Deflection of Beams

Lecture 2 – Macaulay’s Method

Department of Mechanical, Materials & Manufacturing Engineering

MMME2053 – Mechanics of Solids

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Deflection of Beams

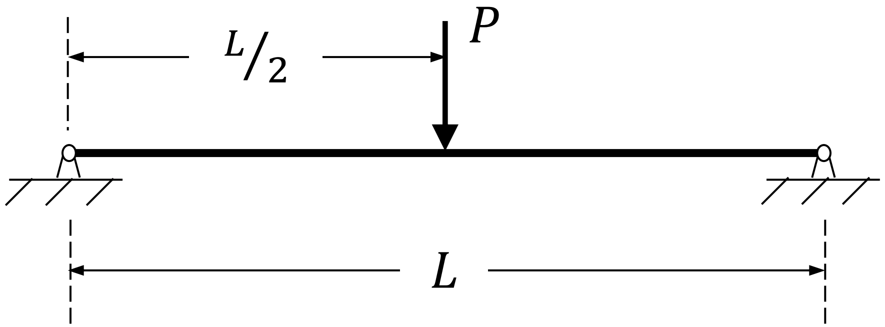
Learning Outcomes

1. Know how to derive the differential equation of the elastic line (i.e. deflection curve) of a beam (synthesis);
2. Employ Macaulay's method, also called the method of singularities, to determine bending moment expressions for beams where there are discontinuities in the bending moment distribution arising from discontinuous loading (application);
3. Be able to solve this equation by successive integration in order to yield the slope, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and the deflection, y, of a beam at any position, x, along its span (application);
4. Recognise and use different singularity functions in the bending moment expression, relating to different loading conditions, including point loads, uniformly distributed loads and point bending moments (comprehension);
5. Employ Macaulay's method for statically indeterminate beam problems (application).

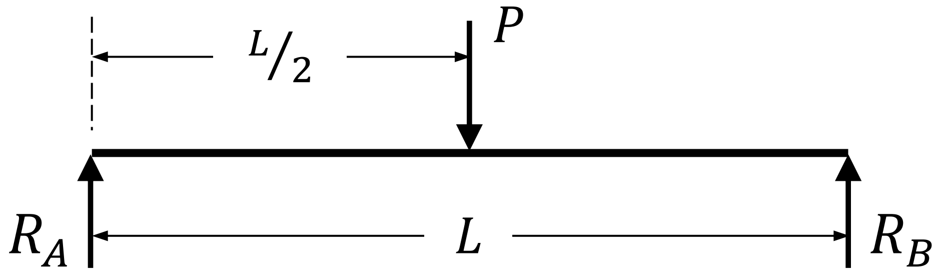
Macaulay’s Method

Named after the mathematician W. H. Macaulay, Macaulay’s Method uses a mathematical technique to deal with discontinuous loading. The bending moment expression $M(x)$, i.e. M as a function of x, is replaced with the step function $M\langle x - a\rangle$, in which a defines the position at which a discontinuity arises.

An example of such discontinuous loading, which gives rise to a discontinuity in the bending moment expression, is the point load applied to the simply supported beam below.



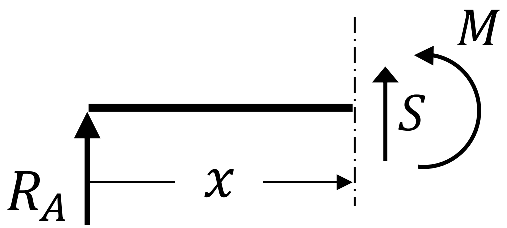
Free body diagram of this beam:



Next, each span between the loading discontinuity is considered separately in order to determine expressions for bending moment, M.

Span 1

Taking the left-hand end as the origin and drawing a free body diagram of the beam sectioned within span 1 (i.e. before the loading discontinuity caused by P).



Taking moments about the section position in order to determine an expression for the bending moment, M, in span 1:

$$M = R_A x$$

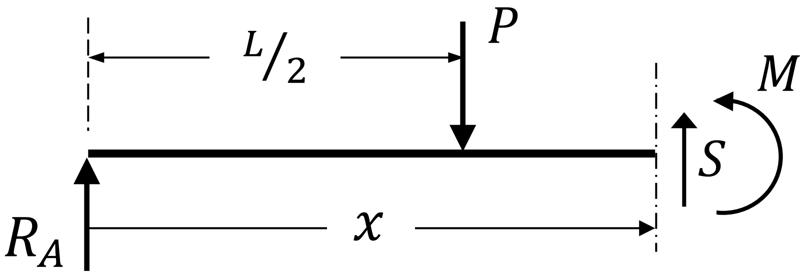
Substituting this into the 2nd order differential equation of the elastic line (equation (8) from lecture 1):

$$EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = R_A x \qquad (9)$$

This expression applies to span 1 of the beam only.

Span 2

Now drawing a free body diagram of the beam sectioned within span 2 (i.e. after the loading discontinuity caused by P).



Taking moments about the section position in order to determine an expression for the bending moment, M, in span 2:

$$M = R_A x - P\left(x - \frac{L}{2}\right)$$

Substituting this into the 2nd order differential equation of the elastic line (equation (8) from lecture 1):

$$EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = R_A x - P\left(x - \frac{L}{2}\right) \qquad (10)$$

This expression applies to span 2 of the beam only.

Application of Macaulay's Method

It is interesting to note that the forms of equations (9) and (10) are similar, in that there is simply an extra term added to take account of the extra span of the beam (as we move past the loading discontinuity). In fact, due to this similarity, equation (10), i.e. the expression derived for the final span of the beam, can be applied to the full length of the beam by rewriting it in a slightly different form as follows:

$$EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = R_A x - P\left\langle x - \frac{L}{2}\right\rangle \qquad (11)$$

Note the change of bracket shape. These ⟨ ⟩ brackets are termed 'Macaulay Brackets' and due to these, equation (11) is now applicable to any position, x, in the entire beam span, if we adopt 'Macaulay's convention'.

Macaulay's convention states that whenever a Macaulay bracketed term becomes negative, the entire term it is part of is set to zero.

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Integration to Solve for Slope and Deflection

Adopting Macaulay's convention, the general 2nd order differential expression for a beam can be integrated with respect to x to give the slope, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and integrated again to give the deflection, y, at any position x, along the length of the beam.

For the single point load example shown in this lecture, equation (11) is integrated to give:

$$EI\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{R_A x^2}{2} - \frac{P\,\langle x - \frac{L}{2}\rangle^2}{2} + \mathrm{A}$$

and again to give:

$$EIy = \frac{R_A x^3}{6} - \frac{P\,\langle x - \frac{L}{2}\rangle^3}{6} + \mathrm{A}x + \mathrm{B}$$

Where $\frac{\mathrm{d}y}{\mathrm{d}x}$ and y represent the slope and deflection at any position, x.

The unknowns in these equations, namely R_A, A and B, can be solved for from equilibrium and boundary conditions.

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https://www.youtube.com/watch?v=s4T_TDB0UpQ