# Minilecture 4D : more examples of applications

**Example** Height of a weighted spring governed by the equation

$$\frac{d^2y}{dt^2} + \omega_0^2 y = f(t)$$

where $f(t)$ is defined by

$$f(t) = \begin{cases} t+\pi & -\pi \le t < 0 \\ \pi - t & 0 \le t < \pi \end{cases} \quad \text{and} \quad f(t+2\pi) = f(t)$$


Even fn ⇒ no sines - cosine series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - t)\,dt$$

$$= \frac{2}{\pi}\left(\pi^2 - \frac{1}{2}\pi^2\right) = \pi$$

$n>0$ $$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - t)\cos nt\,dt$$

$$= -\frac{2}{\pi}\int_0^{\pi} t\cos nt\,dt$$

$$= -\frac{2}{\pi}\int_0^{\pi} t\,\frac{d}{dt}\left(\frac{1}{n}\sin nt\right)dt$$

$$= -\frac{2}{\pi}\left[t\left(\frac{1}{n}\sin nt\right)\right]_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}\sin nt\,dt$$

$0$ $\quad \sin n\pi = \sin 0 = 0$

$$= \frac{2}{n^2\pi}\left[-\cos nt\right]_0^{\pi} = \frac{2}{n^2\pi}\left(1-(-1)^n\right)$$

$$= \frac{4}{n^2\pi}\begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd}\end{cases}$$

Differential equation

$$\frac{d^2y}{dt^2} + \omega_0^2 y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nt$$

$$= \frac{\pi}{2} + \sum_{n\text{ odd}} \frac{4}{n^2\pi}\cos nt$$

For P.I. try

$$y_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n\cos nt + B_n\sin nt\right)$$

$$y_p'(t) = \sum_{n=1}^{\infty}\left(nB_n\cos nt - nA_n\sin nt\right)$$

$$y_p''(t) = \sum_{n=1}^{\infty}\left(-n^2A_n\cos nt - n^2B_n\sin nt\right)$$

$$y_p'' + \omega_0^2 y_p = \omega_0^2 \frac{A_0}{2} + \sum_{n=1}^{\infty} \left( (\omega_0^2 - n^2) A_n \cos nt + (\omega_0^2 - n^2) B_n \sin nt \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

works if

$$A_0 = a_0 / \omega_0^2 = \pi / \omega_0^2$$

$$A_n = \frac{a_n}{\omega_0^2 - n^2} = \frac{4}{\pi} \begin{cases} \frac{1}{n^2(\omega_0^2 - n^2)} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$B_n = 0$$

Complementary function $y_c(t) = C \cos \omega_0 t + D \sin \omega_0 t$

$$y(t) = C \cos \omega_0 t + D \sin \omega_0 t + \frac{\pi}{2\omega_0^2} + \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nt}{n^2(\omega_0^2 - n^2)}$$

$\omega_0 \simeq n$
$\Rightarrow$ resonance

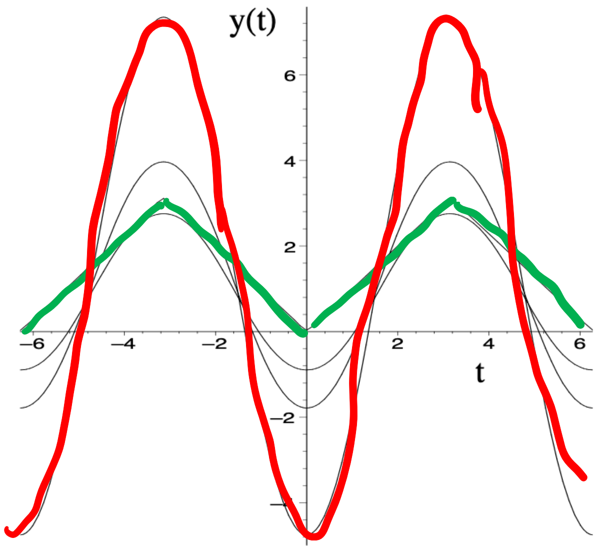
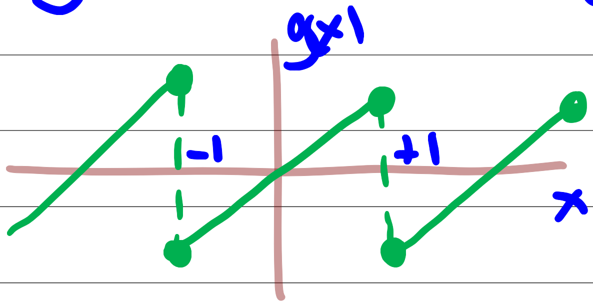



Figure 1: This picture shows $y_p$ for $\omega_0 = 1.1$, $\omega_0 = 1.2$ and $\omega_0 = 1.3$. The closer $\omega_0$ is to one of the values of $n$, the larger the amplitude of $y$ will be. This effect is called *resonance*.

**Example** Find particular integral for

$$\frac{dy}{dx} + y = g(x)$$

$g(x) = x \quad -1 \le x < 1 \;, \quad g(x+2) = g(x)$




$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x$$

$$= \frac{2}{\pi} \left( \sin \pi x - \frac{1}{2} \sin 2\pi x + \dots \right)$$

$$= \sum_{n=1}^{\infty} b_n \sin n\pi x$$

Let P.I. be of the form

$$y_p(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left( A_n \cos n\pi x + B_n \sin n\pi x \right)$$

$$y_p'(x) = \sum_{n=1}^{\infty} \left( n\pi B_n \cos n\pi x - n\pi A_n \sin n\pi x \right)$$

$$y_p' + y_p = \frac{A_0}{2} + \sum \Big( (A_n + n\pi B_n) \cos n\pi x + (B_n - n\pi A_n) \sin n\pi x \Big)$$

$$= \sum_n b_n \sin n\pi x$$

$$\frac{A_0}{2} = 0 \;, \qquad A_n + n\pi B_n = a_n = 0$$

$$\Rightarrow \quad A_n = -n\pi B_n$$

$$B_n - n\pi A_n = (1+n^2\pi^2)B_n = b_n = \frac{2}{n\pi}(-1)^{n+1}$$

$$\Rightarrow B_n = \frac{b_n}{1+n^2\pi^2} = \frac{2}{\pi}\,\frac{(-1)^{n+1}}{n(1+n^2\pi^2)}$$

$$A_n = -n\pi B_n = \frac{2}{\pi}\,\frac{(-1)^n}{1+n^2\pi^2}$$

$$\boxed{y_p(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\,\frac{\cos n\pi x - \frac{1}{n}\sin n\pi x}{1+n^2\pi^2}}$$

Complementary fn $\left(m+1=0 \Rightarrow m=-1 \Rightarrow y_c(x)=Ce^{-x}\right)$

$$\Rightarrow y(x) = Ce^{-x} + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\,\frac{\cos n\pi x - \frac{1}{n}\sin n\pi x}{1+n^2\pi^2}$$