Q1. i)

Part A) solve for $[Z]\{C\} = \{0\}$

① $y(x) = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 \sinh \lambda x + C_4 \cosh \lambda x$

② $\frac{dy}{dx} =$

③ $\frac{d^2y}{dx^2} =$

④ $\frac{d^3y}{dx^3} =$

Given in class

Boundary conditions

At $x=0$

$x=0, \ y=0$ $\qquad \frac{dy}{dx} = 0$

$\therefore$ from eqn ①

$y(0) = 0 = C_1 \times 0 + C_2 \times 1 + C_3 \times 0 + C_4 \times 1$

$C_2 + C_4 = 0$ ⓐ

from eqn ②

$\frac{dy}{dx} = 0 = C_1 \lambda \times 1 - C_2 \lambda \times 0 + C_3 \lambda \times 1 + C_4 \lambda \times 0$

$(C_1 + C_3)\lambda = 0$ ⓑ

At $x = L$

$x = L,\ y = 0$ $\qquad \dfrac{d^2y}{dx^2} = 0$

from eqn (1)

(c) $y(L) = 0 = C_1 \sin \lambda L + C_2 \cos \lambda L + C_3 \sinh \lambda L + C_4 \cosh \lambda L$

from eqn (3)

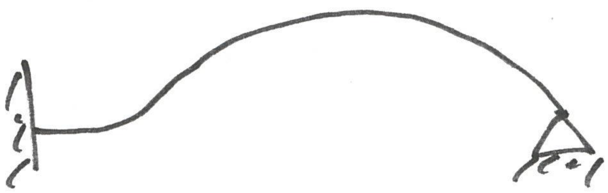
(d) $\dfrac{d^2y}{dx^2} = 0 = -\lambda^2 C_1 \sin \lambda L - \lambda^2 C_2 \cos \lambda L + \lambda^2 C_3 \sinh \lambda L + \lambda^2 C_4 \cosh \lambda L$

combine (a), (b), (c) & (d)

$$[Z] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \lambda & 0 & \lambda & 0 \\ \sin(\lambda L) & \cos(\lambda L) & \sinh(\lambda L) & \cosh(\lambda L) \\ -\lambda^2 \sin(\lambda L) & -\lambda^2 \cos(\lambda L) & \lambda^2 \sinh(\lambda L) & \lambda^2 \cosh(\lambda L) \end{bmatrix}$$

$$\{C\} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \qquad \{0\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Q1. i) Part B



Mode 1

Mode 2

Mode 3

Q1. ii) part A

At $x = 0$

$x = 0,\ y = 0$ $\qquad \dfrac{dy}{dx} = 0$

Same as i)

(a) $C_2 + C_4 = 0$

(b) $(C_1 + C_3)\lambda = 0$

At $x = L$

$x = L,\ \dfrac{\partial y}{\partial x} = 0$ $\qquad \dfrac{\partial^3 y}{\partial x^3} = 0$

from eqn (2)

(c) $\dfrac{dy(L)}{\partial x} = 0 = C_1\ \lambda \cos \lambda L - C_2 \lambda \sin L\lambda + C_3 \lambda \cosh \lambda L + C_4 \lambda \sinh \lambda L$

from eqn (4)

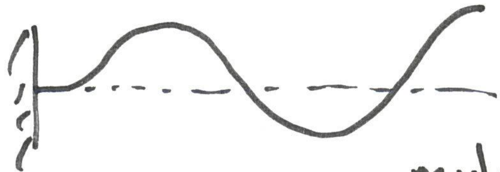
(d) $\dfrac{\partial^3 y}{\partial x^3} = 0 = -C_1 \lambda^3 \cos \lambda L - C_2 \lambda^3 \sin \lambda L + C_3 \lambda^3 \cosh \lambda L$
$+ C_4\ \lambda^3 \sinh \lambda L$

$$[Z] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \lambda & 0 & \lambda & 0 \\ \lambda\cos(\lambda L) & -\lambda\sin(L\lambda) & \lambda\cosh(\lambda L) & \lambda\sinh(\lambda L) \\ -\lambda^3\cos(\lambda L) & -\lambda^3\sin(\lambda L) & \lambda^3\cosh(\lambda L) & \lambda^3\sinh(\lambda L) \end{bmatrix}$$

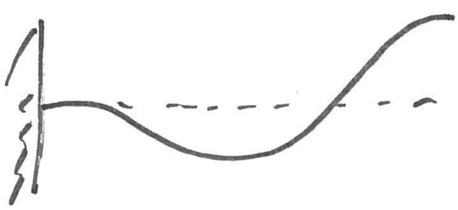
$$\{C\} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \qquad \{0\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Part B

mode 1



mode 3



mode 2

Q1. iii)

Part A)

At $x=0$

$x=0,\ Y=0$ $\qquad$ $\dfrac{\partial^2 y}{\partial x^2}=0$

from (1)

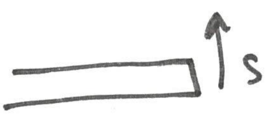
(a) $C_2 + C_4 = 0$

from (3)

(b) $\dfrac{\partial^2 y}{\partial x^2} = 0 = -C_1\lambda^2 \times 0 + C_2\lambda^2 \times 1 + C_3\lambda^2 \times 0 + C_4\lambda^2 \times 1$

$(+C_2 + C_4)\lambda^2 = 0$

At $x = L$



$S = EI \dfrac{\partial^3 Y}{\partial x^3}$

$S = EI\left(\dfrac{d^3 Y}{dx}\right)_{x=L} e^{i\omega t}$

$s \downarrow$ O J $\quad Y(L,t) = Y(L)e^{i\omega t}$

$S = m\ddot{Y}$

$S = -m\dfrac{EI}{\rho A}\omega^2\, Y(L)\, e^{i\omega t}$

where $\omega = \lambda L$

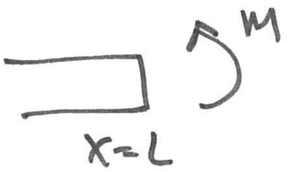
set equal to each other and solve

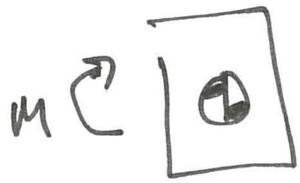
$$\text{(c)} \quad \left(\frac{d^3 \bar{Y}}{dx^3}\right)_{x=L} + \frac{m(\lambda L)^2}{\rho A}\bar{Y}(L) = 0$$

eqn (4)   eqn (5)

Substitute in eqns and solve w/ $x=L$

Also at $x=L$





Slope $\theta = \left(\frac{\partial Y}{\partial x}\right)_{x=L}$

$$M = -EI\frac{\partial^2 y}{\partial x^2}$$

$$y(L,t) = \bar{Y}(L)e^{i\omega t}$$

$$M = -EI\left(\frac{\partial^2 Y}{\partial x^2}\right)_{x=L} e^{i\omega t}$$

~~combine~~

$$M = I_m\left(\frac{\partial^2 \theta}{\partial t^2}\right)_{x=L}$$

$$\theta = \frac{\partial Y}{\partial x} = \frac{\partial \bar{Y}}{\partial \bar{x}}e^{i\omega t}$$

$$\frac{\partial^2 \theta}{\partial t^2} = -\omega^2 \frac{\partial \bar{Y}}{\partial x}e^{i\omega t}$$

$$\omega = \lambda L$$

$$M = -I_m(\lambda L)^2 \frac{\partial Y}{\partial x}e^{i\omega t}$$

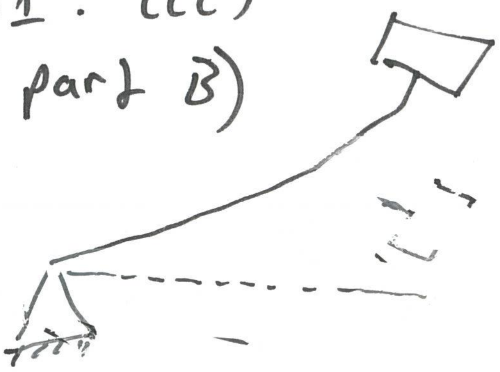
combine

$$\text{(d)} \quad \left(\frac{\partial^2 Y}{\partial x^2}\right)_{x=L} - \frac{I_m(\lambda L)^4}{\rho A L^4}\left(\frac{\partial Y}{\partial x}\right)_{x=L} = 0$$

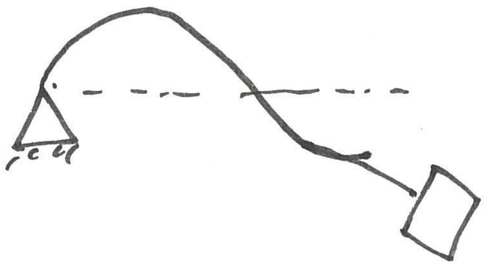
eqn (3)   sub in appropriate eqns   eqn (2)

Combine (a), (b), (c) & (d) into matrix form after subbing in equations (1), (2), (3) & (4) as required

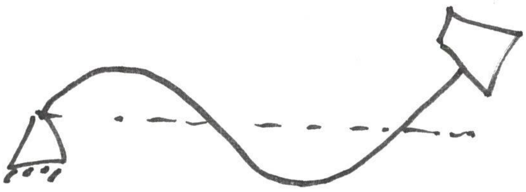
Q1. iii)
part B)



Mode 1



mode 2



mode 3

Q1 iv)

A)

At $x=0$ same as in iii)

ⓐ $C_2 + C_4 = 0$

ⓑ $(C_2 + C_4)\lambda^2 = 0$

At $x = L$

- Shear force summation is same as for part iii)

$$\therefore \text{ⓒ} \quad \left(\frac{d^3 Y}{\partial x^3}\right)_{x=L} + \frac{m(\lambda L)^2}{\rho A} Y(L) = 0$$

eqn ④ eqn ①

- Because no rotation

$$\text{ⓓ} \quad M = -EI\left(\frac{\partial^2 y}{\partial x^2}\right)_{x=L} = 0$$

eqn ③

for [2] combine ⓐ, ⓑ, ⓒ & ⓓ after subing in equation ①, ③, & ④ as required

B)

Mode 1

Mode 2

Mode 3

STRUCTURAL VIBRATION 1 SHEET 3

Q2 The roots $\lambda_r L$ are given on page 5 of the lecture handout.

$$\omega_r = \left(\frac{\lambda_r L}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} = 14.31\,(\lambda_r L)^2 \text{ rad/s}$$

$$= 136.7\,(\lambda_r L)^2 \text{ rev/min}$$

Q3 Free ends $\therefore$ $\dfrac{d^2Y}{dx^2} = \dfrac{d^3Y}{dx^3} = 0$ at $x = 0$ and $x = L$

This leads to the four boundary condition equations:

$$\begin{bmatrix} 0 & -\lambda^2 & 0 & \lambda^2 \\ -\lambda^3 & 0 & \lambda^3 & 0 \\ -\lambda^2.s & -\lambda^2.c & \lambda^2.sh & \lambda^2.ch \\ -\lambda^3.c & \lambda^3.s & \lambda^3.ch & \lambda^3.sh \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

where $s = \sin\lambda L$, $c = \cos\lambda L$, $sh = \sinh\lambda L$, $ch = \cosh\lambda L$

$$\det[Z] = \lambda^{10}\left\{ -(-1).\begin{vmatrix} -1 & 1 & 0 \\ -s & sh & ch \\ -c & ch & sh \end{vmatrix} -1.\begin{vmatrix} -1 & 0 & 1 \\ -s & -c & sh \\ -c & s & ch \end{vmatrix} \right\} = 0$$

$$= \lambda^{10}\left\{ -1.(sh^2 - ch^2) - 1.(-s.sh + c.ch) \right.$$
$$\left. + (-c.ch - s.sh) - 1.(-s^2 - c^2) \right\} = 0$$

Noting that $ch^2 - sh^2 = 1$ and $c^2 + s^2 = 1$, frequency equation becomes

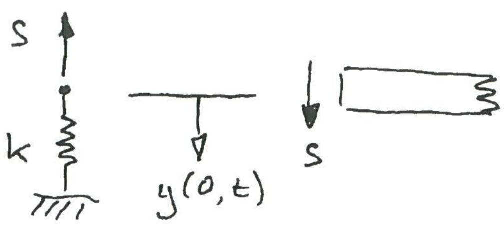
$$\lambda^2 (1 - \cos\lambda L . \cosh\lambda L) = 0$$

A free-free beam has 2 rigid body modes.

FEbeam.m is a Matlab program that uses the finite element method to analyse uniform beams. It's one of the options in the suite of animated help files that can be downloaded from the WebCT site. Follow the instructions to download the programs.

Q4

At $x = 0$:



Check the correct use of the sign convention defining the positive directions for S and y.

Beam : $$S = EI\left(\frac{\partial^3 y}{\partial x^3}\right)_{x=0}$$

Spring : $$S = -k \, . \, y(0,t)$$ [S & y are in <u>opposite</u> directions]

Put $y(x,t) = Y(x)\cos\omega t$ and eliminate S to give

$$EI\left(\frac{d^3Y}{dx^3}\right)_{x=0} + k \, . \, Y(0) = 0$$

Substitute for $Y(x)$ and put $x = 0$. Hence,

$$EI\lambda^3(-c_1 + c_3) + k(c_2 + c_4) = 0$$

Also at $x = 0$, $\frac{d^2Y}{dx^2} = 0$ (No bending moment at pinned end)

$$\therefore \; -\lambda^2 c_2 + \lambda^2 c_4 = 0$$

For the clamped end at $x = L$, $Y(L) = \left(\frac{dY}{dx}\right)_{x=L} = 0$

Q5

Pinned end at $x=0$. $y=0$ and $\frac{\partial^2 y}{\partial x^2}=0$

Free end at $x=L$. $\frac{\partial^2 y}{\partial x^2}=0$ and $\frac{\partial^3 y}{\partial x^3}=0$

This leads to:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -\lambda^2 & 0 & \lambda^2 \\ -\lambda^2.s & -\lambda^2.c & \lambda^2.sh & \lambda^2.ch \\ -\lambda^3.c & \lambda^3.s & \lambda^3.ch & \lambda^3.sh \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \begin{matrix} (a) \\ (b) \\ (c) \\ (d) \end{matrix}$$

or $[Z]\{C\} = \{0\}$

Expanding $\det[Z]=0$ gives

$$2\lambda^7(\sin\lambda L.\cosh\lambda L - \cos\lambda L.\sinh\lambda L) = 0$$

Beam has one rigid body mode. Dividing by $\cos\lambda L.\cosh\lambda L$ gives:

$$\lambda(\tan\lambda L - \tanh\lambda L) = 0$$

Mode shapes:

(a) and (b) imply that $c_2 = c_4 = 0$

(c) gives $c_3 = \frac{\sin\lambda_r L}{\sinh\lambda_r L}.c_1$

With $c_1 = 1$, the mode shape expression is

$$Y_r(x) = \sin\lambda_r x + \frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x$$