T_i} \right] X_i(s) - s F_R(s)$$

Typically integral control action is added to eliminate steady state error. As the error is integrated (i.e. the area under the error curve), even the smallest error eventually produces a corrective signal driving the system to eliminate the error. The steady state error to a ramp signal can now be shown to be:

$$e_{ss} = 0$$

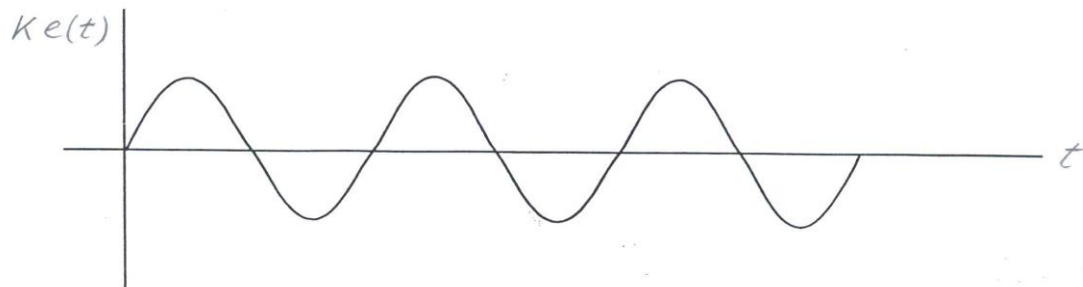
Tuning  $T_i$  will help to reduce the steady state error faster as  $T_i$  decreases. Decreasing by too much could cause the system to become unstable though.

In addition it should be noted that integral action tends to destabilise the system. To explain this, consider a sample time-history of the error:



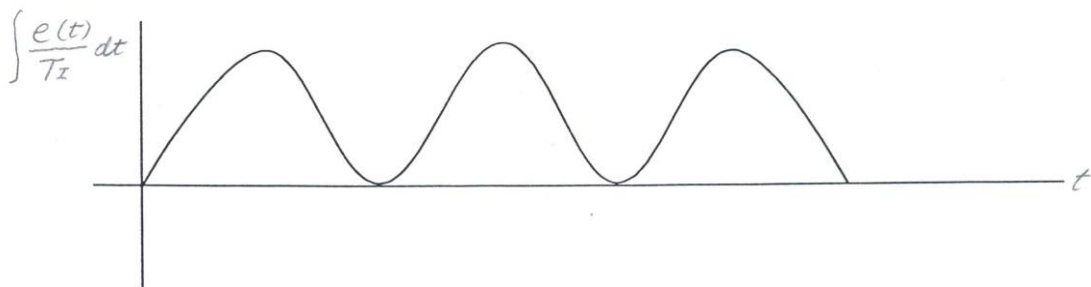


From this the time-history of the restoring action provided by **proportional action** can be calculated (i.e.  $K$  (say) multiplied by the error):



Thus the restoring force provided by proportional action changes sign whenever the error changes.

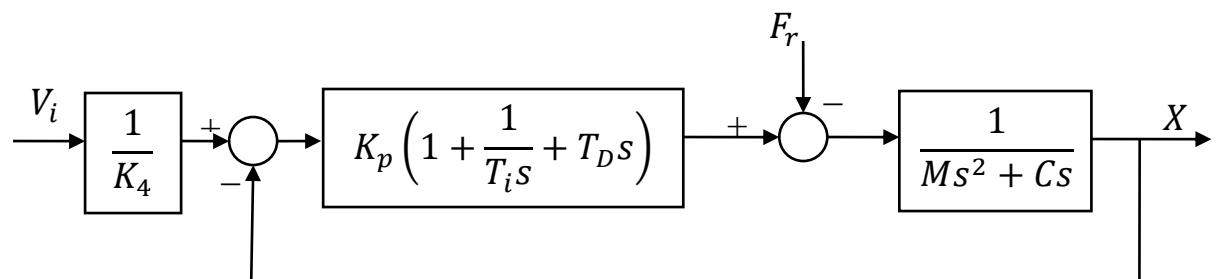
Now consider the restoring action provided by integral action.



It is obvious from this that the integral action does not change sign when the error changes. Thus providing a destabilising effect.

### c) Proportional-Integral-Derivative Control (PID)

The error signal is modified with a three term controller combining the proportional, integral and derivative controllers seen above:



This is easily the most common controller used in industry. The values for  $K_p$ ,  $T_i$  and  $T_D$  are tuned to get some desired transient and steady state response for the system. Note that the P+D and P+I, or even a simple proportional controller can be produced using a PID controller but setting some terms to zero.

Increasing  $K_p$  and  $1/T_i$  tends to reduce the steady state error, but make stability worse. Increasing  $T_D$  tends to improve stability.

However, there are a few possible drawbacks:

- i. The values for  $K_p$ ,  $T_i$  and  $T_D$  interact with each other – if you are getting poor results, it may not be obvious which values need adjusting. For the P+I or P+D controllers optimum values can be found to ensure a given response (beyond this modules scope).
- ii. PID controllers encourage the engineer to use all three terms, when one term may not be adding anything to the response and a simpler controller may be beneficial.
- iii. Established tuning strategies may require the system to be driven to instability, which can be dangerous.

## CONTROL 11

### Transient Response – Third and Higher-order systems

In the general case, the input-output relationship for a system is of the form:

$$X_0(s) = G(s)X_i(s)$$

Where  $G(s) = Q(s)/P(s)$  is the system transfer function.

If  $P(s)$  is a polynomial of order  $N$  then the characteristic equation  $P(s) = 0$  will have  $n$  roots,  $p_1, p_2, \dots, p_N$  and we can write:

$$X_0(s) = \frac{Q(s)X_i(s)}{(s-p_1)(s-p_2)\dots(s-p_N)} \quad (1)$$

The roots  $p_1, p_2, \dots, p_N$  are called the **poles** of the transfer function. These poles are values for which the magnitude of the transfer function will become infinite. The values of  $s$  for which  $Q(s)$  will be zero, which are termed the **zeros** of the transfer function. For real systems, the poles are real ( $s = \sigma_r$ ) or occur in complex conjugate pairs ( $s = \sigma_c \pm j\omega_c$ ).

If the system has distinct poles and is subjected to a unit step input, the output will be of the form:

$$X_0(s) = \frac{1}{s(s-p_1)(s-p_2)\dots(s-p_N)} = \frac{1}{s} + \sum_{r=1}^{N_R} \frac{A_R}{(s-\sigma_r)} + \sum_{c=1}^{N_C} \frac{A_c}{(s-\sigma_c)^2 + \omega_c^2}$$

The time domain response will be given by:

$$x_0(t) = 1 + \sum_{r=1}^{N_R} B_r e^{\sigma_r t} + \sum_{c=1}^{N_C} B_c e^{\sigma_c t} \sin(\omega_c t) \quad (2)$$

As can be seen from equation (2), the transient response of third and higher order systems consists of a summation of exponential terms and exponentially modulated sinusoidal terms similar to those generated by first order and second order systems. The precise form of the transient response will depend on the values of the magnitudes  $B_r$  and  $B_c$  and where these poles lie on the complex  $s$ -plane.

The transient response associated with various pole locations is shown in figure 1. Note that because complex poles occur in conjugate pairs, only the upper half of the  $s$ -plane is shown. Oscillatory responses occur where there is a complex pole, and the natural frequency is  $\omega$ .

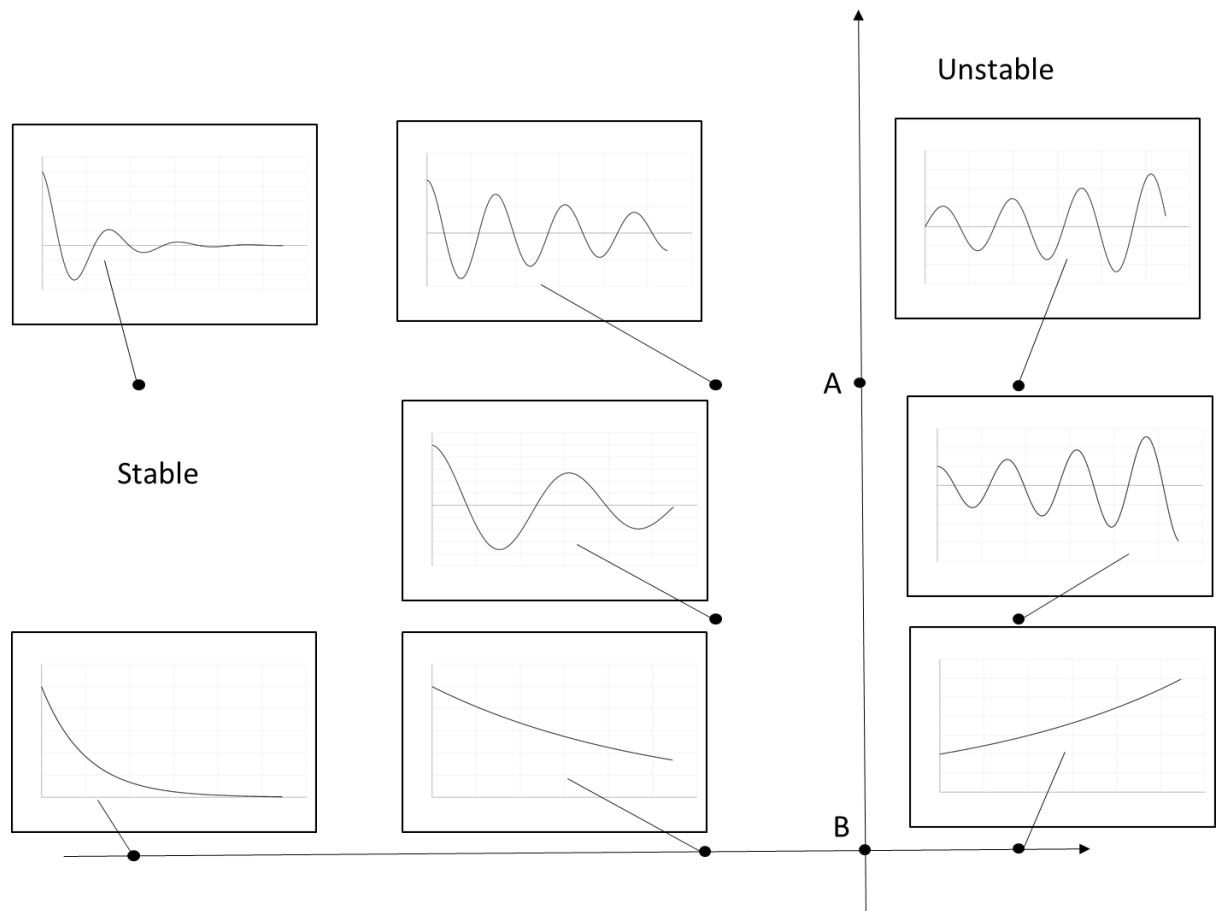


Figure 1. Effect of pole location on transient response

It can be seen that if a pole has a positive real part (it lies in the right hand half of the complex s-plane) the corresponding time response will increase with time and the system would be unstable.

Of the poles on the left hand side of the S-plane, those furthest from the imaginary axis will give contributions to the response that decay most rapidly. The system response will therefore be most strongly influenced by the poles closest to the imaginary axis (dominant poles)

Question: would would the response at A and B look like?

ACR, 27 January 2021

## CONTROL 12

Reading: Nise, Control Systems Engineering, Chapter 6, pp325-332; Dorf and Bishop, Modern Control Systems, Chapter 7, pp408-421.

### Routh-Hurwitz Stability Criteria

It has been shown that the stability of a system is dependent upon whether or not the roots of the characteristic equation (poles of the system transfer function) lie in the right hand half of the s-plane. The presence of a root with a positive real part means that the output of a system will grow indefinitely with time after a disturbance is applied and the system is unstable.

The Routh-Hurwitz criteria provide the simplest method to determine system stability, based on a straightforward algebraic manipulation of the characteristic function. They are presented here without the proof – this can be found in the more advanced textbooks (it's not in Nise).

In general the characteristic equation  $P(s)$  can be written as:

$$P(s) = a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_n = 0 \quad (1)$$

Where  $P(s)$  is an nth order polynomial.

For first and second order systems ( $n=1$  or  $2$ ) the roots of this equation can be calculated easily. However, for higher order systems the task of determining the roots can be more time-consuming. For such systems it is convenient to consider the Routh-Hurwitz criteria which provide a means of determining whether any of the roots lie to the right of the imaginary axis in the s-plane, without having to determine the values of these roots.

The Routh-Hurwitz criteria are:

1. A *necessary* but *not sufficient* condition that no root of equation (1) lies in the right half of the s-plane is that all of the coefficients  $a_0, a_1, \dots, a_n$  are non-zero and have the same sign. Thus, provided that  $a_0$  is positive, if one of the other coefficients is negative, or one of the powers of  $s$  is absent, then at least one root of the characteristic function lies to the right of the imaginary axis and therefore the system is unstable.
2. Provided that condition (1) is satisfied, then the *necessary* and *sufficient* condition that no root of equation (1) lies on the right hand side of the s-plane is that the Hurwitz determinants of the polynomial must be positive, where the Hurwitz determinants are given by:

$$D_1 = a_1$$

$$D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$$

$$D_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 0 & a_2 & a_4 \end{vmatrix}$$

For an equation of order n, there will be n determinant equations. Some of the arithmetic involved in calculating these determinants can be avoided by using the Routh array which is formed as follows:

$$\begin{array}{c|cccccc} s^n & a_0 & a_2 & a_4 & a_6 & \dots \\ s^{n-1} & a_1 & a_3 & a_5 & a_7 & \dots \\ s^{n-2} & b_1 & b_2 & b_3 & \dots & \dots \\ s^{n-3} & c_1 & c_2 & c_3 & \dots & \dots \\ s^{n-4} & d_1 & d_2 & d_3 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ s^0 & \dots & \dots & \dots & \dots & \dots \end{array}$$

The first two rows are formed directly from the coefficients of equation (1), while the values in the third and subsequent rows are calculated as follows:

$$\begin{aligned} b_1 &= \frac{a_1 a_2 - a_0 a_3}{a_1} & b_2 &= \frac{a_1 a_4 - a_0 a_5}{a_1} & b_3 &= \frac{a_1 a_6 - a_0 a_7}{a_1} & \dots \\ c_1 &= \frac{b_1 a_3 - a_1 b_2}{b_1} & c_2 &= \frac{b_1 a_5 - a_1 b_3}{b_1} & \dots & \dots \\ d_1 &= \frac{c_1 b_2 - b_1 c_2}{c_1} & \dots & \dots & \dots & \dots \end{aligned}$$

On completion, the array has  $n + 1$  rows and the last row is always indicated by  $s^0$ .

Every change of sign in the first **column** of the array indicates the presence of a root which lies to the right of the imaginary axis in the s-plane. Hence, for the system to be stable, all values of the first column must be positive.

The procedure breaks down if either:

- a) A zero appears in the first column;
- Or
- b) A complete row of zeros appears so that the array cannot be completed.

For this course, examples will avoid these exceptions; however, there are standard techniques where these problems can be overcome, these can be found in the textbooks (e.g. Nise page 332).

### Example 1

The characteristic equation of a system is:

$$2s^3 + 4s^2 + 4s + 12 = 0$$

Is the system stable or unstable? If it is unstable, how many roots lie in the right half of the s-plane?

Given that the coefficients of the characteristic equation are non-zero and have the same sign, the stability of the system can be investigated using criterion (2):

Provided that condition (1) is satisfied, then the *necessary* and *sufficient* condition that no root of equation (1) lies on the right hand side of the s-plane is that the Hurwitz determinants of the polynomial must be positive.

We begin by constructing the Routh array for the system:

$$\begin{array}{c|cccc}
 s^n & a_0 & a_2 & a_4 & a_6 & \dots \\
 s^{n-1} & a_1 & a_3 & a_5 & a_7 & \dots \\
 s^{n-2} & b_1 & b_2 & b_3 & \dots & \dots \\
 s^{n-3} & c_1 & c_2 & c_3 & \dots & \dots \\
 s^{n-4} & d_1 & d_2 & d_3 & \dots & \dots \\
 \dots & \dots & \dots & \dots & \dots & \dots \\
 \dots & \dots & \dots & \dots & \dots & \dots \\
 s^0 & \dots & \dots & \dots & \dots & \dots
 \end{array}$$

From the characteristic equation, we need a  $3 \times 4$  Routh Table:

$$\begin{aligned}
 b_1 &= \frac{a_1 a_2 - a_0 a_3}{a_1} = \frac{16 - 24}{4} = -2 & b_2 &= \frac{a_1 a_4 - a_0 a_5}{a_1} = 0 & b_3 &= 0 \\
 c_1 &= \frac{b_1 a_3 - a_1 b_2}{b_1} = \frac{-24}{-2} = 12 & c_2 &= \frac{b_1 a_5 - a_1 b_3}{b_1} = 0
 \end{aligned}$$

$$\begin{array}{c|ccc}
 s^3 & 2 & 4 & 0 \\
 s^2 & 4 & 12 & 0 \\
 s & -2 & 0 & 0 \\
 s^0 & 12 & 0 & 0
 \end{array}$$

There are two sign changes in the first column, therefore the system is unstable and two roots of the characteristic equation will lie on the right half of the s-plane.

### Example 2

A 'Three term controller' has the following Transfer Function:

$$\frac{\theta_c(s)}{\theta_\varepsilon(s)} = 20 \left[ 1 + T_D s + \frac{1}{T_i s} \right]$$

Where  $T_D$  = Derivative time and  $T_i$  = Integral time or Reset Time.

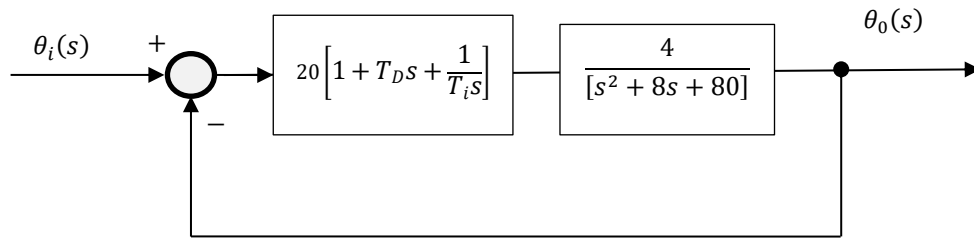
This controller is used to control a process with Transfer Function:

$$\frac{\theta_0(s)}{\theta_c(s)} = \frac{4}{[s^2 + 8s + 80]}$$

Unity feedback is used.

- a) If integral action is NOT employed, find the value of  $T_D$  required to give a closed loop damping ratio of unity.
- b) If this value of  $T_D$  is used, determine the minimum value of  $T_i$  that can be used if the system is to remain stable.
- c) When the values of  $T_D$  and  $T_i$  calculated above are employed, determine the nature of the transient response of the system to an arbitrary input.

The block diagram is:



Thus the overall transfer function for the system is:

$$\begin{aligned} \theta_0(s) &= (\theta_i - \theta_0) 20 \left[ 1 + T_D s + \frac{1}{T_i s} \right] \frac{4}{(s^2 + 8s + 80)} \\ \frac{\theta_0(s)}{\theta_i(s)} &= \frac{\left[ 1 + T_D s + \frac{1}{T_i s} \right] \frac{80}{(s^2 + 8s + 80)}}{1 + \left[ 1 + T_D s + \frac{1}{T_i s} \right] \frac{80}{(s^2 + 8s + 80)}} = \frac{80 \left[ 1 + T_D s + \frac{1}{T_i s} \right]}{(s^2 + 8s + 80) + 80 \left[ 1 + T_D s + \frac{1}{T_i s} \right]} \quad [1] \end{aligned}$$

- a) For the case without integral action, the integrating term  $\frac{1}{T_i s}$  is zero and the transfer function becomes:

$$\frac{\theta_0(s)}{\theta_i(s)} = \frac{80[1 + T_D s]}{(s^2 + 8s + 80) + 80[1 + T_D s]} = \frac{80[1 + T_D s]}{(s^2 + 8(1 + 10T_D)s + 160)}$$

This system will be second order, with  $\omega_n^2 = 160$  ( $\omega_n = 4\sqrt{10} \text{ s}^{-1}$ ) and

$$2\gamma\omega_n = 8(1 + 10T_D).$$

To achieve a unity damping ratio,  $\gamma = 1.0$ ,  $T_D$  is given by:

$$\gamma = \frac{8(1+10T_D)}{2\omega_n} = 1 = \frac{8(1+10T_D)}{2 \times 4\sqrt{10}} \text{ so } (1 + 10T_D) = \sqrt{10} \text{ and}$$

$$T_D = \frac{\sqrt{10} - 1}{10} = 0.216 \text{ s}$$



b) With integral action present, and  $T_D = 0.216$  s, the transfer function becomes:

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{80 \left[ 0.216s^2 + s + \frac{1}{T_i} \right]}{s^3 + 8(1 + 10 \times 0.216)s^2 + 160s + \frac{80}{T_i}} \quad [1b]$$

In order to investigate the stability of the system it is necessary to consider the location of the roots of the characteristic equation on the s-plane. If the system is stable, then the roots will lie on the left half of the s-plane.

From the transfer function (1b), the characteristic equation is:

$$s^3 + 8(1 + 10 \times 0.216)s^2 + 160s + \frac{80}{T_i} = s^3 + 8(3.16)s^2 + 160s + \frac{80}{T_i}$$

$$s^3 + 25.28s^2 + 160s + \frac{80}{T_i} = 0 \quad [2]$$

Note that here the coefficients of the characteristic equation are non-zero and all have the same sign for positive  $T_i$ .

For stability, we will have to use the Routh-Hurwitz criterion: The Routh array is given by:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} = \frac{25.28 \times 160 - \frac{80}{T_i}}{25.28} = 160 - \frac{3.165}{T_i}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} = 0$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} = \frac{b_1 \frac{80}{T_i} - 0}{b_1} = \frac{80}{T_i}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{a_1} = 0$$

$s^3$	1	160	0
$s^2$	25.28	$\frac{80}{T_i}$	0
$s$	$160 - \frac{3.165}{T_i}$	0	0
$s^0$	$\frac{80}{T_i}$	0	0

For the system to be stable, there must be no change of sign in the first column:

Therefore:

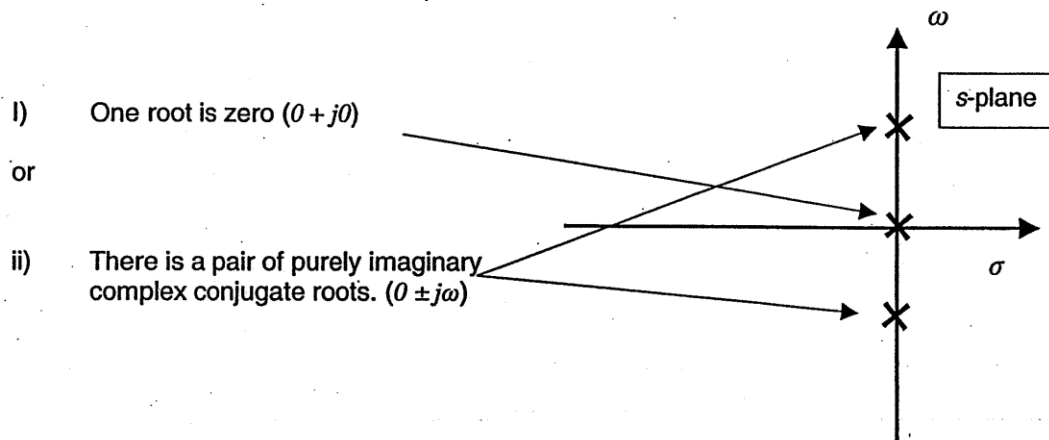
- i)  $\frac{80}{T_i} > 0$  This is satisfied for  $T_i > 0$
- ii)  $160 - \frac{3.165}{T_i} > 0$  i.e.  $T_i > 0.01978$  s

Thus if  $T_i > 0.01978$  then there are no sign changes in the first column and the system will be stable. If  $T_i$  is below this threshold, then there will be two roots lying in the right half of the s-plane.

- c) Using these values for  $T_i$  and  $T_D$  the system will be on the borderline of stability/instability. Putting these values into the characteristic equation gives:

$$s^3 + 25.28s^2 + 160s + \frac{80}{0.01978} = s^3 + 25.28s^2 + 160s + 4044.49 = 0 \quad [3]$$

The roots of the characteristic equation must be such that either:



Substituting  $s = 0$  ( $0 + j0$ ) into equation (3) indicates that the origin is not a root of the characteristic equation, as there is a finite value. There must therefore be a pair of complex conjugate roots to equation (3) of the type ( $s \pm jb$ ).

Therefore:

$$(s + jb)(s - jb)(s + a) = s^3 + 25.28s^2 + 160s + 4044.49 = 0$$

$$(s^2 + b^2)(s + a) = s^3 + as^2 + b^2s + ab^2 = 0$$

Comparing coefficients, we find that  $a = 25.28$  and  $b = \sqrt{160} = 12.65$

With the calculated values of  $T_D$  and  $T_i$ :

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{80[0.216s^2 + s + 50.33]}{(s + 25.28)(s - 12.65j)(s + 12.65j)}$$

And the transient response can be characterised as:

- i) Exponential decay with time constant  $T = 1/25.28 = 0.0396s$
- ii) Constant amplitude sinusoidal response at frequency  $12.65 \text{ rad/s}$

## Bibliography

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