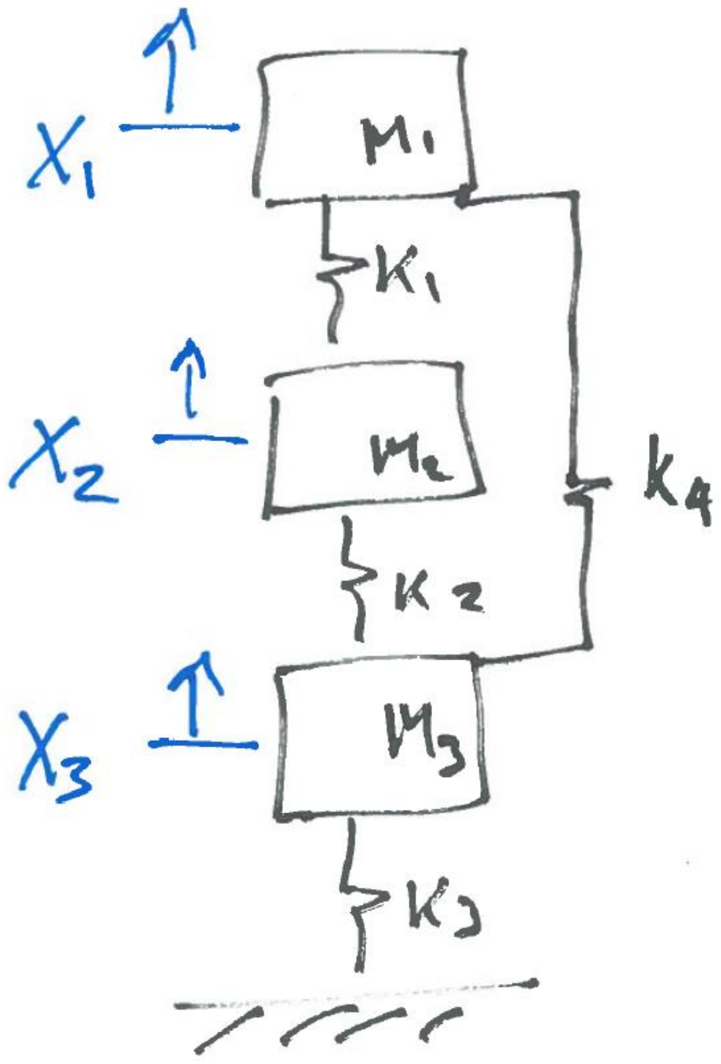
NDOF example problem

- This problem asks you to solve for the [Z] matrices of a 3 DOF mass spring system
- Step 1 is to draw FBDs and EOM for each mass
- Step 2 is to then use these to combine into the final [Z] form
  - This can be done by either
    - Recognizing you're solving for the eigenvector problem and realizing $[Z] = [K] - \omega^2 [M]$
    - Substituting displacement equations into the EOM and combining like terms

$$[Z]\{X\} = \{0\}$$

Where

$$\{X\} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \text{ and } \{0\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$



Assume x_1>> x_2>> x_3

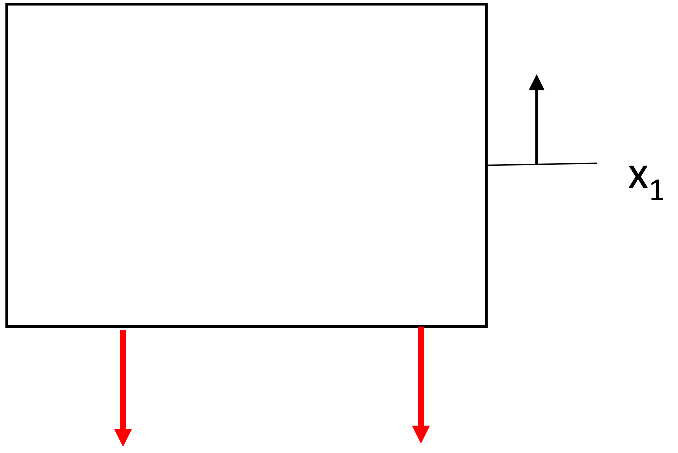
This is required to help you set up the FBD and thus EOM. It's up to you what you choose as being larger than the other, but you have to make a decision here.

Resulting EOM for x_1

$$\sum F_{x1} = m_1\ddot{x}_1$$

$$-k_1(x_1 - x_2) - k_4(x_1 - x_3) = m_1\ddot{x}_1$$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_1x_2 - k_4x_3 = 0 \qquad (1)$$



$k_1(x_1 - x_2)$ $\qquad$ $k_4(x_1 - x_3)$

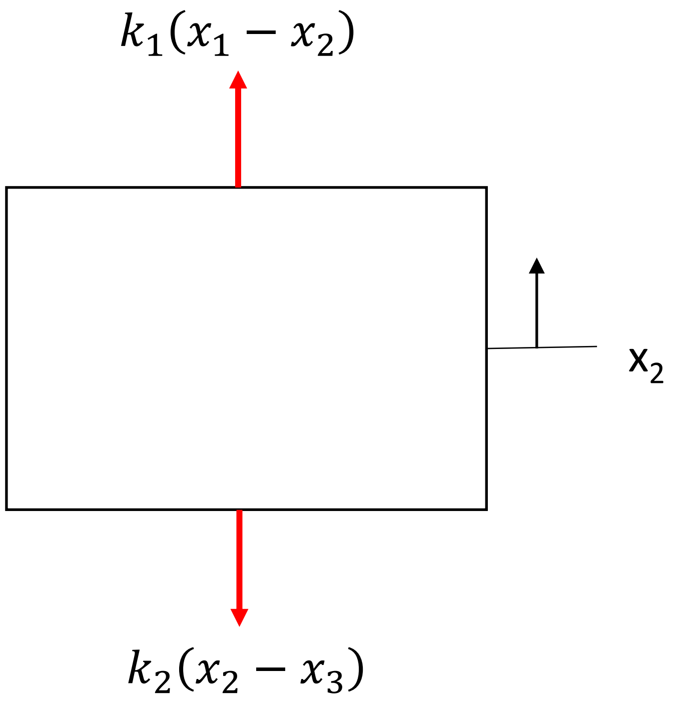
Assume $x_1 >> x_2 >> x_3$

Resulting EOM for x_2

$$\sum F_{x2} = m_2\ddot{x}_2$$

$$+k_1(x_1 - x_2) - k_2(x_2 - x_3) = m_2\ddot{x}_2$$

$$m_2\ddot{x}_2 + (k_1 + k_2)x_2 - k_1x_1 - k_2x_3 = 0 \qquad (2)$$



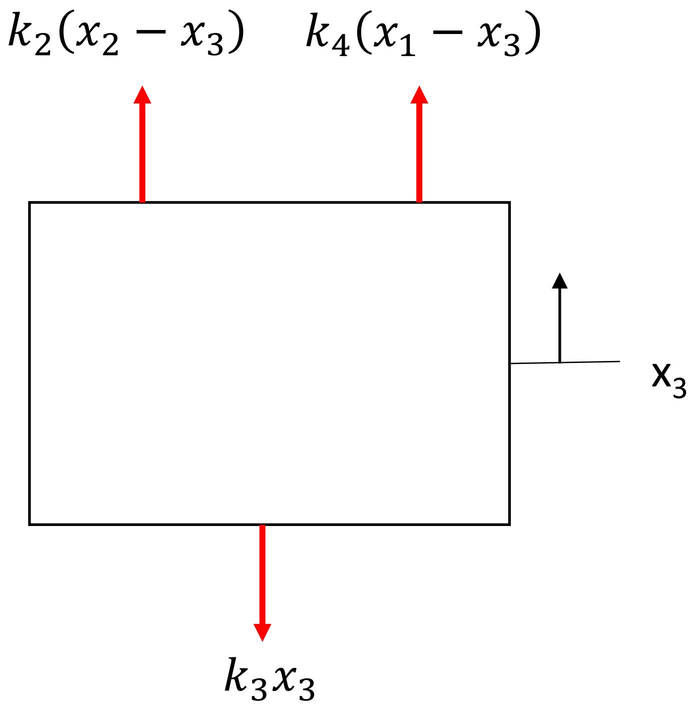
Assume $x_1 >> x_2 >> x_3$

Resulting EOM for x_3

$$\sum F_{x3} = m_3\ddot{x}_3$$

$$+k_4(x_1 - x_3) + k_2(x_2 - x_3) - k_3x_3 = m_3\ddot{x}_3$$

$$m_3\ddot{x}_3 + (k_2 + k_3 + k_4)x_3 - k_4x_1 - k_2x_2 = 0 \qquad (3)$$



$$m_1\ddot{x}_1 + (k_1 + k_4)x_1 - k_1x_2 - k_4x_3 = 0 \quad (1)$$
$$m_2\ddot{x}_2 + (k_1 + k_2)x_2 - k_1x_1 - k_2x_3 = 0 \quad (2)$$
$$m_3\ddot{x}_3 + (k_2 + k_3 + k_4)x_3 - k_4x_1 - k_2x_2 = 0 \quad (3)$$

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_4) & -k_1 & -k_4 \\ -k_1 & (k_1 + k_2) & -k_2 \\ -k_4 & -k_2 & (k_2 + k_3 + k_4) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Check diagonals to ensure nothing wrong there

Solving by recognizing this is an eigenvector

- Simply recognize that

$$[Z] = [K] - \omega^2[M]$$

- Therefore

$$[Z] = \begin{bmatrix} (k_1 + k_4) & -k_1 & -k_4 \\ -k_1 & (k_1 + k_2) & -k_2 \\ -k_4 & -k_2 & (k_2 + k_3 + k_4) \end{bmatrix} - \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \omega^2$$

$$[Z] = \begin{bmatrix} (k_1 + k_4) - m_1\omega^2 & -k_1 & -k_4 \\ -k_1 & (k_1 + k_2) - m_2\omega^2 & -k_2 \\ -k_4 & -k_2 & (k_2 + k_3 + k_4) - m_3\omega^2 \end{bmatrix}$$

Solving through substitution

$$m_1\ddot{x}_1 + (k_1 + k_4)x_1 - k_1x_2 - k_4x_3 = 0 \quad (1)$$
$$m_2\ddot{x}_2 + (k_1 + k_2)x_2 - k_1x_1 - k_2x_3 = 0 \quad (2)$$
$$m_3\ddot{x}_3 + (k_2 + k_3 + k_4)x_3 - k_4x_1 - k_2x_2 = 0 \quad (3)$$

- You can substitute in with $\sin\omega t$ /$\cos\omega t$ if you wish as without damping terms all results will be real, or use the complex numbers method ($e^{i\omega t}$)

$x_1 = X_1 \sin\omega t$ $\qquad$ $x_2 = X_2 \sin\omega t$ $\qquad$ $x_3 = X_3 \sin\omega t$

$\ddot{x}_1 = -\omega^2 X_1 \sin\omega t$ $\qquad$ $\ddot{x}_2 = -\omega^2 X_2 \sin\omega t$ $\qquad$ $\ddot{x}_3 = -\omega^2 X_3 \sin\omega t$

Solving through substitution

$$m_1\ddot{x}_1 + (k_1 + k_4)x_1 - k_1x_2 - k_4x_3 = 0 \qquad (1)$$

$$m_1(-\omega^2X_1 \sin \omega t) + (k_1 + k_4)X_1 \sin \omega t - k_1X_2 \sin \omega t - k_4X_3 \sin \omega t = 0$$

$$-m_1\omega^2X_1 + (k_1 + k_4)X_1 - k_1X_2 - k_4X_3 = 0$$

$$\left((k_1 + k_4) \ -m_1 \ \omega^2\right)X_1 - k_1X_2 - k_4X_3 = 0 \qquad (1b)$$

Solving through substitution

$$m_2\ddot{x}_2 + (k_1 + k_2)x_2 - k_1x_1 - k_2x_3 = 0 \qquad (2)$$

$$m_2(-\omega^2 X_2 \sin\omega t) + (k_1 + k_2)X_2 \sin\omega t - k_1X_1 \sin\omega t - k_2X_3 \sin\omega t = 0$$

$$-m_2\omega^2 X_2 + (k_1 + k_2)X_2 - k_1X_1 - k_2X_3 = 0$$

$$\left((k_1 + k_2) - m_2\,\omega^2\right)X_2 - k_1X_1 - k_2X_3 = 0 \qquad (2b)$$

Solving through substitution

$$m_3\ddot{x}_3 + (k_2 + k_3 + k_4)x_3 - k_4x_1 - k_2x_2 = 0 \quad (3)$$

$$m_3(-\omega^2 X_3 \sin\omega t) + (k_2 + k_3 + k_4)X_3 \sin\omega t - k_4X_1 \sin\omega t - k_2X_2 \sin\omega t = 0$$

$$-m_3\omega^2 X_3 + (k_2 + k_3 + k_4)X_3 - k_4X_1 - k_2X_2 = 0$$

$$\left((k_2 + k_3 + k_4) - m_3\,\omega^2\right)X_3 - k_4X_1 - k_2X_2 = 0 \quad (3b)$$

Solving through substitution

- You end up with

$$\big((k_1 + k_4)\ - m_1\ \omega^2\big)X_1 - k_1 X_2 - k_4 X_3 = 0$$
$$\big((k_1 + k_2)\ - m_2\ \omega^2\big)X_2 - k_1 X_1 - k_2 X_3 = 0$$
$$\big((k_2 + k_3 + k_4)\ - m_3\ \omega^2\big)X_3 - k_4 X_1 - k_2 X_2 = 0$$

- Which in matrix form is the same as before

$$[Z]\{X\} = \{0\}$$

$$\begin{bmatrix} (k_1 + k_4) - m_1\omega^2 & -k_1 & -k_4 \\ -k_1 & (k_1 + k_2) - m_2\omega^2 & -k_2 \\ -k_4 & -k_2 & (k_2 + k_3 + k_4) - m_3\omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$