Strain Energy Methods

Lecture 3 – Castigliano’s Theorem

Department of Mechanical, Materials & Manufacturing Engineering

MMME2053 – Mechanics of Solids

University of Nottingham

UK | CHINA | MALAYSIA

Strain Energy Methods

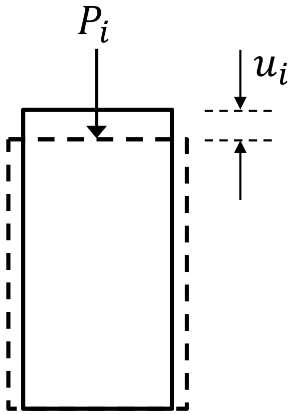
Learning Outcomes

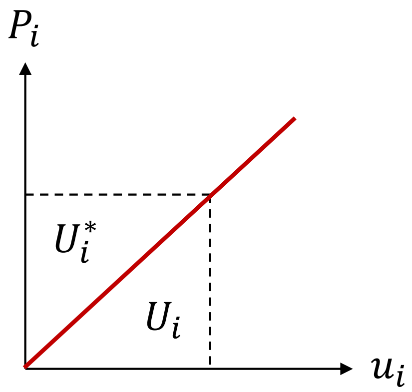
1. Know the basic concept of strain energy stored in a material body under loading (knowledge);
2. Be able to calculate strain energy in an elastic body/structure arising from various types of loading, including tension/compression, bending and torsion (application);
3. Be able to apply Castigliano's theorem for linear elastic bodies in order to enable the deflection, change of slope and/or rotation of a body, or structure, to be calculated from strain energy expressions (application).

Castigliano’s Theorem

Consider a linear elastic body loaded by a force, P_i, as shown in the figure below, noting that it may be the case that $i \neq 1$, i.e. there may be more than one force acting on the body.

The corresponding displacement at the location, and in the direction of P_i, is u_i.





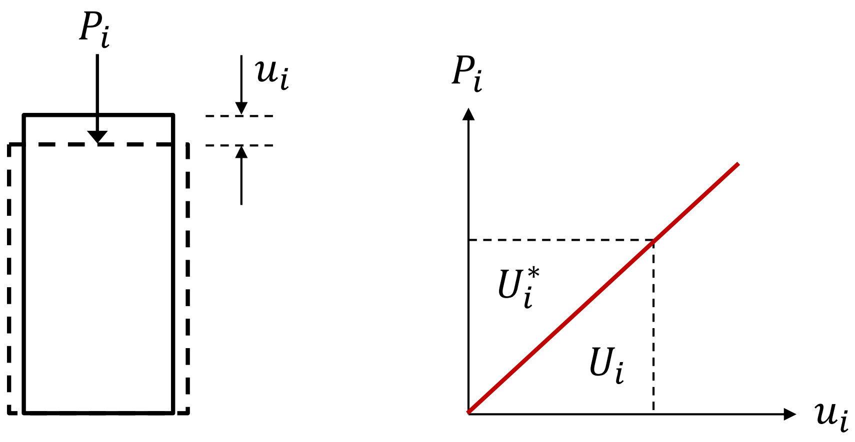
As previously defined, the general expression for strain energy, shown as the area under the load-displacement plot above, is:.

$$U_i = \int_0^{u_i} P_i \mathrm{d}u_i$$

We can also define the complimentary strain energy, U_i^*, as:

$$U_i^* = \int_0^{P_i} u_i \mathrm{d}P_i$$

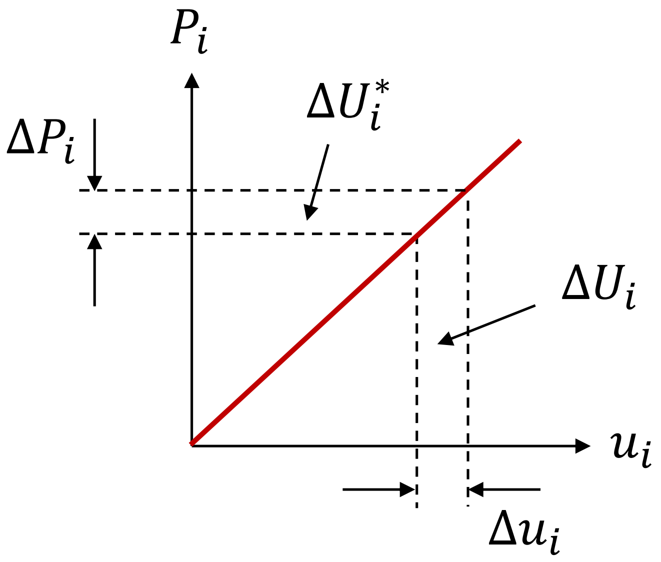
where U_i^* is the area above the load-displacement plot.



Now, consider a small increment of the load ΔP_i, while any other loads (assuming $i \neq 1$), remain constant. This causes an increment of the complementary strain energy, shown in the figure below, as:

$$\Delta U_i^* = \int_{P_i}^{P_i+\Delta P_i} u_i \mathrm{d}P_i \approx u_i \Delta P_i$$

$$\therefore u_i = \frac{\Delta U_i^*}{\Delta P_i}$$



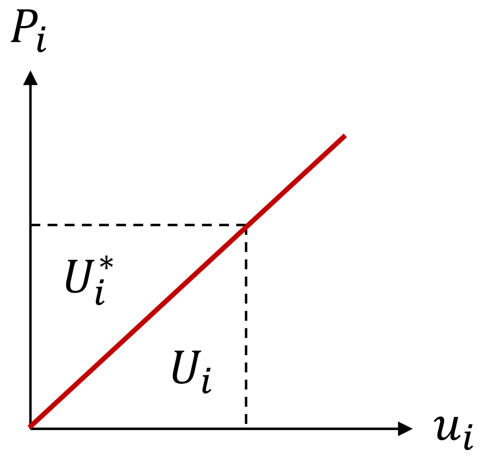
In the limit, as $\Delta P_i \to 0$, the above expression can be rewritten as:

$$u_i = \frac{\partial U_i^*}{\partial P_i}$$

Note that this is a partial derivative as, if $i \neq 1$, then all loads except P_i were kept constant in the derivation of the above.

For linear elastic bodies, the strain energy is equal to the complementary strain energy as shown in the figure below, thus:

$$U_i = U_i^*$$



And therefore: $u_i = \dfrac{\partial U_i}{\partial P_i}$

Castigliano's theorem may be also be extended to beam bending and torsion as follows,

$$\phi_i = \frac{\partial U_i}{\partial M_i} \qquad\qquad \theta_i = \frac{\partial U_i}{\partial T_i}$$

Castigliano's Theorem

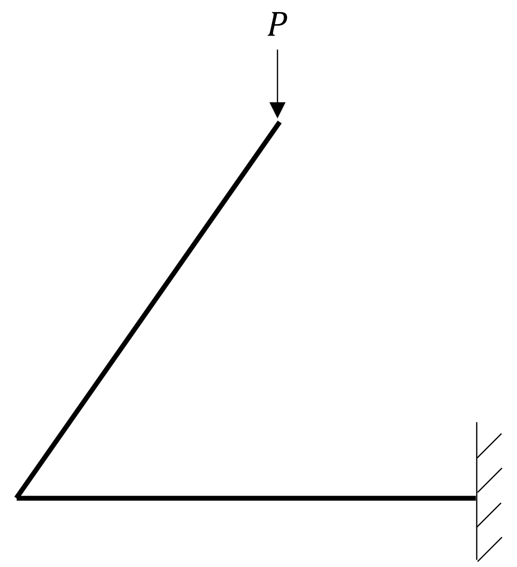
Dummy Loads

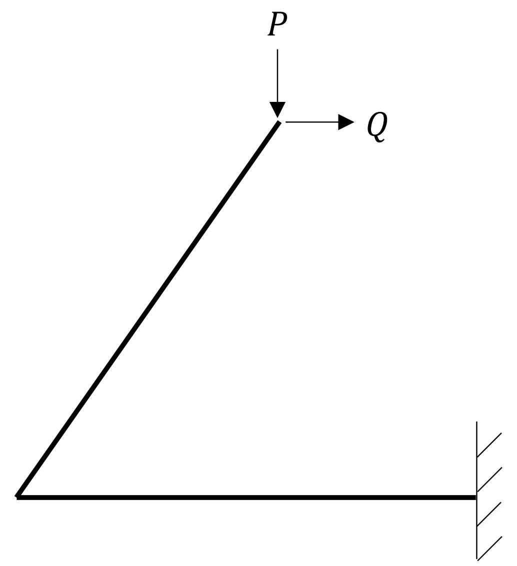
In order to determine the deflection at a point in a structure where a load is not applied, a dummy load is added at the point of and in the direction that the deflection is required.

The expression for strain energy is then obtained incorporating the dummy load and differentiated with respect to this dummy load in order to obtain the required deflection.

The dummy load is then set to zero when numerically evaluating the deflection.

For example, to determine the horizontal deflection at the tip of the structure shown in the top figure (this is the same as the worked example), it is necessary to add the horizontal dummy load, Q, at this position, as shown in the bottom figure.





Following the processed previously described, the total strain energy expression for this structure can be calculated to be:

$$U = \frac{L^3}{6EI}(P^2\cos^2\theta + Q^2\sin^2\theta + PQ\cos\theta\sin\theta) + \frac{L^3}{2EI}\left(\frac{P^2}{3} - P^2\cos\theta + P^2\cos^2\theta - PQ\sin\theta + Q^2\sin^2\theta + 2PQ\cos\theta\sin\theta\right)$$

The vertical deflection of the tip of the beam can be calculated by differentiating this with respect to the vertical applied load, P, to give:

$$u_{v_A} = \frac{\partial U}{\partial P} = \frac{L^3}{6EI}(2P\cos^2\theta + Q\cos\theta\sin\theta) + \frac{L^3}{2EI}\left(\frac{2P}{3} - 2P\cos\theta + 2P\cos^2\theta - Q\sin\theta + 2Q\cos\theta\sin\theta\right)$$

The horizontal deflection of the tip of the beam can be calculated by differentiating the above expression for strain energy with respect to the horizontal dummy load, Q, to give:

$$u_{h_A} = \frac{\partial U}{\partial Q} = \frac{L^3}{6EI}(2Q\sin^2\theta + P\cos\theta\sin\theta) + \frac{L^3}{2EI}(-P\sin\theta + 2Q\sin^2\theta + 2P\cos\theta\sin\theta)$$

Setting dummy load to zero and substituting values for P, L, E, I and θ (as given in the worked example) into these expressions for u_{v_A} and u_{h_A} gives:

$$\boldsymbol{u_{v_A} = 47.38\ \mathrm{mm}} \qquad \therefore \boldsymbol{u_{h_A} = 33.09\ \mathrm{mm}}$$

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