9. Elastic Instability (Buckling)

Learning Summary

1. Be able to apply Macaulay's method for determining beam deflection in situations with axial loading (application);
2. Know the meanings of and the differences between stable, unstable and neutral equilibria (knowledge);
3. Be able to determine the buckling loads for ideal struts (application);
4. Be able to include the interaction of yield behaviour with buckling and how to represent this interaction graphically (knowledge/application).

9.1 Introduction

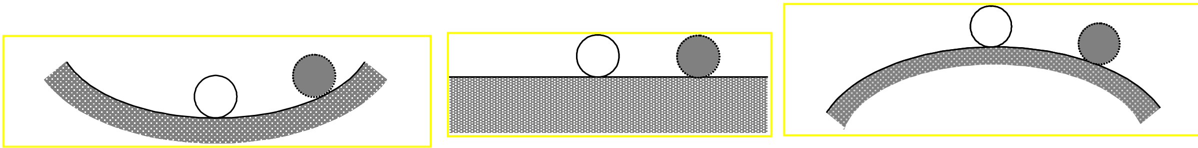
For many structural problems, it is reasonable to assume that the system is in stable equilibrium. However, not all structural arrangements are stable. For example, consider a one-meter long stick with the cross-sectional area of a pencil. If this stick were stood on its end, the axial stress would be small, but the stick could easily topple over sideways. This simple example demonstrates that in some configurations, stability considerations can be primary.

This section is concerned with the stability of struts. Struts are compression members with cross-sectional dimensions which are small compared to the length, i.e., they are slender. If a circular rod of, say, 5mm diameter, which has its ends machined flat and perpendicular to the axis, were made 10mm long to act as a column, there would not be a problem of instability and it could carry considerable force. However, if the same rod were made a meter long, the rod would become laterally unstable at a much smaller applied force and could collapse.

Buckling also occurs in many other situations with compressive forces. Examples include thin sheets which have no problem carrying tensile loads and vacuum tanks, as well as submarine hulls. Thin-walled tubes can wrinkle like paper when subjected to torque.

9.2 Buckling Phenomenon

Consider the response of a marble when subjected to disturbances from an initial equilibrium position on different types of surfaces, as shown in Figure 9.1. If the surface is concave, the marble will return to its original equilibrium position and the marble is said to be in a stable equilibrium position; if the surface is flat the marble will move to another equilibrium position and the marble is said to be in a neutral equilibrium position. Finally, if the surface is convex, the marble will roll off uncontrollably in an unstable fashion and the marble is said to be in an unstable equilibrium position.



Stable equilibrium position Neutral equilibrium position Unstable equilibrium position

Figure 9.1. Equilibrium states for a marble on various surfaces

This analogy is useful for understanding the energy approach to buckling problems. Every deformed structure has a potential energy associated with it, which depends on the strain energy stored in the structure and the work done by the external loads. A concave potential energy function at equilibrium gives a stable equilibrium while a convex potential energy function gives unstable equilibrium.

Alternatively, buckling problems may be treated as bifurcation problems. Referring to Figure 9.2(a), it is clear that the tensile force will tend to restore the bar to equilibrium if there is a slight displacement to the right. However, the same bar under the action of a compressive force, Figure 9.2(b), will continue to fall when subjected to a slight displacement. This illustrates unstable equilibrium.

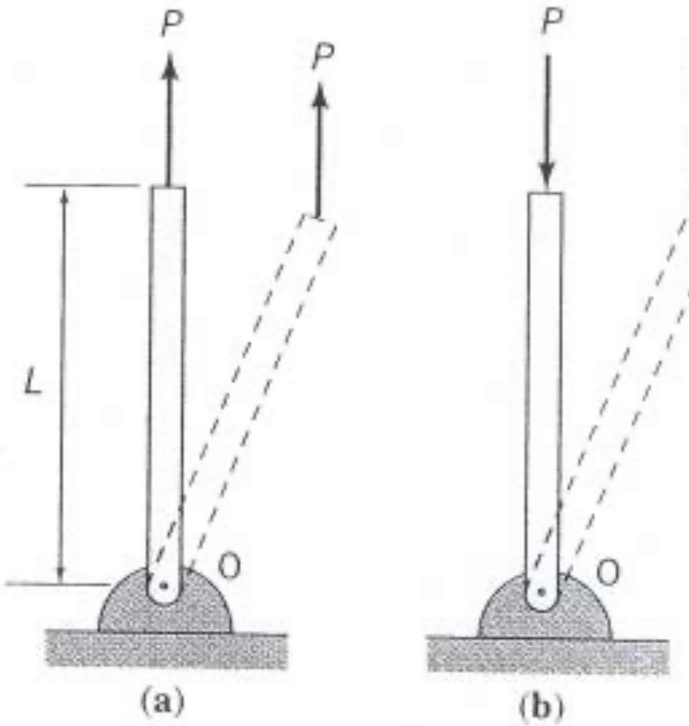


Figure 9.2. Examples of stable and unstable equilibrium

Figure 9.3 illustrates a slightly more complicated example of the same phenomenon. The vertical bar is supported horizontally by two springs of stiffness, *k*. If the bar of length L, is displaced a small amount, *x*, horizontally, there is a displacing moment of *Px* about O and a restoring moment 2*kxL*. Hence we get $Px < 2kxL$ for stable equilibrium and $Px > 2kxL$ for unstable equilibrium.

The critical condition occurs when $Px = 2kxL$ or $P_c = 2kL$, where P_c is termed the critical load between stable and unstable equilibrium.

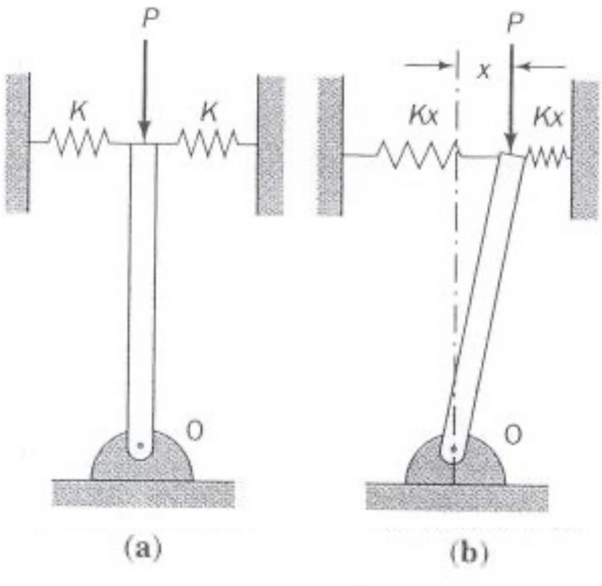


Figure 9.3. Axially loaded rigid bar with transverse springs

Figure 9.4 shows a rigid bar subjected to a compressive axial load with a torsional spring at its base; a free body diagram of the problem is also shown. Taking moments about the point O, gives:

$$PL\sin\theta = K_\theta \theta \qquad \therefore \frac{PL}{K_\theta} = \frac{\theta}{\sin\theta}$$

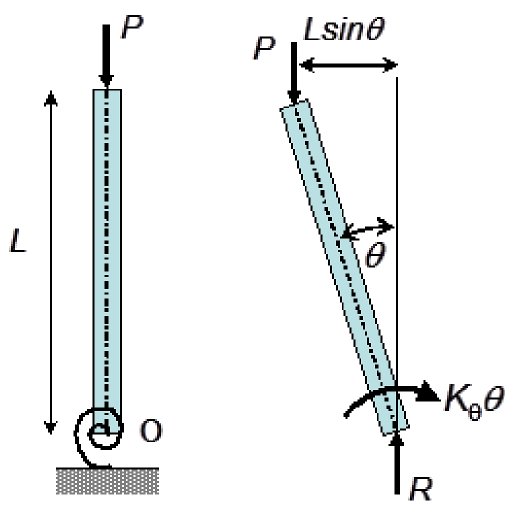


Figure 9.4. Rigid bar supported by a torsional spring

Figure 9.5 shows a graph of $\frac{PL}{K_\theta}$ versus θ. There is a stable region for low loads and an unstable region for high loads. Below point A, the bar will return to its equilibrium position if rotated slightly to either the right or left. Once the load exceeds the value at point A, then any disturbance will cause the bar to rotate along either the right branch or the left branch of the bifurcation curve. Point A is called the bifurcation point, at which there are three possible solutions. The associated load at point A is called the critical (buckling) load, $\theta = 0^\circ$ is the trivial solution. Only non-trivial solutions are generally of interest, i.e. for $\theta \neq 0^\circ$.

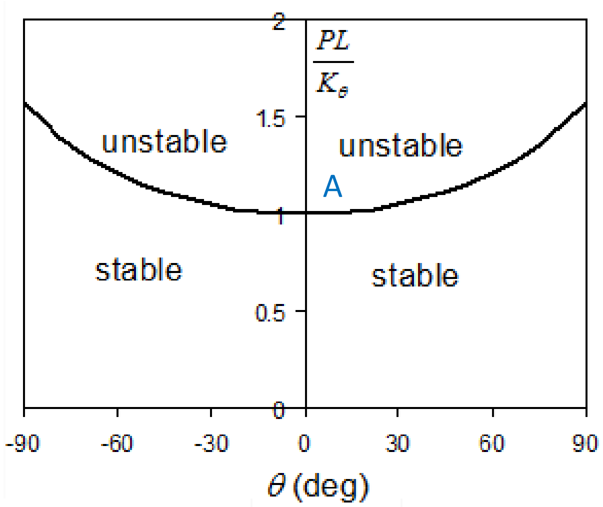


Figure 9.5. Variation of PL/K$_\sigma$ with θ, indicating stable and unstable regions

9.3 Ideal struts

Ideal struts are assumed to be initially perfectly straight and of uniform section properties, and subjected to purely axial loading. Expressions will be developed relating the critical buckling load to the applied load, the material properties and the member dimensions, for different support conditions of the struts. At a critical load, members which are circular or tubular in cross-section will buckle sideways in any direction. Often, compression members do not have equal flexural rigidity, *EI*, in all directions and there will be one axis about which the flexural rigidity is a minimum, depending on the dimensions. The member will therefore buckle about this axis and the *I*-value (second moment of area) referred to in the present section is assumed to be the minimum value, based on the nominal dimensions of the member.

9.3.1 Case 1: Hinged-Hinged

Consider an initially straight strut with its ends free to rotate around frictionless pins, as shown in Figure 9.6, which will be referred to as a hinged-hinged case. The dashed line represents the initially straight strut. The strut is now considered to be perturbed, from its initially straight position, as shown in Figure 9.6. This perturbation is equivalent to the movement of the marble in Figure 9.1.

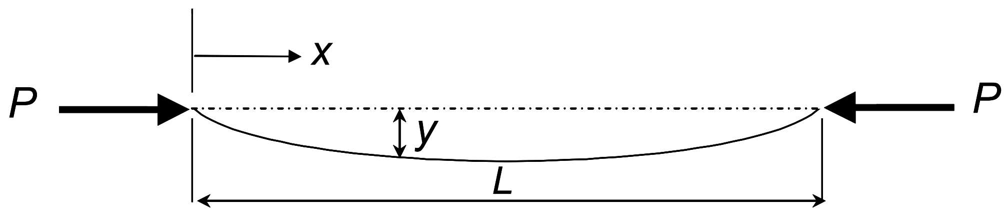


Figure 9.6. A hinged-hinged strut

The bending moment, *M*, depends on the deflection, y, and is hence a function of position, x.

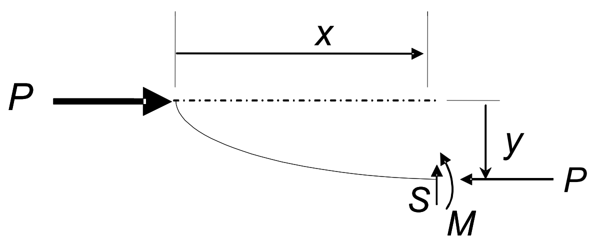


Figure 9.7. Free body diagram for the left hand portion of the strut

The deflection, *y*, is related to the moment *M*, by the relationships:

$$EI\,\frac{d^2 y}{dx^2} = -M \quad \text{and} \quad M = Py$$

Therefore, $\dfrac{d^2y}{dx^2}+\dfrac{P}{EI}y=0$ or $\dfrac{d^2y}{dx^2}+\alpha^2 y=0$, where $\alpha^2=\dfrac{P}{EI}$.

The solution to a second order differential equation of this form is:

$$y = A\sin\alpha x + B\cos\alpha x$$

In order to determine *A* and *B*, we need two "boundary conditions", i.e.

1. at $x = 0$ $y = 0$, and
2. at $x = l$, $y = 0$

Therefore, B = 0 and $A\sin(\alpha l) = 0$

The condition $A = 0$ results in a trivial solution, i.e. $y = 0$, which is the case for an un-deflected strut. Hence, the non-trivial solution is $\sin(\alpha l) = 0$, which gives $\alpha l = n\pi$, where $n = 0, 1, 2...$

$$\therefore \alpha^2 l^2 = n^2\pi^2$$

$$\frac{P}{EI}l^2 = n^2\pi^2$$

$$\text{or } P = \frac{n^2\pi^2 EI}{l^2}$$

$n = 0$ gives another trivial solution, i.e. $P = 0$,

$n = 1$ gives:

$$\boxed{P_c = \frac{\pi^2 EI}{l^2}}$$

This is called the **Euler buckling (or crippling) load**, P_c, it is the lowest load at which buckling can occur (Euler solved this problem in 1757).

$n = 2$ gives:

$$\boxed{P = \frac{4\pi^2 EI}{l^2}}$$

and this corresponds to a different deflected (buckling) shape of the strut.

For n = 1, $y = y_{\max}$ at $x = \frac{l}{2}$ and therefore $A = y_{\max}$ and the deflected shape of the strut is given by the following expression:

$$y = y_{\max}\sin(\alpha x) = y_{\max}\sin\left(\frac{n\pi x}{l}\right)$$

The magnitude of y_{max} cannot be determined from the boundary conditions and it can become arbitrarily large, leading to elastic instability of the structure. The first three buckling mode shapes are shown in Figure 9.8. If buckling mode I is prevented from occurring by installing a restraint (support), then the column would buckle at the next higher mode at critical load values that are higher than for the lower mode. The inflexion points, *I*, for each deflection curve has zero deflection. Recalling that the curvature $\frac{d^2y}{dx^2}$ at an inflexion point is zero indicates that the internal moment at these points is zero. If roller supports are put at any other point than Point *I*, the boundary value problem must be solved for new eigenvalues (buckling loads) and eigenvectors (mode shapes).

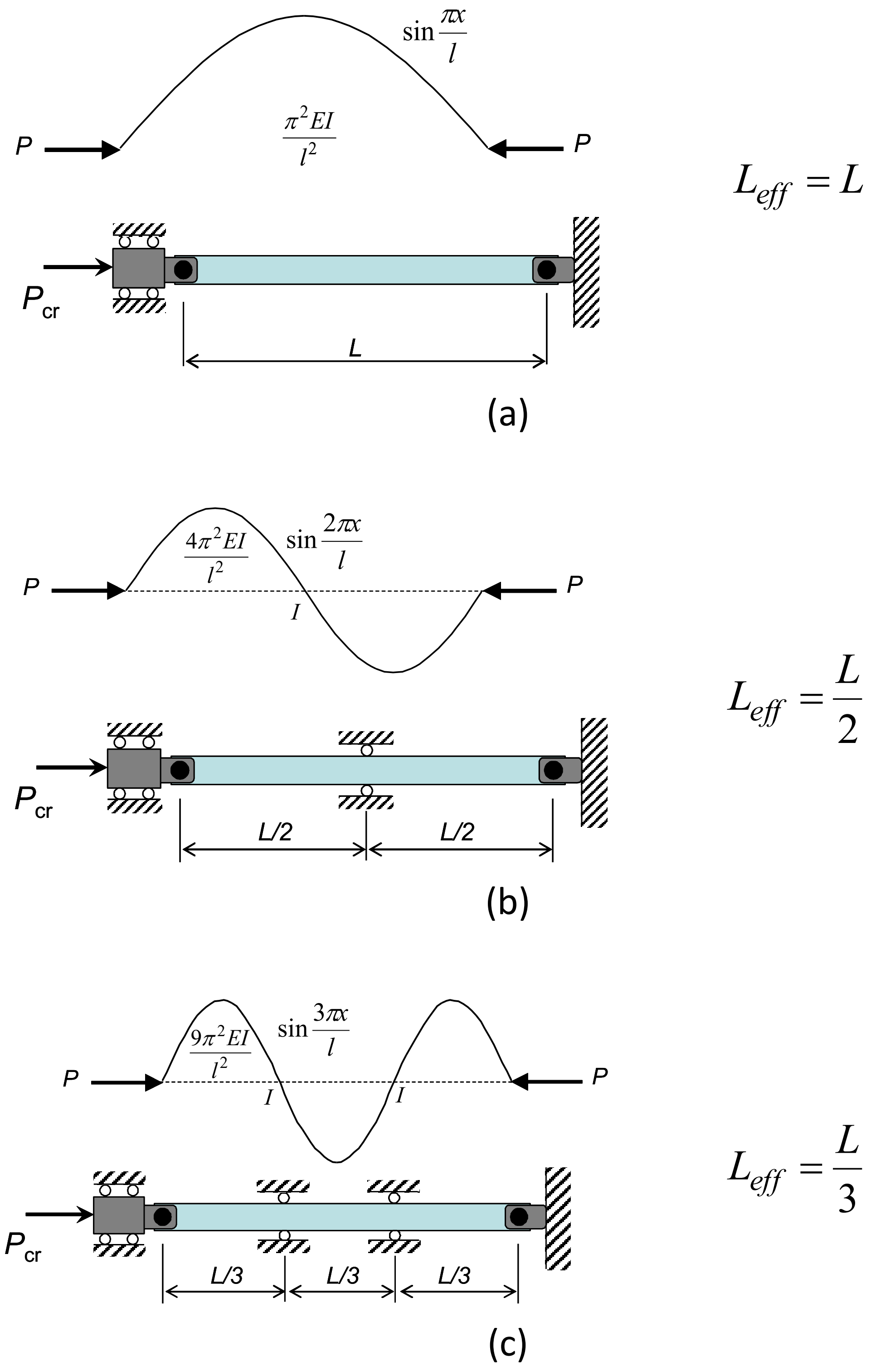


Figure 9.8. Buckling mode shapes for a hinged-hinged strut with n=1, n=2 and n=3

9.3.2 Case 2: Free-Fixed

Figure 9.9 shows the deflected shape and free-body diagram for a fixed-free strut.

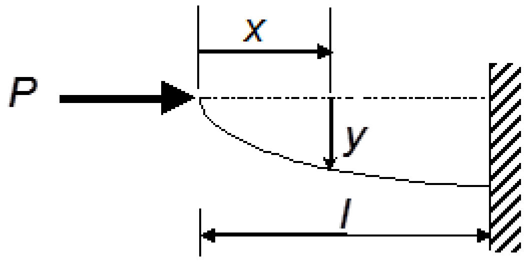


Figure 9.9. A fixed free strut

$$EI\frac{d^2y}{dx^2} = -M \quad \text{and} \quad M = Py$$

$$\therefore\ EI\frac{d^2y}{dx^2} + Py = 0$$

$$\text{or} \quad \frac{d^2y}{dx^2} + \alpha^2 y = 0$$

$$\text{where}\ \alpha^2 = \frac{P}{EI}$$

The solution to this differential equation is:

$$y = A\sin\alpha x + B\cos\alpha x$$

Boundary conditions:

1. At $x = 0$, $y = 0$ *and therefore* $B = 0$, and
2. at $x = l$, $\frac{dy}{dx} = 0$ *and therefore* $A\alpha\cos(\alpha l) = 0$

So far, the mathematical solution is identical to that of a free-free strut. However, the boundary conditions are different, i.e. in this case, $A = 0$ or $\alpha = 0$, leading to trivial solutions (as before). The non-trivial solution results from taking $\cos(\alpha l) = 0$, which implies that $\alpha l = \frac{n\pi}{2}$, i.e.

$$\frac{P}{EI}l^2 = \frac{n^2\pi^2}{4} \text{ where } n = 1, 3, \ldots$$

The smallest, non-trivial, value of *P* occurs with *n* = 1, i.e.

$$\boxed{P_c = \frac{\pi^2 EI}{4l^2}}$$

By comparison with Case 1, i.e., the hinged-hinged case, it can be seen that the solution is the same except that "*l*" is replaced by "*2l*", i.e. the fixed-free case can be treated as the hinged-hinged case for a strut with an equivalent length of 2*l*.

9.3.3 Case 3: Fixed-Fixed

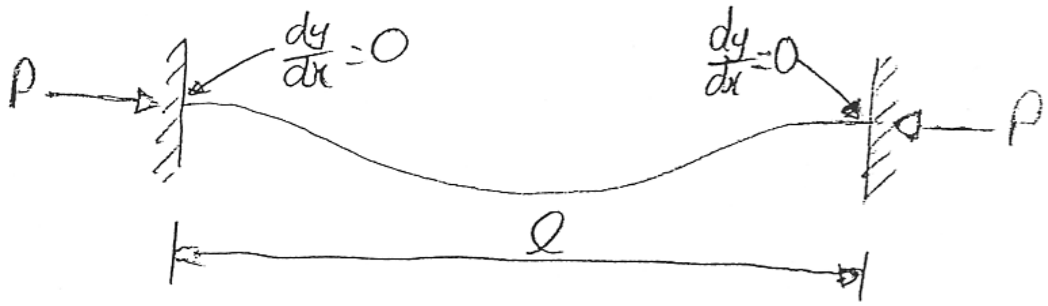


Figure 9.10. A fixed-fixed strut

The solution procedure is the same as that for Cases 1 and 2, leading to:-

$$\boxed{P_c = \frac{4\pi^2 EI}{l^2}}$$

The fixed-fixed case shows a significant increase in the buckling capacity relative to the hinged-hinged case.

9.3.4 Case 4: Fixed-Hinged

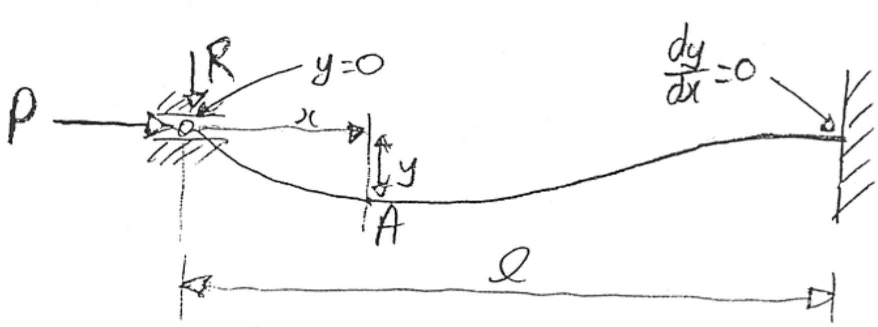


Figure 9.11. A fixed-hinged strut

$$\boxed{P_c = \frac{2.045\pi^2 EI}{l^2} \quad \left(\approx \frac{2\pi^2 EI}{l^2} \right)}$$

This case differs from the previous examples in that a transverse force, R, shown in Figure 9.11, is necessary to create this mode of deformation.

$$EI\frac{d^2y}{dx^2} = -M$$

and

$$M + Rx = Py$$

$$\therefore EI\frac{d^2y}{dx^2} + Py = Rx$$

$$\therefore \frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{R}{EI}x$$

$$\therefore \frac{d^2y}{dx^2} + \alpha^2 y = \frac{R}{EI}x$$

where:

$$\alpha^2 = \frac{P}{EI}$$

The solution to this type of differential equation consists of two parts, a homogenous solution and a particular integral. The homogeneous solution is given by the following:

$$y = A\sin\alpha x + B\cos\alpha x$$

The particular integral for such 2nd order differential equations is generally obtained by taking:

$$P.I. = y = C.f(x)$$

where:

$$\frac{d^2 y}{dx^2} + \alpha^2 y = \text{const.} f(x)$$

And substituting for *y* in the differential equation gives the solution for *C*. In this particular case, *f*(*x*) = *x*, so that:

$$\alpha^2 C.f(x) = \frac{R}{EI} f(x)$$
$$C = \frac{R}{\alpha^2 EI} = \frac{R}{P}$$
$$P.I. = y = \frac{R}{P}.f(x) = \frac{R}{P}x$$

Hence, the complete solution is:

$$y = \underbrace{A\sin\alpha x + B\cos\alpha x}_{\text{Homogeneous}} + \underbrace{\frac{R}{P}x}_{\text{Particular integral}}$$

There are three unknowns this time, i.e. *A*, *B* and *R*. Therefore, we need three boundary conditions, i.e.:

1. at $x = 0$, $y = 0$ and so $B = 0$,
2. $at\ x = l, y = 0, and\ so, 0 = A\sin\alpha l + \frac{R}{P}l$

   $Therefore\ A = -\frac{Rl}{P\sin\alpha l}$

3. at $x = l$, $\frac{dy}{dx} = 0, \quad -\frac{Rl\alpha}{P\sin\alpha l}\cos\alpha l + \frac{R}{P} = 0$

   $Therefore\ \tan\alpha l = \alpha l$

The smallest non-trivial root to this equation is

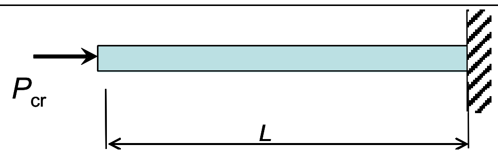
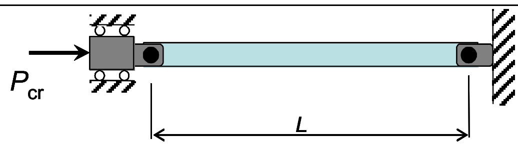
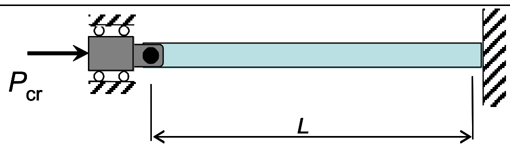
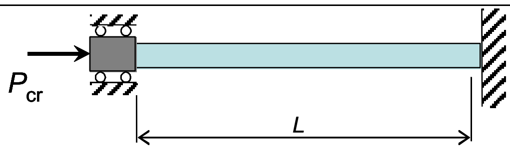
$$\alpha l = 4.493 (\approx 1.43\pi)$$

$$\therefore \ \alpha^2 l^2 = \frac{P}{EI} l^2 = 1.43^2 \pi^2$$

$$\text{i.e. } P_c = \frac{2.045\pi^2 EI}{l^2}$$

9.4 Summary of Euler buckling loads of struts

General formula: $P_c = \dfrac{\pi^2 EI}{L_{eff}^{\ 2}}$

Description	Schematic	Critical buckling load, P_c	Effective length, L_{eff}
Free-fixed		$P_c = \dfrac{\pi^2 EI}{4l^2}$	$2l$
Hinged-hinged		$P_c = \dfrac{\pi^2 EI}{l^2}$	l
Fixed-hinged		$P_c = \dfrac{2.045\pi^2 EI}{l^2}$	$0.7l$
Fixed-fixed		$P_c = \dfrac{4\pi^2 EI}{l^2}$	$\dfrac{l}{2}$

The effective length, L_{eff}, is a measure of how much longer (and thus more unstable) a given strut configuration appears to be in terms of critical buckling load, relative to the hinged-hinged case. Thus, the fixed-fixed case, for example, has a shorter effective length because it is more stable and thus appears to be shorter with respect to buckling.

9.5 Some important notes

1) In contrast to the classical cases considered here, actual compression members are seldom truly pinned or completely fixed against rotation at the ends. Because of this uncertainty regarding the fixity of the ends, struts or columns are often assumed to be pin-ended. This procedure is conservative.
2) The above equations are not applicable in the inelastic range, i.e. for $\sigma > \sigma_y$, and must be modified.
3) The critical load formulae for struts or columns are remarkable in that they do not contain any strength property of the material and yet they determine the load carrying capacity of the member. The only material property required is the elastic modulus, *E*, which is a measure of the stiffness of the strut.

9.6 Compressive loading of rods

If we assume that the rod loading is perfectly axial, and the material can be represented by an elastic-perfectly plastic stress-strain curve (see Figure 9.12), then the plastic collapse failure would occur in compression if $\sigma\left(=\frac{-P}{A}\right)$ reaches $-\sigma_y$ before the buckling load is reached.

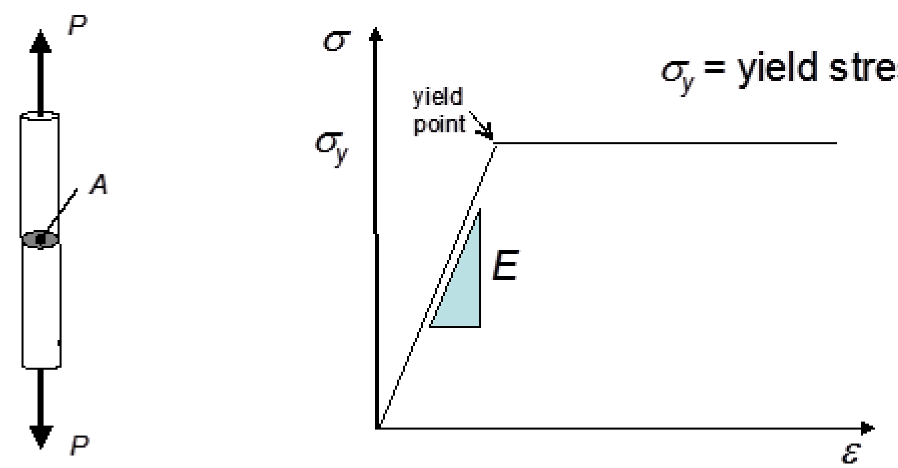


Figure 9.12. Tensile test specimen and elastic perfectly plastic stress-strain behaviour

Now

$$\boxed{P_c = \frac{\pi^2 EI}{l^2}}$$

and defining the second moment of area, I, as: $I = Ak^2$

where *k* is the radius of gyration, gives:

$$P_c = \frac{\pi^2 EAk^2}{l^2}$$

and $\sigma = \frac{P_c}{A} = \frac{\pi^2 Ek^2}{l^2} = \frac{\pi^2 E}{(l/k)^2}$

l/*k* is the slenderness ratio.

Therefore, buckling will occur if $\sigma = \frac{\pi^2 E}{(l/k)^2}$, whereas plastic collapse will occur if $\sigma = \sigma_y$

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This can be represented diagrammatically as shown in Figure 9.13.

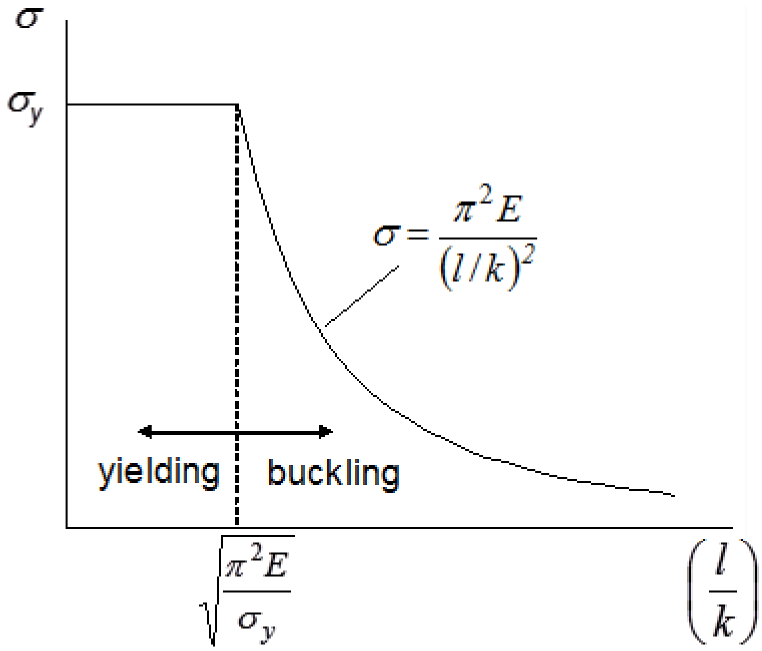

Figure 9.13. Plot of σ versus l/k indicating the buckling and plastic collapse regions.