MTHS2007 Advanced Mathematics and Statistics for Mechanical Engineers

Chapter 3: Fourier series

School of Mathematical Sciences

The University of Nottingham

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3 Fourier series and periodic functions

A function of one variable x is said to be periodic if there exists a fixed number (called the *period*) L such that

$$f(x) = f(x+L) \quad \text{for all } x.$$

For example, $\sin\left(\frac{\pi}{\ell}x\right)$ and $\cos\left(\frac{\pi}{\ell}x\right)$ have period $L = 2\ell$: we can say they are 2ℓ-periodic.

In Engineering, such functions are extremely important: Oscillations of all kinds are described by periodic functions.

Idea: Approximate an arbitary 2ℓ-periodic function by a sum of simpler trigonometric funcitons such as $\sin\left(\frac{\pi}{\ell}x\right)$ and $\cos\left(\frac{\pi}{\ell}x\right)$. In acoustics, this would correspond to decomposing a complicated time series into its individual musical notes.

Symmetries

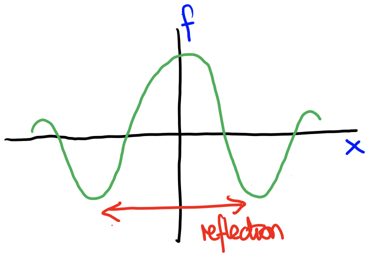
A function is **even** if

$$f(-x) = f(x)$$

Example The (periodic) function

$$f(x) = \cos\left(\frac{\pi x}{\ell}\right)$$

is even.



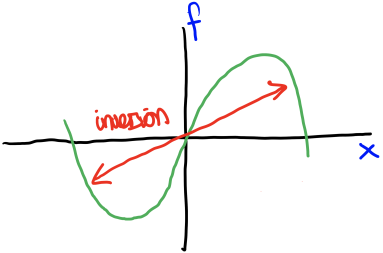
A function is **odd** if

$$f(-x) = -f(x)$$

Example The (periodic) function

$$f(x) = \sin\left(\frac{\pi x}{\ell}\right)$$

is odd.



Key idea of Fourier series

We know that functions $\sin\left(\frac{n\pi}{\ell}x\right)$ and $\cos\left(\frac{n\pi}{\ell}x\right)$, where n is any integer, are 2ℓ-periodic. We can try to use these for an approximation of any other 2ℓ-periodic function:

$$f(x) \approx f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{n\pi}{\ell}x\right) + b_n \sin\left(\frac{n\pi}{\ell}x\right)\right). \tag{1}$$

(Treating the coefficient a_0 differently to the coefficients a_n with $n > 0$ is done to make some general formulas coming later in the discussion easier to write down!)

We want to choose the coefficients such that the approximation gets more and more accurate the more terms we include in the sum, ie

$$f(x) = \lim_{N\to\infty} f_N(x).$$

In other words,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{\ell}x\right) + b_n \sin\left(\frac{n\pi}{\ell}x\right)\right).$$

How can we find coefficients a_n and b_n such that this is achieved?

Let us try something. Let us calculate

$$\int_{-\ell}^{\ell} f(x)\,\mathrm{d}x.$$

For $n \geq 1$, we have

$$\int_{-\ell}^{\ell} \cos\left(\frac{n\pi}{\ell}x\right) \mathrm{d}x = 0 \quad \text{and} \quad \int_{-\ell}^{\ell} \sin\left(\frac{n\pi}{\ell}x\right) \mathrm{d}x = 0.$$

This means

$$\int_{-\ell}^{\ell} f(x)\,\mathrm{d}x = \int_{-\ell}^{\ell} \frac{a_0}{2}\,\mathrm{d}x = a_0\ell.$$

But that gives us a way to calculate a_0!

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,\mathrm{d}x$$

What about a_n (and b_n)for $n \geq 1$?

Here it helps to know

$$\int_{-\ell}^{\ell} \cos\left(\frac{m\pi}{\ell}x\right) \cos\left(\frac{n\pi}{\ell}x\right) \mathrm{d}x = \begin{cases} \ell & m = n \\ 0 & m \neq n. \end{cases}$$

$$\int_{-\ell}^{\ell} \sin\left(\frac{m\pi}{\ell}x\right) \sin\left(\frac{n\pi}{\ell}x\right) \mathrm{d}x = \begin{cases} \ell & m = n \\ 0 & m \neq n. \end{cases}$$

$$\int_{-\ell}^{\ell} \sin\left(\frac{m\pi}{\ell}x\right) \cos\left(\frac{n\pi}{\ell}x\right) \mathrm{d}x = 0$$

Let us try to calculate (where $m \geq 1$)

$$\begin{aligned}
\int_{-\ell}^{\ell} f(x) \cos\left(\frac{m\pi}{\ell}x\right) \mathrm{d}x &= \frac{a_0}{2}\int_{-\ell}^{\ell} \cos\left(\frac{m\pi}{\ell}x\right) \mathrm{d}x \\
&\quad + \sum_{n=1}^{\infty} a_n \int_{-\ell}^{\ell} \cos\left(\frac{m\pi}{\ell}x\right) \cos\left(\frac{n\pi}{\ell}x\right) \mathrm{d}x \\
&\quad + \sum_{n=1}^{\infty} b_n \int_{-\ell}^{\ell} \cos\left(\frac{m\pi}{\ell}x\right) \sin\left(\frac{n\pi}{\ell}x\right) \mathrm{d}x \\
&= 0 + a_m + 0 = a_m.
\end{aligned}$$

Therefore

$$a_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x) \cos\left(\frac{m\pi}{\ell}x\right) \mathrm{d}x$$

or

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x) \cos\left(\frac{n\pi}{\ell}x\right) \mathrm{d}x$$

and similarly

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi}{\ell}x\right) \mathrm{d}x.$$

Let us summarise what we have just found.

Fourier series for a 2ℓ-periodic function f

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{\ell} x\right) + b_n \sin\left(\frac{n\pi}{\ell} x\right) \right)$$

$$a_0 = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x)\, \mathrm{d}x$$

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos\left(\frac{n\pi}{\ell} x\right) \mathrm{d}x \qquad (n \geq 1)$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi}{\ell} x\right) \mathrm{d}x \qquad (n \geq 1)$$

Example

Obtain the Fourier series for the 2π -periodic triangle wave

$$f(x) = \begin{cases} x + \dfrac{\pi}{2} & \text{for } -\pi < x < 0, \\ \dfrac{\pi}{2} - x & \text{for } 0 \leq x \leq \pi. \end{cases}$$

In this example, $\ell = \pi$.

All we need to do is to substitute $f(x)$ into the equations for a_0, a_n and b_n on the previous slide!

$$
\begin{aligned}
a_0 &= 0, \\
a_n &= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} \frac{\pi}{2}\cos(nx)\,\mathrm{d}x + \int_{-\pi}^{0} x\cos(nx)\,\mathrm{d}x \right. \\
&\qquad \left. + \int_{0}^{\pi} (-x)\cos(nx)\,\mathrm{d}x\right) \qquad \text{for} \quad n \geq 1, \\
b_n &= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} \frac{\pi}{2}\sin(nx)\,\mathrm{d}x + \int_{-\pi}^{0} x\sin(nx)dx \right. \\
&\qquad \left. + \int_{0}^{\pi} (-x)\sin(nx)\,\mathrm{d}x\right) = 0.
\end{aligned}
$$

Note that the first integrals in the expressions for both a_n and b_n are zero and that the second and third integrals in the expression for a_n take the same value whilst those in the expression for b_n cancel. Thus

$$
\begin{aligned}
a_n &= -\frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,\mathrm{d}x = -\frac{2}{\pi}\left[\frac{x\sin(nx)}{n}+\frac{\cos(nx)}{n^2}\right]_0^{\pi} \\
\Rightarrow a_n &= \frac{2}{n^2\pi}\left(1-(-1)^n\right),
\end{aligned}
$$

and the Fourier series for $f(x)$ is

$$
f(x) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left(1-(-1)^n\right)\cos(nx).
$$

Plots of the first 10 terms in the Fourier series are shown in figure (1).

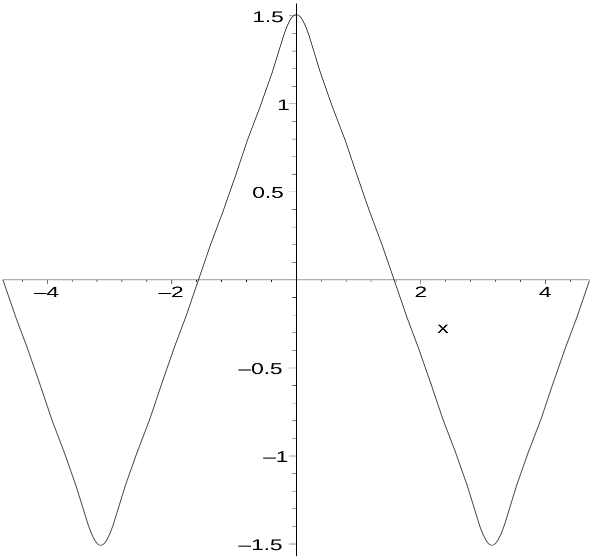


Figure 1: The first 10 terms in the Fourier series of f.

Remark

Note that f is an even function and furthermore that its Fourier series contains no terms involving $\sin(nx)$. It is in general true that the Fourier series for an even function f takes the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{\ell}x\right),$$

while that for an odd function g takes the form

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}x\right),$$

(convince yourself that these statements are true). Remembering these facts can save time when evaluating the coefficients in a Fourier series.

Example

Obtain the Fourier series for the periodic function g given by

$$g(x) = x \qquad \text{for} \quad -1 < x < 1.$$

Note that this is an odd function in x and hence $a_n = 0$ for all n.

$$\begin{aligned} b_n = \int_{-1}^{1} x \sin(n\pi x)\, \mathrm{d}x &= \left[-\frac{x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{(n\pi)^2} \right]_{-1}^{1} \\ \Rightarrow b_n &= -\frac{2}{n\pi}(-1)^n. \end{aligned}$$

Substituting this expression for b_n back into the Fourier series gives

$$g(x) = -\sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^n \sin(n\pi x).$$

The first 5 and first 40 terms of this series are plotted in figure 2.

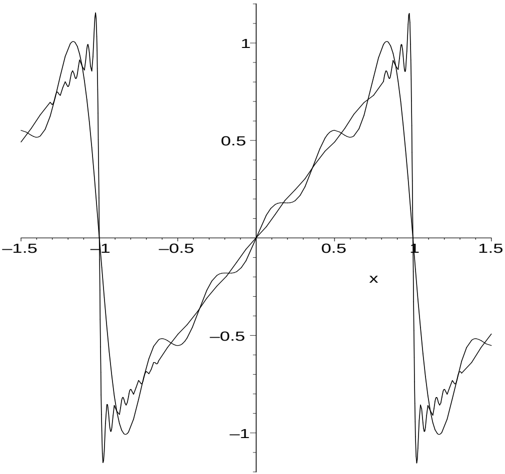


Figure 2: The first 5 terms and the first 40 terms in the Fourier series of g.

Remark

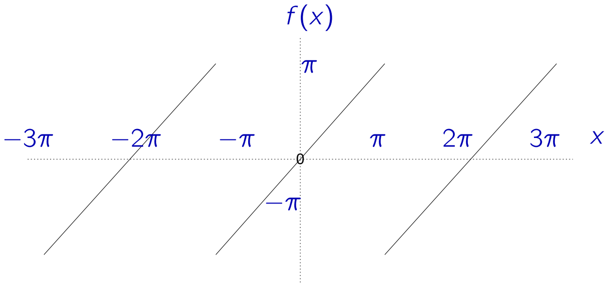
The Fourier series converges much more slowly for the function g (as we take more terms in the series) than it did for the 2π-periodic function f. This is because the periodic function g has a discontinuity in contrast to f which is continuous. In particular note that the Fourier series has difficulty approximating g close to the discontinuity where it has many finescale oscillations; this is termed Gibb's phenomenon. We consider these aspects in more detail after some more examples.

Example

Find the Fourier coefficients of the function f given by

$$f(x) = x \quad \text{for} \quad -\pi < x \leq \pi \quad \text{and} \quad f(x + 2\pi) = f(x)$$

The graph of the function in the region $-3\pi < x < 3\pi$ is



Note:

$$
\begin{aligned}
\int_{-\pi}^{\pi} x\cos(nx)\,\mathrm{d}x &= \left[\frac{x\sin(nx)}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin(nx)}{n}\,\mathrm{d}x \\
&= 0 + \left[\frac{\cos(nx)}{n^2}\right]_{-\pi}^{\pi} \\
&= 0 \\
\int_{-\pi}^{\pi} x\sin(nx)\,\mathrm{d}x &= \left[-\frac{x\cos(nx)}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(nx)}{n}\,\mathrm{d}x \\
&= -\frac{2\pi\cos(n\pi)}{n} + \left[\frac{\sin(nx)}{n^2}\right]_{-\pi}^{\pi} \\
&= -\frac{2\pi(-1)^n}{n} + 0 \\
&= \frac{2\pi(-1)^{n+1}}{n}
\end{aligned}
$$

since $\cos(n\pi) = (-1)^n$ and $-\cos(n\pi) = (-1)^{n+1}$.

The Fourier coefficients are

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \, \mathrm{d}x, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos(nx) \, \mathrm{d}x, \qquad n = 1, 2, \ldots, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx) \, \mathrm{d}x, \qquad n = 1, 2, \ldots \;. \end{aligned}$$

Evaluating the integrals we obtain the results

$$a_0 = 0, \;\; a_n = 0, \;\; b_n = \frac{2(-1)^{n+1}}{n} \;\; \text{for } n = 1, 2, \ldots \;.$$

Hence the required Fourier series is

$2\left(\sin x - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x)\ldots\right),$

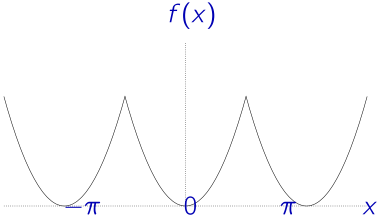
$$\text{ie.} \quad f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(nx)}{n}.$$

Another example

Find the Fourier coefficients of the function f defined by

$$f(x) = x^2 \quad \text{for} \quad -\pi < x \le \pi, \text{ and } f(x+2\pi) = f(x).$$

The graph of the function in the region $-3\pi < x < 3\pi$ is



The Fourier coefficients are

$$
\begin{aligned}
a_0 &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} x^2\,\mathrm{d}x \\
a_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,\mathrm{d}x, \qquad n = 1, 2, \ldots. \\
b_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)\,\mathrm{d}x, \qquad n = 1, 2, \ldots.
\end{aligned}
$$

Evaluating the integrals we obtain the results

$$a_0 = \frac{2\pi^2}{3},\ a_n = \frac{4(-1)^n}{n^2},\ b_n = 0 \text{ for } n = 1, 2, \ldots.$$

Hence the required Fourier series is

$$\tfrac{\pi^2}{3} + 4\left(-\cos x + \tfrac{1}{4}\cos(2x) - \tfrac{1}{9}\cos(3x) + \tfrac{1}{16}\cos(4x) + \ldots\right)$$

$$\text{ie} \quad \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx)}{n^2}.$$

And one more example

If $g(x) = x^2$ for $-\ell \le x < \ell$ and

$$g(x + 2\ell) = g(x),$$

the Fourier series for $g(x)$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)$$

$$\text{where } a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} x^2\,dx = \frac{2\ell^2}{3}$$

$$\text{and } a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} x^2 \cos\left(\frac{n\pi x}{\ell}\right) dx = \frac{4\ell^2(-1)^n}{n^2\pi^2}$$

Note:

(1) $b_n = 0$ for $n = 1, 2, \ldots$, because g is an even function

(2) If $\ell = \pi$, the series reduces to that from the previous example.

Still not running out of examples

Find the Fourier series of the periodic function f given by

$$f(x) = e^x \qquad \text{for} \quad -2 \leq x < 2, \quad \text{and}$$

$$f(x+4) = f(x).$$

This function is neither even nor odd, so we have to find all coefficients a_n and b_n. Let's start with a_0.

$$a_0 = \frac{1}{2}\int_{-2}^{2} e^x \, \mathrm{d}x = \frac{1}{2}\left[e^x\right]_{-2}^{2} = \frac{1}{2}\left(e^2 - e^{-2}\right) = \sinh(2).$$

Now let $n \geq 1$.

$$
\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} e^x \cos\left(\frac{n\pi}{2}x\right)\,\mathrm{d}x \\
&= \frac{1}{2}\left[e^x \cos\left(\frac{n\pi}{2}x\right)\right]_{-2}^{2} + \frac{n\pi}{4}\int_{-2}^{2} e^x \sin\left(\frac{n\pi}{2}x\right)\,\mathrm{d}x \\
&= \frac{(-1)^n}{2}\left(e^2 - e^{-2}\right) + \frac{n\pi}{4}\left[e^x \sin\left(\frac{n\pi}{2}x\right)\right]_{-2}^{2} \\
&\quad -\frac{n^2\pi^2}{8}\int_{-2}^{2} e^x \cos\left(\frac{n\pi}{2}x\right)\,\mathrm{d}x \\
&= (-1)^n \sinh(2) - \frac{n^2\pi^2}{4}a_n
\end{aligned}
$$

Solving for a_n gives

$$\left(1 + \frac{n^2\pi^2}{4}\right)a_n = (-1)^n \sinh(2), \quad \text{ie.} \quad a_n = \frac{4}{4+n^2\pi^2}(-1)^n \sinh(2).$$

Time to find b_n.

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} e^x \sin\left(\frac{n\pi}{2}x\right)\,\mathrm{d}x \\ &= \frac{1}{2}\left[e^x \sin\left(\frac{n\pi}{2}x\right)\right]_{-2}^{2} - \frac{n\pi}{4}\int_{-2}^{2} e^x \cos\left(\frac{n\pi}{2}x\right)\,\mathrm{d}x \\ &= -\frac{n\pi}{4}\int_{-2}^{2} e^x \cos\left(\frac{n\pi}{2}x\right)\,\mathrm{d}x \end{aligned}$$

Now there are (at least) two ways to solve this. We could integrate by parts again, and then solve for b_n as we did for a_n. Alternatively, we can spot that

$$b_n = -\frac{n\pi}{2}a_n, \quad \text{ie} \quad b_n = -\frac{2n\pi}{4+n^2\pi^2}(-1)^n \sinh(2).$$

Now we've got our desired Fourier series,

$$
\begin{aligned}
f(x) \quad = \quad & \sinh(2)\left[\frac{1}{2} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{4}{4+n^2\pi^2}\cos\left(\frac{n\pi}{2}x\right)\right.\right. \\
& \left.\left. - \frac{2n\pi}{4+n^2\pi^2}\sin\left(\frac{n\pi}{2}x\right)\right)\right].
\end{aligned}
$$

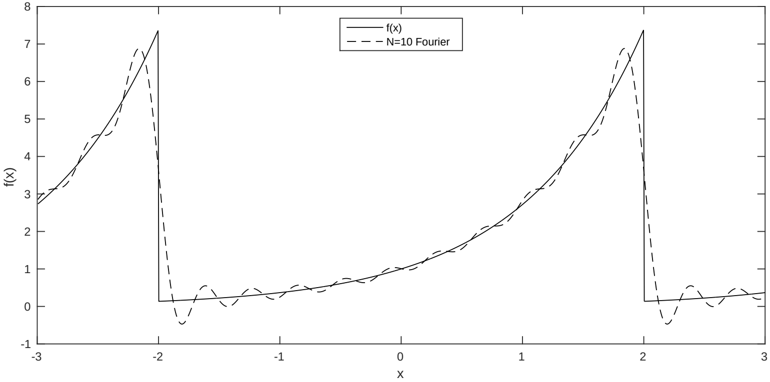


Figure 3: The first 10 terms in the Fourier series of f.

A final example

Find the Fourier series for the function g given by

$$g(x) = \begin{cases} 0 & (x = -\pi) \\ -1 + \sin(3x) & (-\pi < x < 0) \\ 0 & (x = 0) \\ 1 + \sin(3x) & (0 < x < \pi) \end{cases}$$

and

$$g(x + 2\pi) = g(x).$$

Fortunately, the function is a lot simpler than it looks: It is a sum of an odd step function, and $\sin(3x)$.

Let's find the Fourier series for the step function first.

$$h(x) = \begin{cases} 0 & (x = -\pi) \\ -1 & (-\pi < x < 0) \\ 0 & (x = 0) \\ 1 & (0 < x < \pi) \end{cases} \qquad h(x + 2\pi) = h(x)$$

As h is an odd function, we have $a_n = 0$ for all $n \geq 0$.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\sin(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi} h(x)\sin(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi} \sin(nx)\,\mathrm{d}x \\ &= -\frac{2}{n\pi}\left[\cos(nx)\right]_0^{\pi} \\ &= \frac{2}{n\pi}\left(1-(-1)^n\right) \end{aligned}$$

Now the Fourier series for g is obtained by adding $\sin(3x)$ to the Fourier series for h, ie

$$g(x) = \sum_{n=0}^{\infty} \left[\frac{2}{n\pi} (1 - (-1)^n) + \delta_{n3} \right] \sin(nx)$$

$$= \frac{4}{\pi} \sin(x) + \left(\frac{4}{3\pi} + 1 \right) \sin(3x) + \frac{4}{5\pi} \sin(5x) + \frac{4}{7\pi} \sin(7x) + \ldots.$$

See the next slide for the graph of g and the Fourier series up to and including $n = 19$.

We can see that the first terms of the Fourier series do not approximate the function g very well, especially at the points $x = \pm\pi$ and $x = 0$, where the function g is discontinuous.

We have to understand whether the Fourier series really converges to $g(x)$ at all points x!

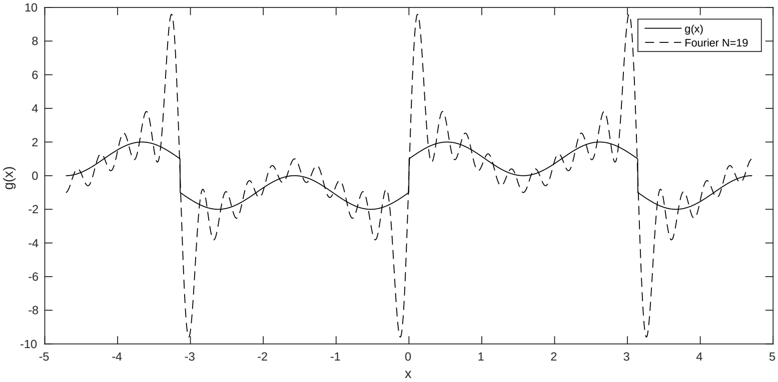


Figure 4: The first 19 terms in the Fourier series of g.

Properties of Fourier series

A Fourier series is an infinite series used to represent function values and therefore we must ask the following questions

- Is it convergent?
- If so, to what does it converge?
- Can we apply differentiation to the terms in the series to represent $f'(x)$?
- Can we apply integration to the terms in the series to represent $\int f(x)\,\mathrm{d}x$?

Continuity

A function f is continuous at $x = a$, if

(1) the limit as $x \to a$ from values of $x > a$, written $\lim_{x \to a^+} f(x)$

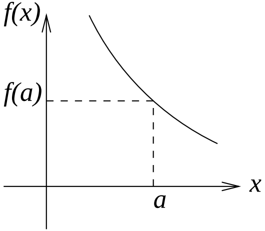
(2) the limit as $x \to a$ from values of $x < a$, written, $\lim_{x \to a^-} f(x)$

are equal and are *also* equal to $f(a)$:

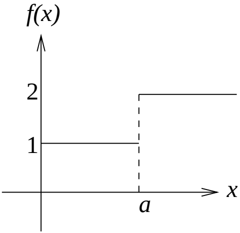
$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = f(a).$$

Examples

(1) a continuous function

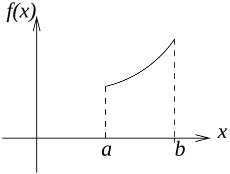


(2) a discontinuous function



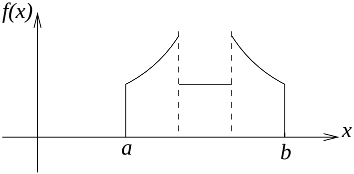
- A function is continuous in the interval $a < x < b$ if it is continuous at every point of the interval $a < x < b$.

Example



- A function is said to be *piecewise continuous* in an interval $a < x < b$ if it has at most a finite number of finite discontinuities in the interval $a < x < b$.

Example



(Periodic) piecewise continuous functions can be represented by Fourier series!

Example

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x \leq 0 \\ 1 & \text{for } 0 < x \leq \pi \end{cases}$$

$$f(x + 2\pi) = f(x)$$

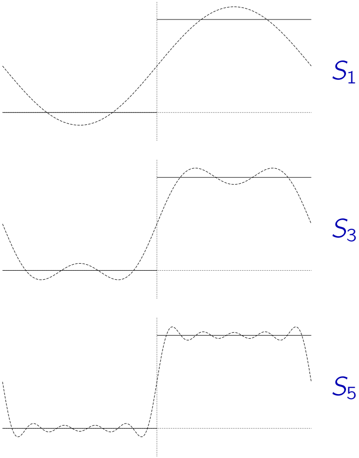
The Fourier coefficients are

$$a_0 = 1, \;\; a_n = 0, \;\; b_n = \frac{1 - (-1)^n}{n\pi} \qquad n = 1, 2, \ldots$$

The Fourier series is therefore

$$\frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin(3x) + \frac{2}{5\pi} \sin(5x) \ldots.$$

$$\begin{aligned} \text{Define} \qquad S_1 &= \tfrac{1}{2} + \tfrac{2}{\pi} \sin x \\ S_3 &= S_1 + \tfrac{2}{3\pi} \sin 3x \\ S_5 &= S_3 + \tfrac{2}{5\pi} \sin 5x \quad \text{etc.} \end{aligned}$$



S_1
S_3
S_5

In this example, it "looks like"

- at points where f is continuous, say $x = a$, the sequence of Fourier sums tends to $f(a)$,
- at points where f is discontinuous, say $x = b$, the sequence of Fourier sums tends to

$$\frac{1}{2}\left[f(b^+) + f(b^-)\right],$$

ie mid-way between the two values.

This result is called *Fourier's theorem*.

Differentiability

- A function f is differentiable at $x = a$ if

$$\lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \right] \quad \text{exists.}$$

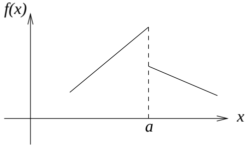
  If it exists, the derivative is written as $f'(a)$.

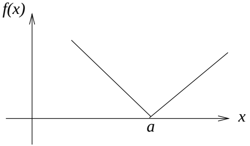
- This limit may not exist at $x = a$, that is $f(x)$ may not be differentiable at $x = a$, if

  (1) f is discontinuous at $x = a$,

  or

  (2) the curve $y = f(x)$ does not have a tangent at $x = a$.





A Fourier series can be used for integrating functions.

Theorem
If f is piecewise continuous in $-\ell < x < \ell$ and periodic with period 2ℓ then the Fourier series can be integrated term by term.

The Fourier series for *some* functions may be differentiated term by term.

Theorem
The Fourier series for $f'(x)$ can be obtained by differentiating the Fourier series for $f(x)$

ONLY

if

- f is differentiable,

and

- f' is piecewise continuous.