Strain Energy Methods

Lecture 2 – Application to Elastic Material Behaviour

Department of Mechanical, Materials & Manufacturing Engineering

MMME2053 – Mechanics of Solids

University of Nottingham
UK | CHINA | MALAYSIA

Strain Energy Methods

Learning Outcomes

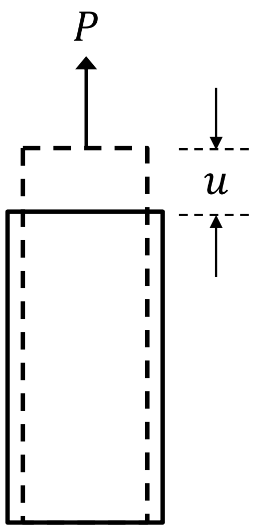
1. Know the basic concept of strain energy stored in a material body under loading (knowledge);
2. Be able to calculate strain energy in an elastic body/structure arising from various types of loading, including tension/compression, bending and torsion (application);
3. Be able to apply Castigliano’s theorem for linear elastic bodies to enable the deflection or rotation of a body at a point to be calculated from strain energy expressions (application).

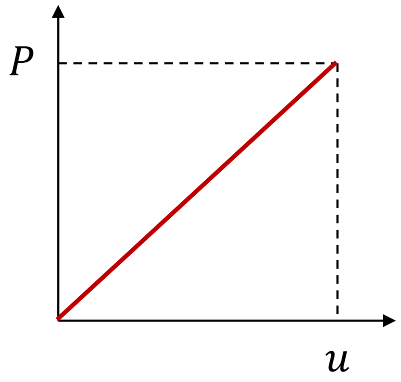
Strain Energy for Elastic Material Behaviour

Axial Loading

If the material body shown in the left-hand figure below is subjected to an axial load, P, behaves linear elastically, as shown in the right-hand figure below, it can be seen that the work done, or strain energy, can be expressed simply as:

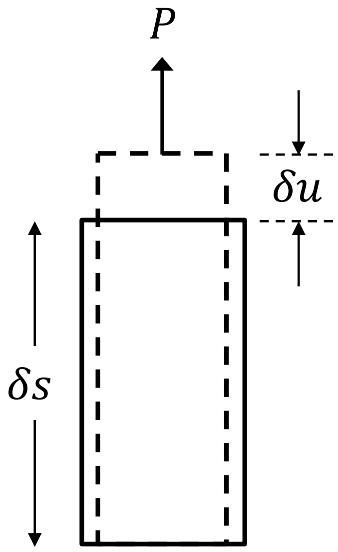
$$U = \frac{1}{2}Pu$$





If this material body represents an element, of length δs, of a larger beam, of length L, and the change in length of this element due to the applied load, P, is δu, then the strain energy within this element is:

$$\delta U = \frac{1}{2} P \delta u$$



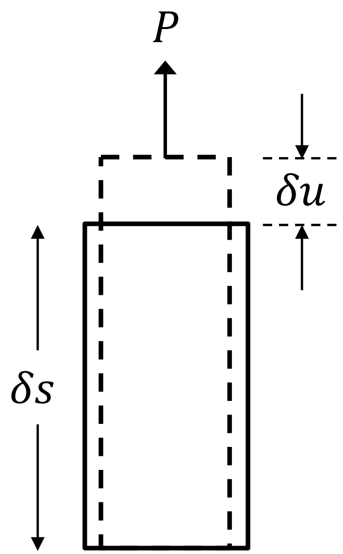
It is important to note that there are transverse strains/displacements due to Poisson's effects but there are no transverse stresses/loads. Thus, there is no work done in the transverse direction.

Axial strain in the element is: $\varepsilon = \dfrac{\delta u}{\delta s}$

and as the material behaves linear elastically, Hooke's law applies. Therefore: $\varepsilon = \dfrac{\sigma}{E}$

Eliminating ε from the above two equations: $\dfrac{\delta u}{\delta s} = \dfrac{\sigma}{E} = \dfrac{P}{EA}$ $\quad \therefore \delta u = \dfrac{P}{EA} \delta s$

Substituting this into the above equation for δU: $\delta U = \dfrac{P^2}{2EA} \delta s$



$$\delta U = \frac{P^2}{2EA}\delta s$$

As this is the expression for strain energy in the element of beam δs, integrating this expression over the full length of the beam, in order to give the total strain energy in the beam:

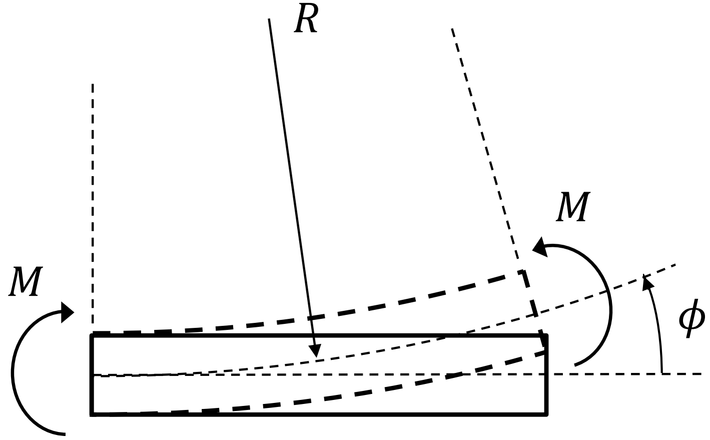
$$\boldsymbol{U = \int_0^L \frac{P^2}{2EA}\delta s}$$

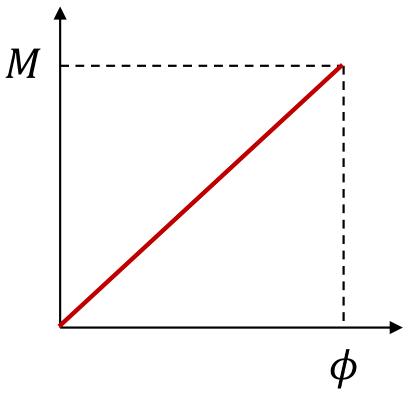
Strain Energy for Elastic Material Behaviour

Bending

If the material body shown in the left-hand figure below is subjected to a pure bending moment, M, and behaves linear elastically, as shown in the right-hand figure below, it can be seen that the work done, or strain energy, can be expressed simply as:

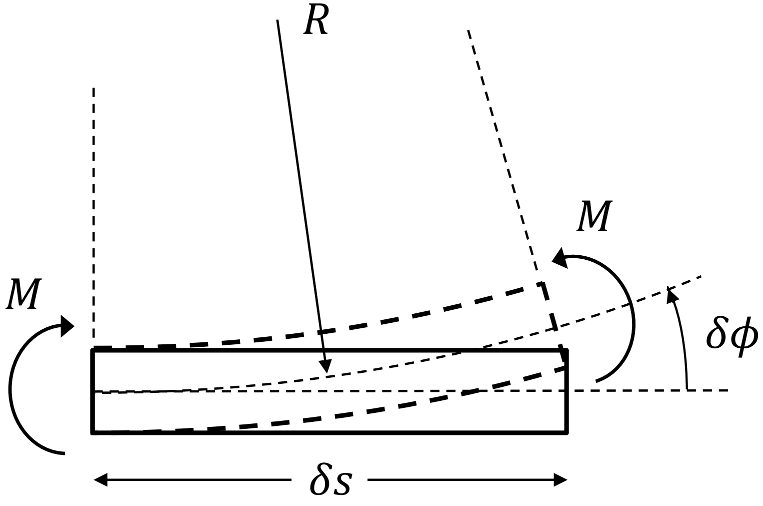
$$U = \frac{1}{2} M\phi$$





If this material body represents an element, of length δs, of a larger beam, of length L, which bends to curvature R, giving a change in slope of this element due to the applied bending moment M, of $d\phi$, then strain energy within this element is:

$$\delta U = \frac{1}{2} M \delta \phi$$

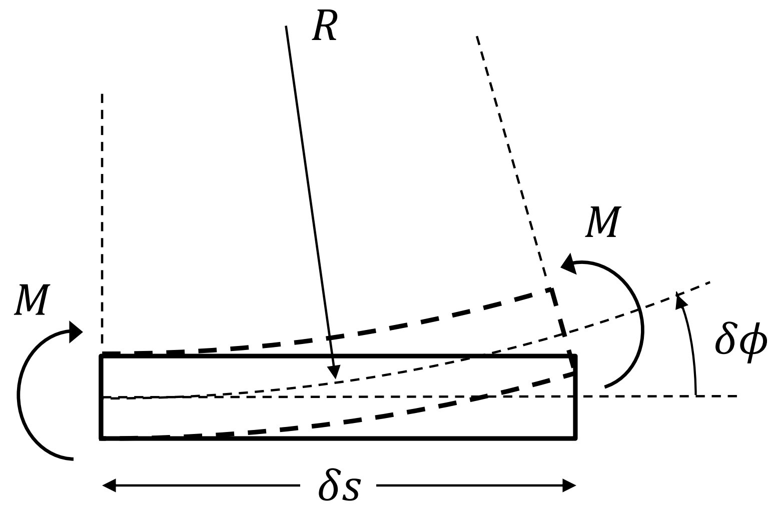


From the elastic beam bending equation: $\dfrac{M}{I} = \dfrac{E}{R}$

and as the angle subtended by the element is equal to the change in slope, the expression for the arc created by the element is: $\delta s = R\delta\phi$

Eliminating R from the above two equations: $\dfrac{M}{I} = \dfrac{E}{\frac{\delta s}{\delta \phi}}$ $\quad \therefore \delta\phi = \dfrac{M}{EI}\delta s$

Substituting this into the above equation for δU: $\delta U = \dfrac{M^2}{2EI}\delta s$



$$\delta U = \frac{M^2}{2EI}\delta s$$

As this is the expression for strain energy in the element of beam δs, integrating this expression over the full length of the beam, in order to give the total strain energy in the beam:

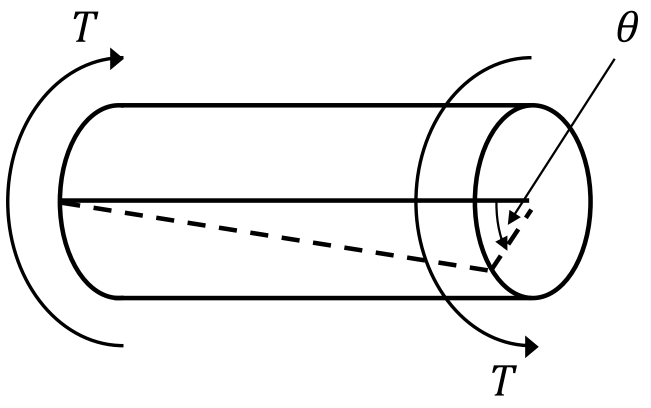
$$\boldsymbol{U = \int_0^L \frac{M^2}{2EI}\delta s}$$

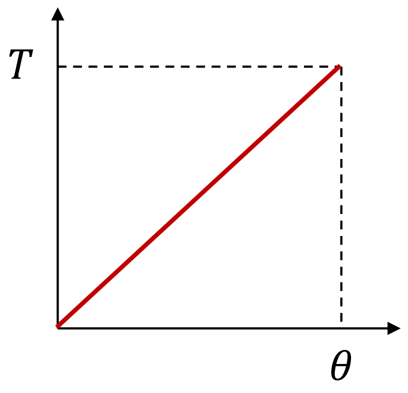
Strain Energy for Elastic Material Behaviour

Torsion

If the material body shown in the left-hand figure below is subjected to a torque, T, and behaves linear elastically, as shown in the right-hand figure below, it can be seen that the work done, or strain energy, can be expressed simply as:

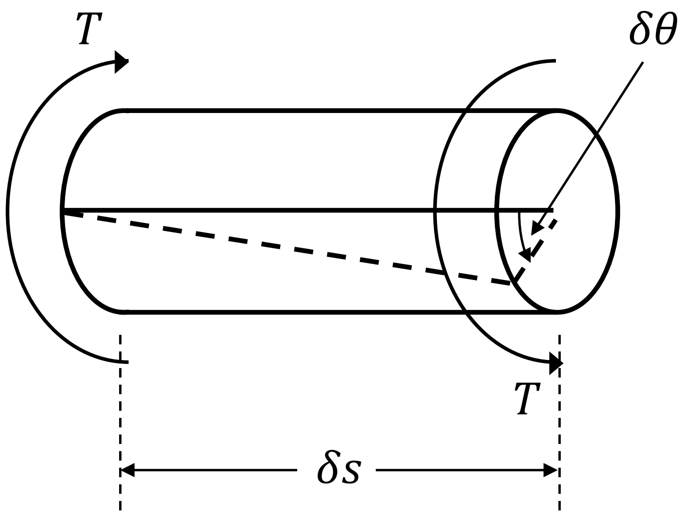
$$U = \frac{1}{2}T\theta$$





If this material body represents an element, of length δs, of a larger beam, of length L, , and the twist of this element due to the applied torque, T, is $\delta\theta$, then the strain energy within this element is:

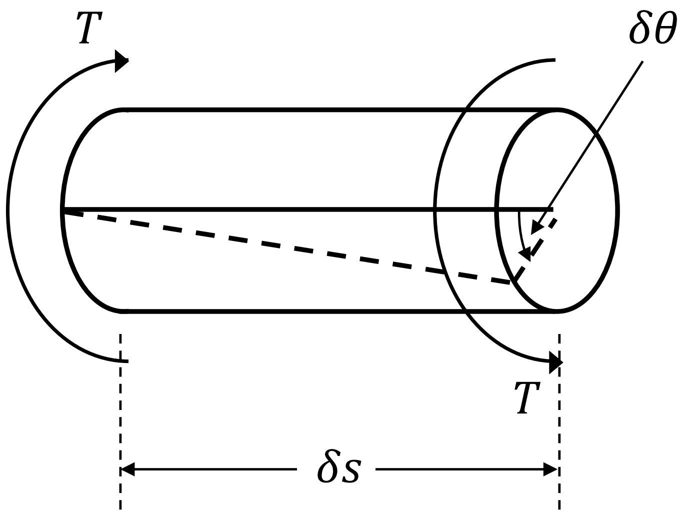
$$\delta U = \frac{1}{2} T\delta\theta$$



From the elastic torsional equation: $\frac{T}{J} = \frac{G\theta}{L}$

For the element, δs, this can be rewritten as: $\frac{T}{J} = \frac{G\delta\theta}{\delta s}$ $\therefore \delta\theta = \frac{T}{GJ}\delta s$

Substituting this into the above equation for δU: $\delta U = \frac{T^2}{2GJ}\delta s$



$$\delta U = \frac{T^2}{2GJ}\delta s$$

As this is the expression for strain energy in the element of beam δs, integrating this expression over the full length of the beam, in order to give the total strain energy in the beam:

$$\boldsymbol{U = \int_0^L \frac{T^2}{2GJ}\delta s}$$

Strain Energy for Elastic Material Behaviour

Summary

Axial Loading

$$U = \int_0^L \frac{P^2}{2EA} \delta s$$

Bending

$$U = \int_0^L \frac{M^2}{2EI} \delta s$$

Torsion

$$U = \int_0^L \frac{T^2}{2GJ} \delta s$$

The above equations summarise the strain energy expressions for elastic bodies for axial, bending and torsional loading types, respectively. In practical engineering structures, where members are relatively long and slender, strain energy due to axial loading can usually be neglected with bending usually being dominant. Strain energy due to shear deflections can also exist but, again, can normally be neglected.

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