University of Nottingham
Department of Mechanical Engineering

MM2MS3 Mechanics of Solids 3
Exercise Sheet 6 – Strain Energy Methods Solutions

1. Using strain energy, derive an expression for the end deflection of the cantilever beam shown in Fig Q1. I = Second Moment of Area of cross-section and E = Young's Modulus of the beam.

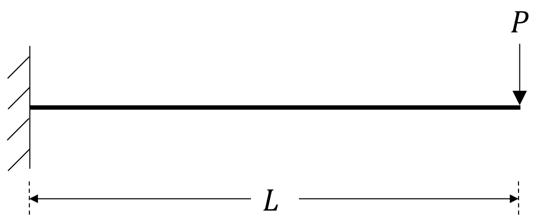
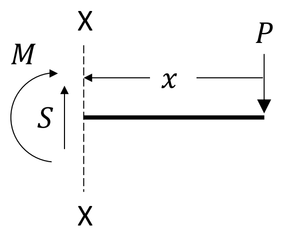


Fig Q1

[Ans: $\boldsymbol{u_v = \frac{PL^3}{3EI}}$***]***

Solution 1

Free Body Diagram:



Taking moments about X-X:

$$M = Px$$

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U = \int \frac{M^2}{2EI} ds = \int_0^L \frac{P^2 x^2}{2EI} dx = \frac{P^2}{2EI} \int_0^L x^2 dx = \frac{P^2}{2EI} \left[\frac{x^3}{3}\right]_0^L = \frac{P^2}{2EI} \left(\frac{L^3}{3} - 0\right)$$

$$\therefore U = \frac{P^2 L^3}{6EI}$$

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End Deflection Calculation for a Cantilever Beam using Castigliano's Theorem

$$u_v = \frac{\delta U}{\delta P}$$

$$\boxed{\therefore u_v = \frac{PL^3}{3EI}}$$

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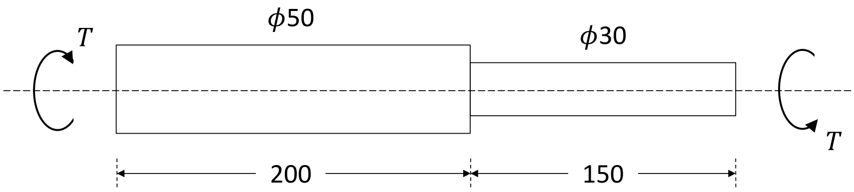
2. The stepped steel shaft shown in Fig Q2 carries a uniform torque of 500Nm. Determine the total torsional strain energy stored in the shaft. Assume G_{steel} = 70GPa.



All dimensions in mm

Fig Q2

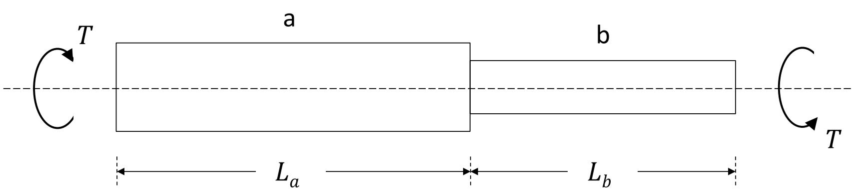
[Ans: 3.95J]

Solution 2

Torsional Strain Energy is given as,

$$U = \int \frac{T^2}{2GJ} ds$$

Labelling each section of the shaft (a and b) and the lengths of the sections, L_a and L_b, respectively, as follows,



Section a

Strain Energy for section a,

$$U_a = \int_0^{L_a} \frac{T^2}{2GJ_a} dx$$

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Section b

Strain Energy for section b,

$$U_b = \int_0^{L_b} \frac{T^2}{2GJ_b} dx$$

Total Strain Energy

$$U = U_a + U_b = \int_0^{L_a} \frac{T^2}{2GJ_a} dx + \int_0^{L_b} \frac{T^2}{2GJ_b} dx$$

$$= \frac{T^2}{2GJ_a} \int_0^{L_a} 1dx + \frac{T^2}{2GJ_b} \int_0^{L_b} 1dx = \frac{T^2}{2GJ_a} [x]_0^{L_a} + \frac{T^2}{2GJ_b} [x]_0^{L_b} = \frac{T^2}{2GJ_a} (L_a - 0) + \frac{T^2}{2GJ_b} (L_b - 0)$$

$$\therefore U = \frac{T^2 L_a}{2GJ_a} + \frac{T^2 L_b}{2GJ_b} \quad (1)$$

Where,

$$J_a = \frac{\pi {d_a}^4}{32} = \frac{\pi \times 50^4}{32} = 613{,}592.32mm^4$$

and,

$$J_b = \frac{\pi {d_b}^4}{32} = \frac{\pi \times 30^4}{32} = 79{,}521.56mm^4$$

Substituting values for T, L_a, L_b, G, J_a and J_b into (1),

$$\boxed{U = 3950.41Nmm = 3.95Joules}$$

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3. Using strain energy, derive an expression for the deflection at the load point of the beam shown in Fig Q3.

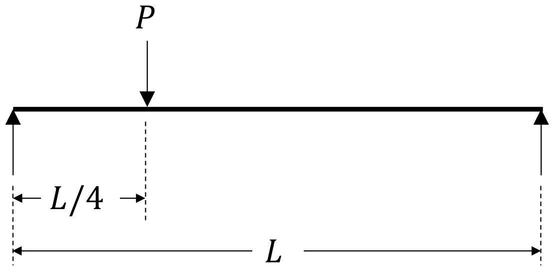
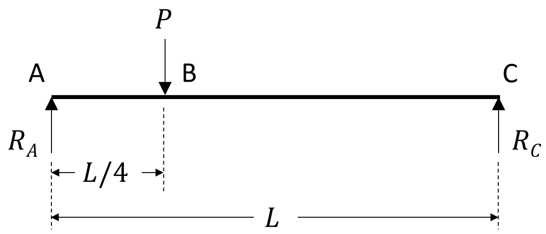


Fig Q3

[Ans: $\frac{9PL^3}{768EI}$]

Solution 3

Find reaction forces as follows:



Vertical equilibrium:

$$R_A + R_C = P \quad \text{(i)}$$

Taking moments about A:

$$\frac{PL}{4} = R_C L$$

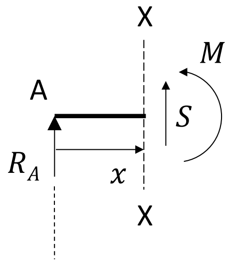
$$\therefore R_C = \frac{P}{4} \quad \text{(ii)}$$

Substituting (ii) into (i) gives:

$$R_A = \frac{3P}{4} \qquad \text{(iii)}$$

Section AB (*bending only*)

Free Body Diagram:



Taking moments about X-X:

$$M_{AB} = R_A x$$

Substituting (iii) into this gives,

$$M_{AB} = \frac{3Px}{4}$$

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{AB} = \int \frac{{M_{AB}}^2}{2EI} ds = \int_0^{L/4} \frac{\left(\frac{3Px}{4}\right)^2}{2EI} dx = \frac{9P^2}{32EI} \int_0^{L/4} x^2 dx = \frac{9P^2}{32EI} \left[\frac{x^3}{3}\right]_0^{L/4} = \frac{9P^2}{32EI} \left(\frac{L^3}{192} - 0\right)$$

Substituting (iii) into this gives,

$$U_{AB} = \frac{9P^2 L^3}{6144EI}$$

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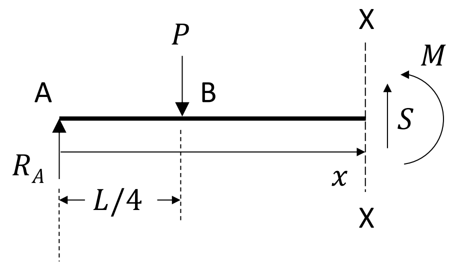
Exercise Sheet 6 – Strain Energy Methods Solutions

Section BC (*bending only*)

Free Body Diagram:



Taking moments about X-X:

$$M_{BC} + P\left(x - \frac{L}{4}\right) = R_A x$$

$$\therefore M_{BC} = R_A x - P\left(x - \frac{L}{4}\right)$$

Substituting (iii) into this gives,

$$M_{BC} = \frac{3Px}{4} - Px + \frac{PL}{4} = \frac{P}{4}(L - x)$$

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{BC} = \int \frac{{M_{BC}}^2}{2EI} ds = \frac{P^2}{32EI}\int_{L/4}^{L}(L-x)^2 dx = \frac{P^2}{32EI}\int_{L/4}^{L}(L^2 - 2Lx + x^2)dx = \frac{P^2}{32EI}\left[L^2x - Lx^2 + \frac{x^3}{3}\right]_{L/4}^{L}$$

$$= \frac{P^2}{32EI}\left(L^3 - L^3 + \frac{L^3}{3} - \frac{L^3}{4} + \frac{L^3}{16} - \frac{L^3}{192}\right)$$

$$\therefore U_{BC} = \frac{27P^2L^3}{6144EI}$$

Total Strain Energy

$$U = U_{AB} + U_{BC} = \frac{9P^2L^3}{6144EI} + \frac{27P^2L^3}{6144EI} = \frac{36P^2L^3}{6144EI}$$

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Exercise Sheet 6 – Strain Energy Methods Solutions

Deflection Calculation using Castigliano's Theorem

$$u_{vB} = \frac{\delta U}{\delta P}$$

$$\therefore u_{vB} = \frac{72PL^3}{6144EI} = \frac{9PL^3}{768EI}$$

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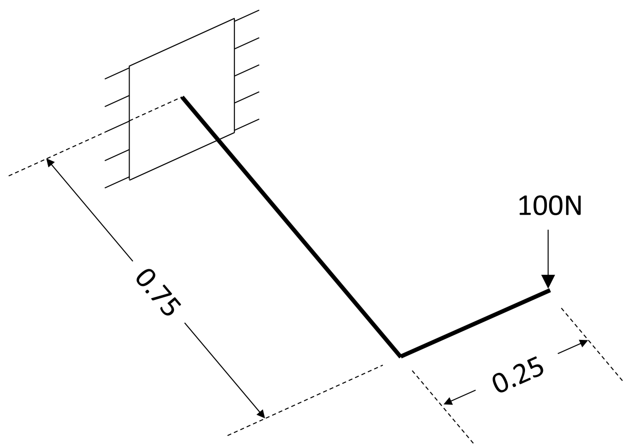
4. Calculate the deflection beneath the force for the cantilevered bracket shown in Fig Q4. The bar is circular in cross section with a diameter, ϕ of 20mm, a Young's Modulus, E of 200GPa and a Shear Modulus, G of 80GPa.



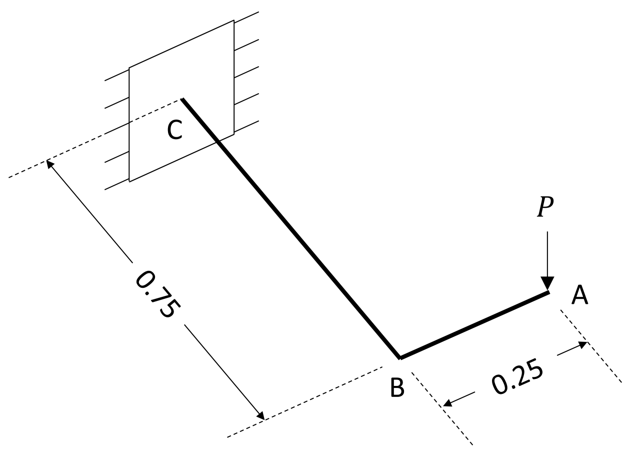
All dimensions in meters

Fig Q4

[Ans: 13mm]

Solution 4

Label lengths and ends of each section:



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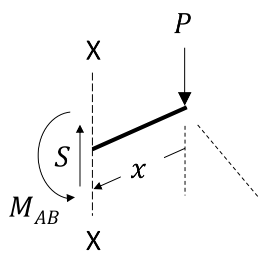
Exercise Sheet 6 – Strain Energy Methods Solutions

Section AB (*bending only*)

Free Body Diagram:



Taking moments about X-X:

$$M_{AB} = Px$$

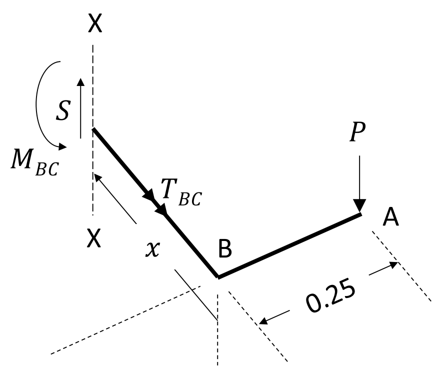
Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{AB} = \int \frac{{M_{AB}}^2}{2EI} ds = \int_0^a \frac{(Px)^2}{2EI} dx = \frac{P^2}{2EI}\int_0^a x^2 dx = \frac{P^2}{2EI}\left[\frac{x^3}{3}\right]_0^a = \frac{P^2}{2EI}\left(\frac{a^3}{3} - 0\right)$$

$$\therefore U_{AB} = \frac{P^2 a^3}{6EI}$$

Section BC

Free Body Diagram:



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Exercise Sheet 6 – Strain Energy Methods Solutions

Bending

Taking moments about X-X:

$$M_{BC} = Px$$

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{BC}^{bending} = \int \frac{{M_{BC}}^2}{2EI} ds = \int_0^b \frac{(Px)^2}{2EI} dx = \frac{P^2}{2EI} \int_0^b x^2 dx = \frac{P^2}{2EI} \left[\frac{x^3}{3}\right]_0^b = \frac{P^2}{2EI} \left(\frac{b^3}{3} - 0\right)$$

$$\therefore U_{BC}^{bending} = \frac{P^2 b^3}{6EI}$$

Torsion

Taking moments about X-X:

$$T_{BC} = Pa$$

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{BC}^{torsion} = \int \frac{{T_{BC}}^2}{2GJ} ds = \int_0^b \frac{(Pa)^2}{2GJ} dx = \frac{P^2 a^2}{2GJ} \int_0^b 1 dx = \frac{P^2 a^2}{2GJ} [x]_0^b = \frac{P^2 a^2}{2GJ} (b - 0)$$

$$\therefore U_{BC}^{torsion} = \frac{P^2 a^2 b}{2GJ}$$

$$U_{BC} = U_{BC}^{bending} + U_{BC}^{torsion} = \frac{P^2 b^3}{6EI} + \frac{P^2 a^2 b}{2GJ}$$

Total Strain Energy

$$U = U_{AB} + U_{BC} = \frac{P^2 a^3}{6EI} + \frac{P^2 b^3}{6EI} + \frac{P^2 a^2 b}{2GJ}$$

$$\therefore U = \frac{P^2}{2} \left(\frac{a^3}{3EI} + \frac{b^3}{3EI} + \frac{a^2 b}{GJ}\right)$$

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Deflection Calculation using Castigliano's Theorem

$$u_{vA} = \frac{\delta U}{\delta P} = P\left(\frac{a^3}{3EI} + \frac{b^3}{3EI} + \frac{a^2 b}{GJ}\right)$$

Where,

$$I = \frac{\pi d^4}{64} = \frac{\pi \times 0.02^4}{64} = 7.85 \times 10^{-9} m^4$$

and,

$$J = \frac{\pi d^4}{32}(= 2I) = 1.57 \times 10^{-8} m^4$$

$$\therefore u_{vA} = 0.013m = 13mm$$

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5. Derive an expression for the increase in distance between the ends A and D of a thin bar of uniform cross-section consisting of a semi-circular portion BC and two straight portions AB and CD as shown in Fig Q5.

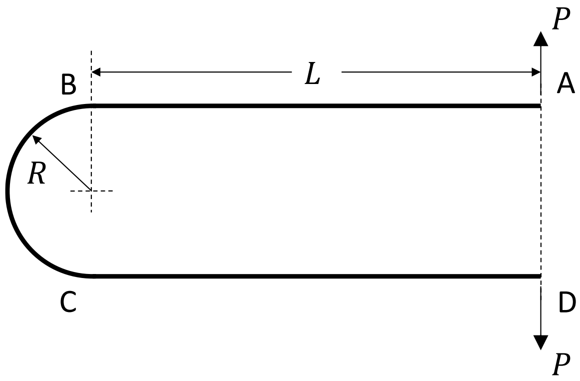


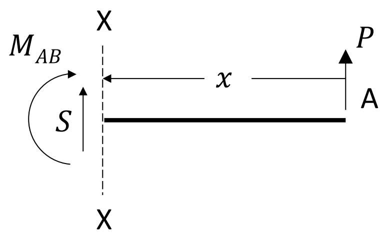
Fig Q5

If the bar is of diameter 6mm, R is 40mm and is to have a spring stiffness, P/δ of 100kg/m, show that the necessary length for L, is approximately 210mm. The bar is made from mild steel with Young's modulus, E = 210GPa.

Solution 5

Section AB (*bending only*)

Free Body Diagram:



Taking moments about X-X:

$$M_{AB} = Px$$

Substituting this into the equation for Strain Energy in a beam under bending gives,

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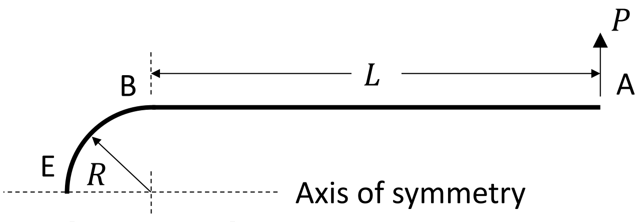
$$U_{AB} = \int \frac{{M_{AB}}^2}{2EI} ds = \int_0^L \frac{P^2 x^2}{2EI} dx = \frac{P^2}{2EI} \int_0^L x^2 dx = \frac{P^2}{2EI} \left[\frac{x^3}{3}\right]_0^L = \frac{P^2}{2EI} \left(\frac{L^3}{3} - 0\right)$$

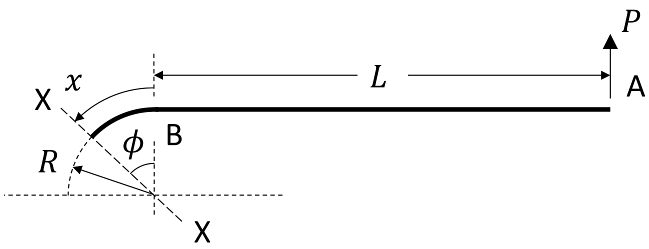
$$\therefore U_{AB} = \frac{P^2 L^3}{6EI}$$

Section BC (*bending only*)

Due to symmetry, only half of this section must be considered as shown in the diagram below,



For section BE, at angle ϕ, the following Free Body Diagram can be drawn,



Taking moments about X-X:

$$M_{BE} = P(L + Rsin\phi)$$

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{BE} = \int \frac{{M_{BE}}^2}{2EI} ds = \int_0^{\pi/2} \frac{\left(P(L + Rsin\phi)\right)^2}{2EI} Rd\phi$$

where,

$$dx = Rd\phi$$

Therefore,

$$U_{BE} = \frac{P^2R}{2EI}\int_0^{\pi/2} (L^2 + 2LRsin\phi + R^2sin^2\phi)d\phi \tag{1}$$

Trigonometric Identities:

$$sin^2\phi + cos^2\phi = 1 \tag{2}$$

and,

$$cos2\phi = cos^2\phi - sin^2\phi \tag{3}$$

Rearranging (2) gives,

$$cos^2\phi = 1 - sin^2\phi$$

Substituting this into (3) gives,

$$cos2\phi = 1 - 2sin^2\phi$$

$$\therefore sin^2\phi = \frac{1}{2} - \frac{cos2\phi}{2}$$

Substituting this into (1) gives,

$$U_{BE} = \frac{P^2R}{2EI}\int_0^{\pi/2}\left(L^2 + 2LRsin\phi + \frac{R^2}{2}(1 - cos2\phi)\right)d\phi = \frac{P^2R}{2EI}\left[L^2\phi - 2LRcos\phi + \frac{R^2}{2}(\phi - sin\phi cos\phi)\right]_0^{\pi/2}$$

$$= \frac{P^2R}{2EI}\left(\left(\frac{\pi L^2}{2} - 2LRcos\left(\frac{\pi}{2}\right) + \frac{R^2}{2}\left(\frac{\pi}{2} - sin\left(\frac{\pi}{2}\right)cos\left(\frac{\pi}{2}\right)\right)\right) - \left(-2LRcos(0)\right)\right)$$

$$\therefore U_{BE} = \frac{P^2R}{2EI}\left(\frac{\pi L^2}{2} + \frac{\pi R^2}{4} + 2LR\right)$$

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Total Strain Energy

The strain energy above the axis of symmetry is:

$$U_{above} = U_{AB} + U_{BE} = \frac{P^2L^3}{6EI} + \frac{P^2R}{2EI}\left(\frac{\pi L^2}{2} + \frac{\pi R^2}{4} + 2LR\right) = \frac{P^2}{EI}\left(\frac{L^3}{6} + \frac{\pi L^2 R}{4} + \frac{\pi R^3}{8} + LR^2\right)$$

As the strain energy below the axis of symmetry is equal to that above, the total strain energy is,

$$U = 2(U_{AB} + U_{BE}) = \frac{P^2}{EI}\left(\frac{L^3}{3} + \frac{\pi L^2 R}{2} + \frac{\pi R^3}{4} + 2LR^2\right)$$

Vertical Deflection Calculation using Castigliano's Theorem

$$u_{vA} = \delta = \frac{\delta U}{\delta P} = \frac{2P}{EI}\left(\frac{L^3}{3} + \frac{\pi L^2 R}{2} + \frac{\pi R^3}{4} + 2LR^2\right)$$

Rearranging this for stiffness gives,

$$\frac{P}{\delta} = \frac{EI}{2\left(\frac{L^3}{3} + \frac{\pi L^2 R}{2} + \frac{\pi R^3}{4} + 2LR^2\right)} \qquad (4)$$

where,

$$I = \frac{\pi d^4}{64} = \frac{\pi \times 6^4}{64} = 63.62mm^4$$

Putting L = 210mm into (4) gives,

$$\therefore \frac{P}{\delta} = 1.02\frac{N}{mm} = 0.104\frac{kg}{mm} \approx 104\frac{kg}{m}$$

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6. Considering the effect of the bending only, determine the horizontal deflection of the point A of the frame shown in Fig Q6 due to the force P.

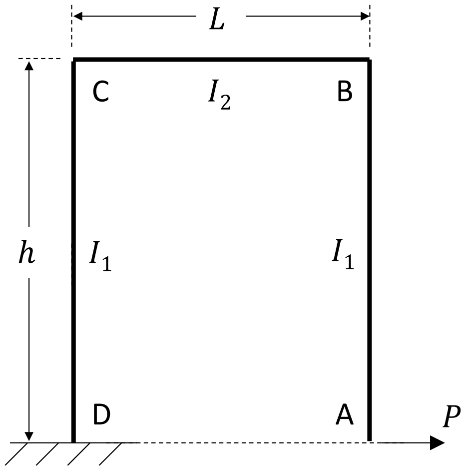


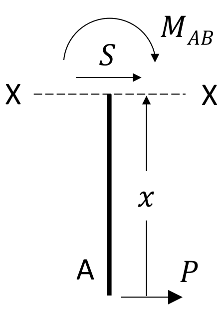
Fig Q6

[Ans: $\frac{Ph^2}{E}\left(\frac{2h}{3I_1}+\frac{L}{I_2}\right)$***]***

Solution 6

Section AB (*bending only*)

Free Body Diagram:



Taking moments about X-X:

$$M_{AB} = Px$$

Substituting this into the equation for Strain Energy in a beam under bending gives,

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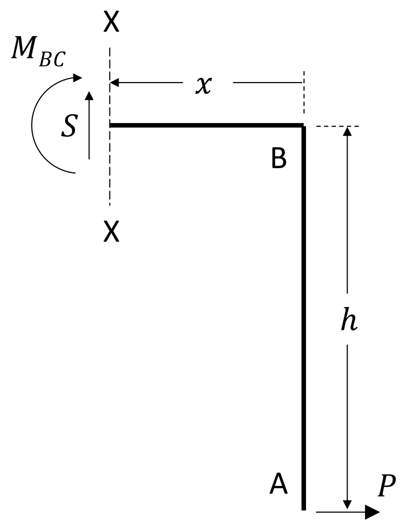
$$U_{AB} = \int \frac{{M_{AB}}^2}{2EI_1} ds = \int_0^h \frac{P^2 x^2}{2EI_1} dx = \frac{P^2}{2EI_1} \int_0^h x^2 dx = \frac{P^2}{2EI_1} \left[\frac{x^3}{3}\right]_0^h = \frac{P^2}{2EI_1}\left(\frac{h^3}{3} - 0\right)$$

$$\therefore U_{AB} = \frac{P^2 h^3}{6EI_1}$$

Section BC (*bending only*)

Free Body Diagram:



Taking moments about X-X:

$$M_{BC} = Ph$$

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{BC} = \int \frac{{M_{BC}}^2}{2EI_2} ds = \int_0^L \frac{P^2 h^2}{2EI_2} dx = \frac{P^2 h^2}{2EI_2} \int_0^L 1 dx = \frac{P^2 h^2}{2EI_2} [x]_0^L = \frac{P^2 h^2}{2EI_2} (L - 0)$$

$$\therefore U_{BC} = \frac{P^2 h^2}{2EI_2} L$$

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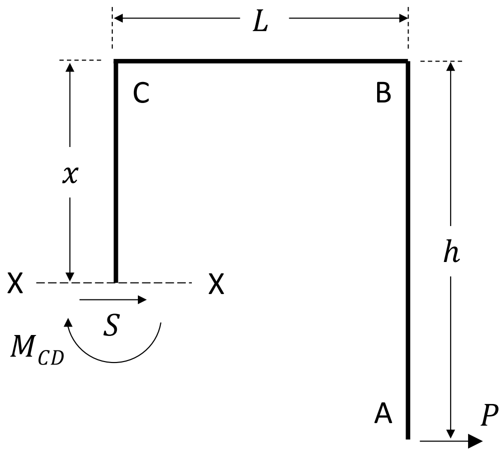
Section CD (*bending only*)

Free Body Diagram:



Taking moments about X-X:

$$M_{CD} = P(h-x)$$

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{CD} = \int \frac{{M_{CD}}^2}{2EI_1} ds = \int_0^h \frac{\left(P(h-x)\right)^2}{2EI_1} dx = \int_0^h \frac{P^2(h-x)^2}{2EI_1} dx = \frac{1}{2EI_1} \int_0^h P^2(h^2 - 2hx + x^2) dx$$

$$= \frac{P^2}{2EI_1}\left[h^2 x - hx^2 + \frac{x^3}{3}\right]_0^h = \frac{P^2}{2EI_1}\left(h^3 - h^3 + \frac{h^3}{3}\right)$$

$$\therefore U_{CD} = \frac{P^2 h^3}{6EI_1}$$

Total Strain Energy

$$U = U_{AB} + U_{BC} + U_{CD} = \frac{P^2 h^3}{6EI_1} + \frac{P^2 h^2}{2EI_2} L + \frac{P^2 h^3}{6EI_1} = \frac{P^2 h^2}{E}\left(\frac{h}{3I_1} + \frac{L}{2I_2}\right)$$

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Exercise Sheet 6 – Strain Energy Methods Solutions

Deflection Calculation using Castigliano's Theorem

$$u_{vA} = \frac{\delta U}{\delta P}$$

$$\therefore u_{vA} = \frac{Ph^2}{E}\left(\frac{2h}{3I_1} + \frac{L}{I_2}\right)$$

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7. The cranked rod ABCD in Fig Q7 is built-in at end A and carries a transverse force P, perpendicular to the plane ABCD at D. Assuming that the rod is made from round bar of uniform section, obtain (a) the deflection of D in the direction of P and (b) the angular rotation of the end D about axis CD.

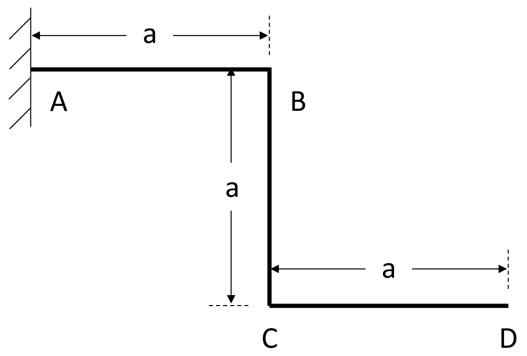
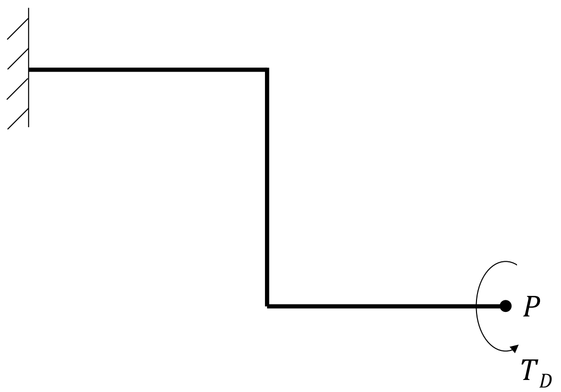


Fig Q7

[Ans:** $\frac{Pa^3\left(3+\frac{E}{G}\right)}{EI}$, $\frac{Pa^2\left(1+\frac{E}{G}\right)}{2EI}$**]

Solution 7

Since the angular displacement is required at position D, a dummy torque must be applied at this position (as well as the applied point load) as follows,



Section CD (*bending and torsion*)

At x from D in CD,

$$M_{CD} = Px$$

and,

$$T_{CD} = T_D$$

$$U_{CD} = U_{CD}^{bending} + U_{CD}^{torsion} = \int \frac{{M_{CD}}^2}{2EI} ds + \int \frac{{T_{CD}}^2}{2GJ} ds = \int_0^a \frac{(Px)^2}{2EI} dx + \int_0^a \frac{{T_D}^2}{2GJ} dx$$

$$= \frac{P^2}{2EI} \int_0^a x^2 dx + \frac{{T_D}^2}{2GJ} \int_0^a 1 dx = \frac{P^2}{2EI} \left[\frac{x^3}{3}\right]_0^a + \frac{{T_D}^2}{2GJ} [x]_0^a = \frac{P^2}{2EI} \left(\frac{a^3}{3} - 0\right) + \frac{{T_D}^2}{2GJ} (a - 0)$$

$$\therefore U_{CD} = \frac{P^2 a^3}{6EI} + \frac{{T_D}^2 a}{2GJ}$$

Section BC (*bending and torsion*)

At x from C in BC,

$$M_{BC} = Px + T_D$$

and,

$$T_{BC} = Pa$$

$$U_{BC} = U_{BC}^{bending} + U_{BC}^{torsion} = \int \frac{{M_{BC}}^2}{2EI} ds + \int \frac{{T_{BC}}^2}{2GJ} ds = \int_0^a \frac{(Px + T_D)^2}{2EI} dx + \int_0^a \frac{(Pa)^2}{2GJ} dx$$

$$= \frac{1}{2EI} \int_0^a (Px + T_D)^2 dx + \frac{P^2 a^2}{2GJ} \int_0^a 1 dx = \frac{1}{2EI} \int_0^a \left(P^2 x^2 + 2PxT_D + {T_D}^2\right) dx + \frac{P^2 a^2}{2GJ} \int_0^a 1 dx$$

$$= \frac{1}{2EI} \left[\frac{P^2 x^3}{3} + Px^2 T_D + {T_D}^2 x\right]_0^a + \frac{P^2 a^2}{2GJ} [x]_0^a$$

$$\therefore U_{BC} = \frac{1}{2EI} \left(\frac{P^2 a^3}{3} + Pa^2 T_D + {T_D}^2 a\right) + \frac{P^2 a^3}{2GJ}$$

Section AB (*bending and torsion*)

At x from b in AB,

$$M_{AB} = P(x + a)$$

and,

$$T_{AB} = T_D + Pa$$

$$U_{AB} = U_{AB}^{bending} + U_{AB}^{torsion} = \int \frac{{M_{AB}}^2}{2EI} ds + \int \frac{{T_{AB}}^2}{2GJ} ds$$

$$= \int_0^a \frac{\left(P(x+a)\right)^2}{2EI} dx + \int_0^a \frac{(T_D + Pa)^2}{2GJ} dx = \frac{P^2}{2EI} \int_0^a (x+a)^2 dx + \frac{(T_D + Pa)^2}{2GJ} \int_0^a 1dx$$

$$= \frac{P^2}{2EI} \int_0^a (x^2 + 2ax + a^2) dx + \frac{{T_D}^2 + 2T_D Pa + P^2 a^2}{2GJ} \int_0^a 1dx$$

$$= \frac{P^2}{2EI} \left[\frac{x^3}{3} + ax^2 + a^2 x\right]_0^a + \frac{{T_D}^2 + 2T_D Pa + P^2 a^2}{2GJ} [x]_0^a$$

$$\therefore U_{AB} = \frac{7P^2 a^3}{6EI} + \frac{{T_D}^2 a + 2T_D P a^2 + P^2 a^3}{2GJ}$$

Total Strain Energy

$$U = U_{AB} + U_{BC} + U_{CD}$$

$$= \frac{P^2 a^3}{6EI} + \frac{{T_D}^2 a}{2GJ} + \frac{1}{2EI}\left(\frac{P^2 a^3}{3} + Pa^2 T_D + T_D^2 a\right) + \frac{P^2 a^3}{2GJ} + \frac{7P^2 a^3}{6EI} + \frac{{T_D}^2 a + 2T_D P a^2 + P^2 a^3}{2GJ}$$

(a) Deflection at D Calculation using Castigliano's Theorem

$$u_{vD} = \frac{\delta U}{\delta P} = \frac{Pa^3}{3EI} + \frac{1}{2EI}\left(\frac{2Pa^3}{3} + a^2 T_D + {T_D}^2 a\right) + \frac{Pa^3}{GJ} + \frac{7Pa^3}{3EI} + \frac{T_D a^2 + Pa^3}{GJ}$$

Setting dummy torque, T_D, to zero gives,

$$u_{vD} = \frac{3Pa^3}{EI} + \frac{2Pa^3}{GJ}$$

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(b) Angular Rotation at D Calculation using Castigliano's Theorem

$$\theta_D = \frac{\delta U}{\delta T_D} = \frac{{T_D}^2 a}{2GJ} + \frac{Pa^2 + 2T_D a}{2EI} + \frac{T_D a + Pa^2}{GJ}$$

Setting dummy torque, T_D, to zero gives,

$$\theta_D = \frac{Pa^2}{2EI} + \frac{Pa^2}{GJ}$$

For a circular cross section,

$$I = \frac{\pi d^4}{64} = \frac{\pi \times 0.02^4}{64} = 7.85 \times 10^{-9} m^4$$

and,

$$J = \frac{\pi d^4}{32} (= 2I) = 1.57 \times 10^{-8} m^4$$

$$\therefore J = 2I$$

Therefore,

$$\boxed{u_{vD} = \frac{Pa^3}{EI}\left(3 + \frac{E}{G}\right)}$$

and,

$$\boxed{\theta_D = \frac{Pa^2}{2EI}\left(1 + \frac{E}{G}\right)}$$