MMME2046 Dynamics and Control

SYSTEMS MODELLING AND CONTROL

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Semester 2 2022-23

- Control Laboratory: 20/2/2023 to 10/3/2023 (check your laboratory timetable)
  - Hand in of report – Moodle, 2 weeks after the lab.
- 2 Bank holidays in May:
  - Lecture on 1st May will be on the 3rd of May
  - Lecture on 8th May will be in the Wednesday seminar slot

Easter Break			
33	01/05/2022	Review Lecture	Vibrations Review (This week has a Bank Holiday Monday so lectures will be on Wednesday, but in different rooms; 1000-1100 Pope C17, 1100-1200 Pope C15 & 1200-1300 Physics B1)
34	08/05/2022	Baby Crashing Lecture (material is not on exam; This will be held druing the Wednesday Seminar as there is a Bank Holiday Monday this week)	
35	15/05/2022	Review Lecture	Open Topic Questions
Spring Exams			

CONTROL TOPICS

Handout	Title	Lecture no.
1	Introduction	1
2	Representation of Control Systems	1
3	Laplace Transforms	1
4	Modelling of simple components	1
5	Non-linearity and linearisation	2
6	Block Diagram manipulation	3
7	Introduction to transient and steady state response	3
8	Hydraulic Position control System	3
9	Electro-mechanical position control system	4
10	Improving Transient and Steady State performance	4
11	The stability of feedback systems	4
Laboratory	Control for a fluid flow system	

Learning Outcomes

- At the end of this lecture, you should:
  - Appreciate the uses of system and control modelling
  - Understand the purpose and concept of a system model
  - Understand the *block diagram* representation of control systems
  - Know how to use Laplace Transforms to solve differential equations
  - Know how to model simple components of control systems

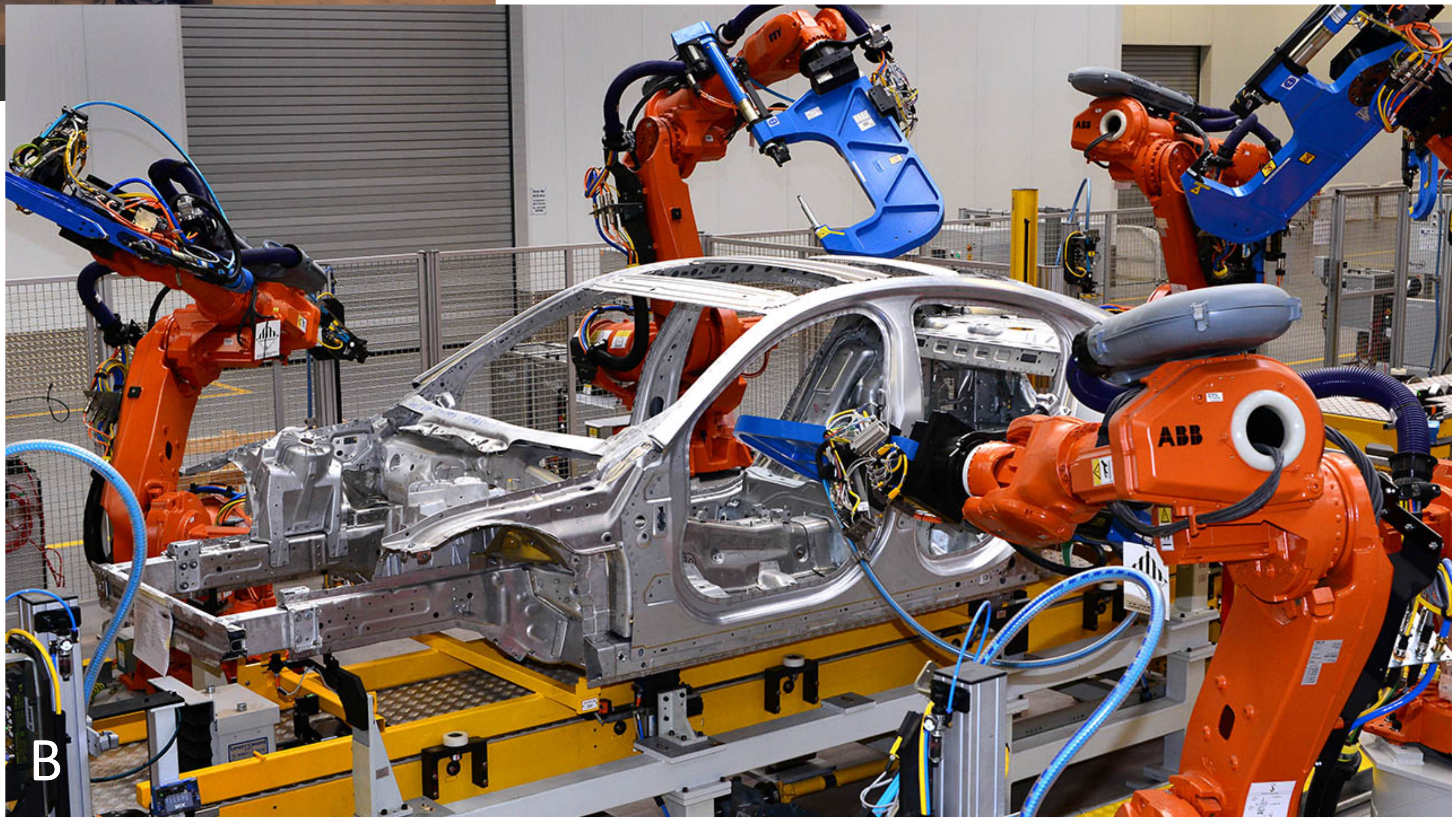
Why model control?

- Driverless trains

- Fly by wire aircraft

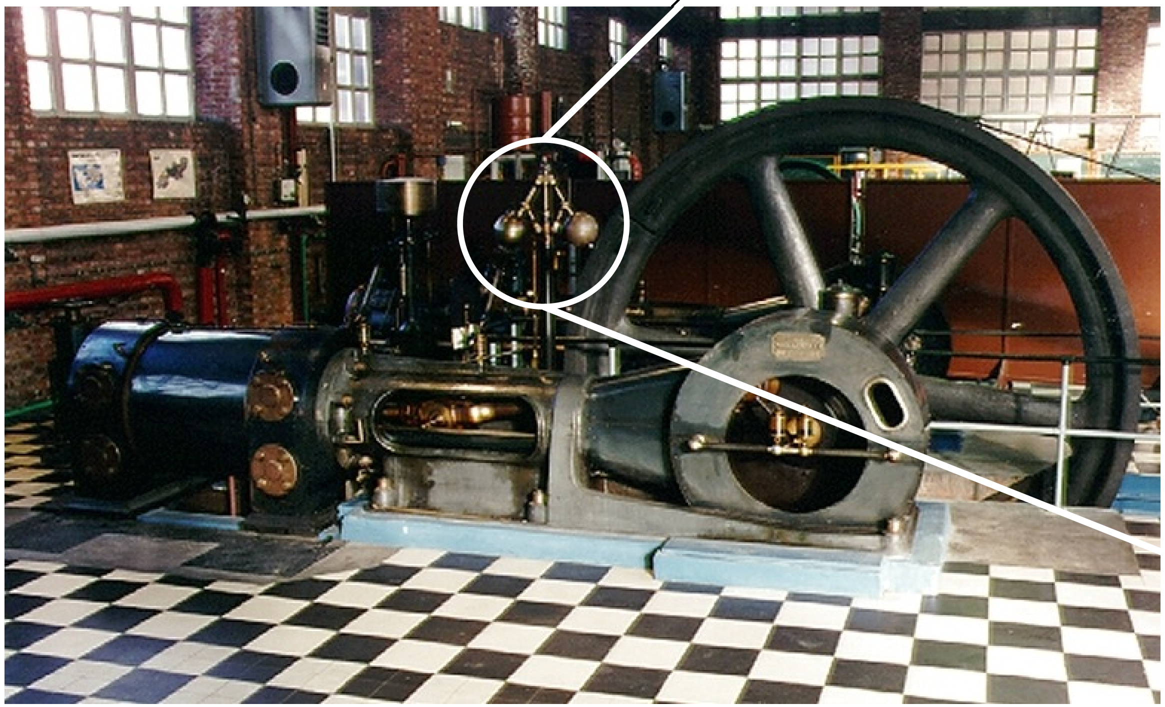
Which one needs a control system?

https://www.youtube.com/watch?v=ArxzMqf3aZg

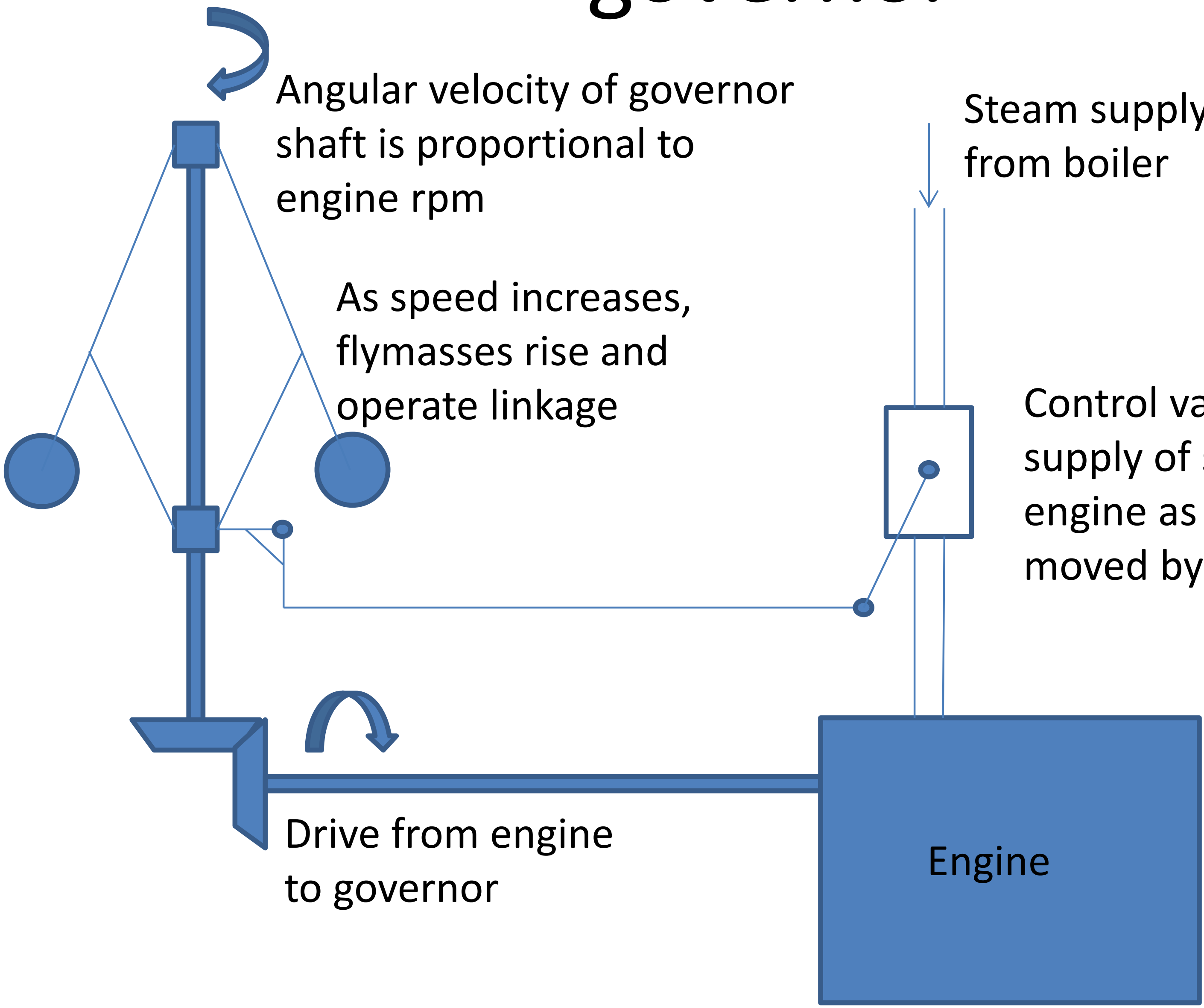


Brilliant idea no. 1

- Centrifugal governor
  - Patented 1788 by James Watt



Method of operation for centrifugal governor



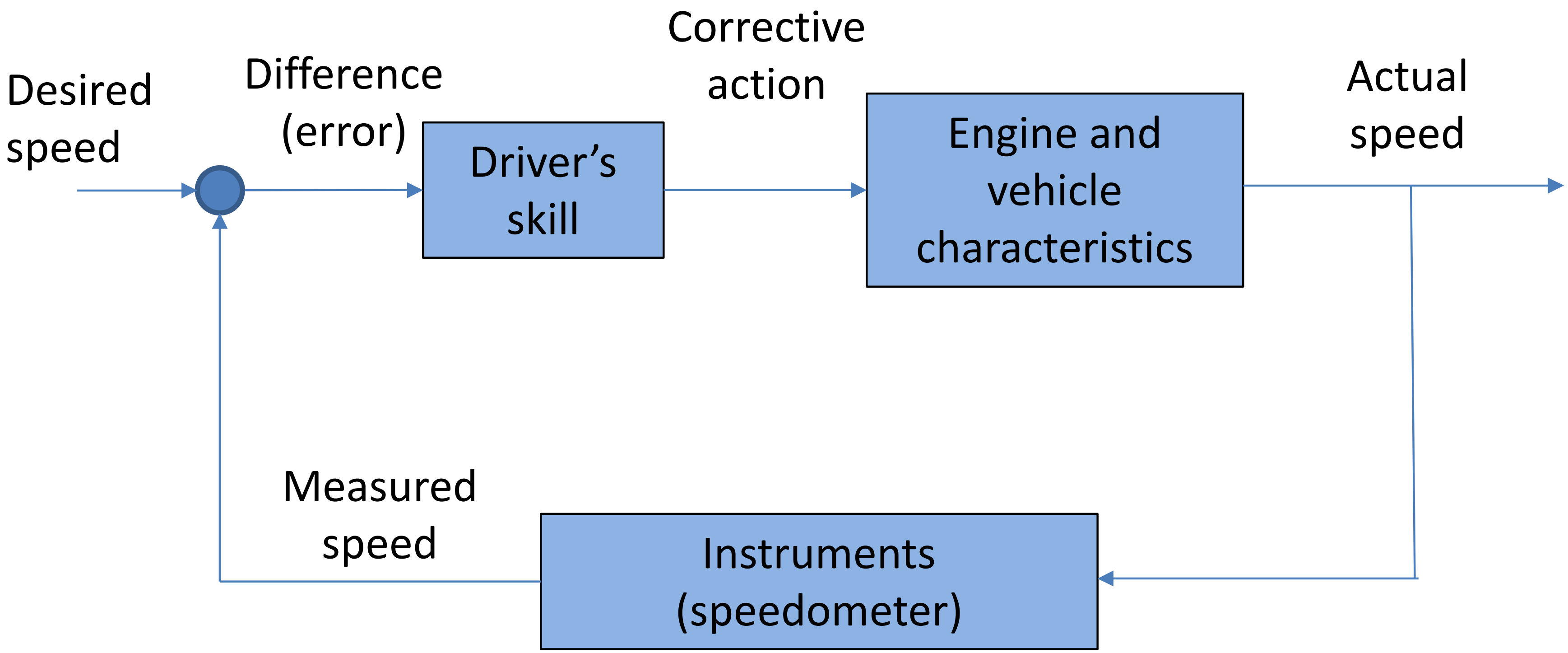
Video: https://www.youtube.com/watch?v=OG1AiaNTT6s

Why did you need this?

- 19th century mills – all machines run by mechanical linkage
- Continually varying load
- Governor keeps shaft going at constant speed

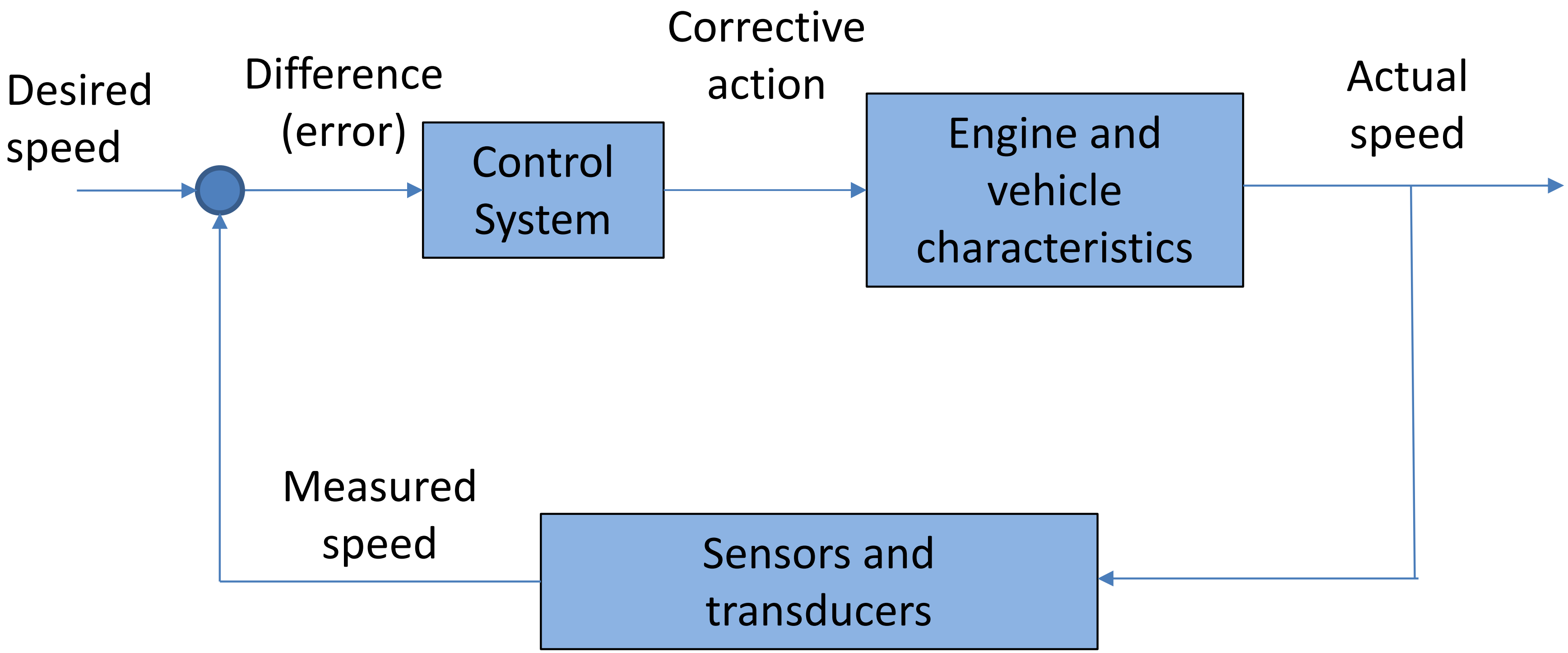
Examples of feedback control

- Driving a car (speed control)



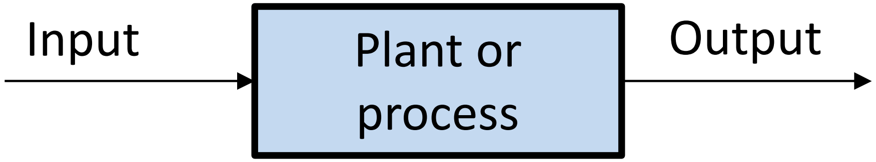
Examples of feedback control

- Car with cruise control

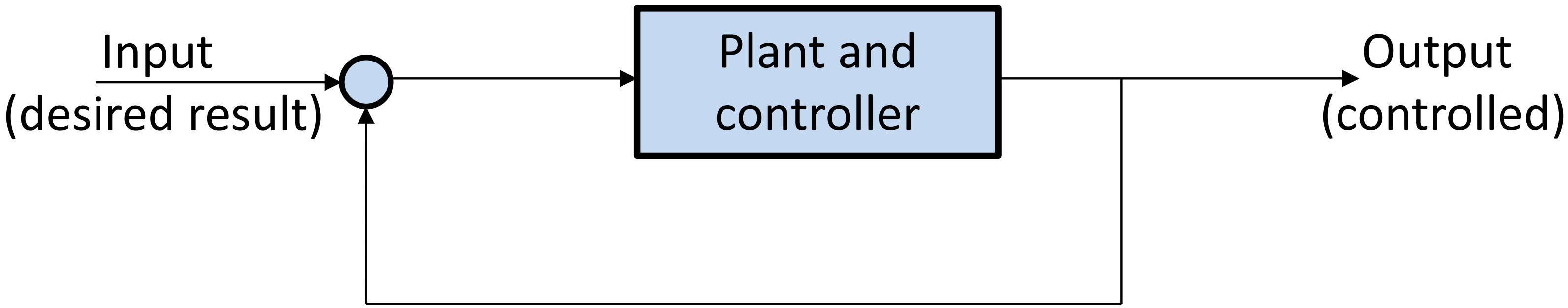


Systems and block diagrams

- Open-Loop system

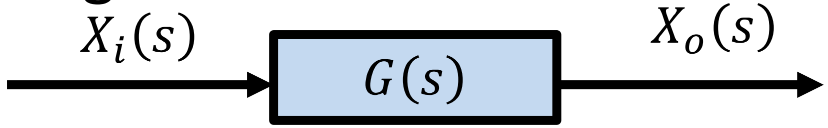


- Closed-Loop (feedback) system



Representation of control systems

- The transfer function of a linear system is formally defined as the ratio of the Laplace transform of the output to the Laplace transform of the input, where the initial conditions are zero.
- The block diagram for an element is drawn as follows:



$$X_o(s) = G(s)\, X_i(s)$$

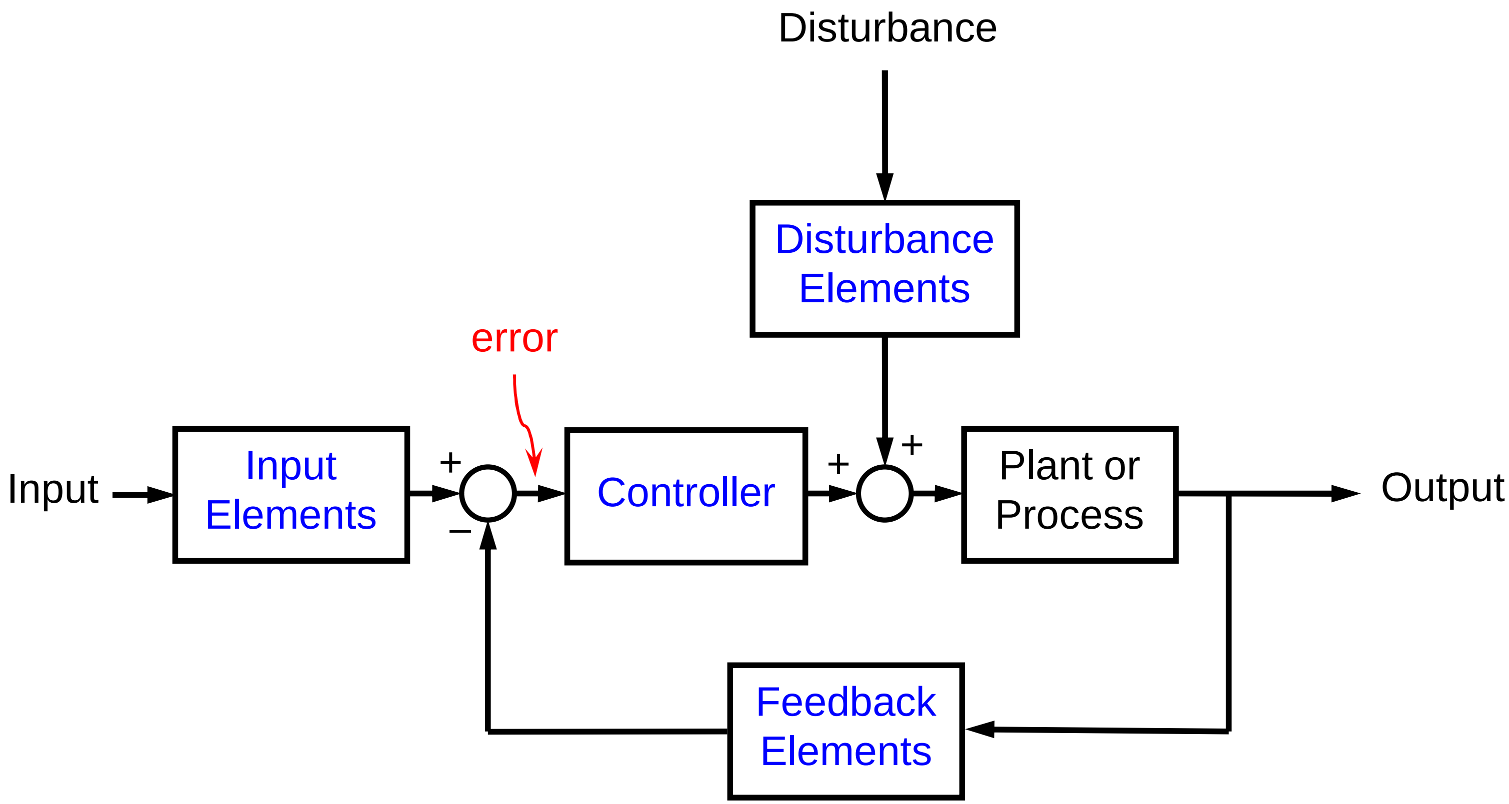
- The transfer function $G(s)$ is thus given by:

$$G(s) = \frac{X_o(s)}{X_i(s)} = \frac{P(s)}{Q(s)}$$

- Where the **denominator** $Q(s)$ is known as the *characteristic function*, and $Q(s) = 0$ is the *characteristic equation*.

Representation of Control Systems

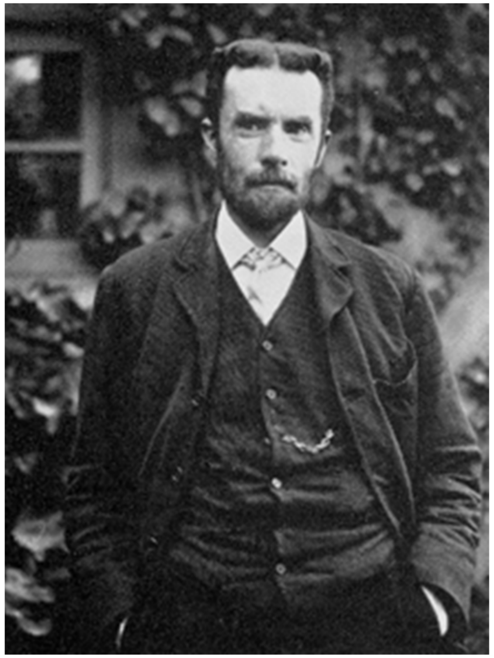
A typical system has a block diagram of the following form



Each box will then contain the transfer function of the element contained in the box.

Laplace Transforms

- Don’t Panic!



Oliver Heaviside
1850-1925

Pierre Simon de Laplace
1749-1827

Laplace Transforms

- The definition of the Laplace transform is:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

  Where $s = \alpha + j\omega$

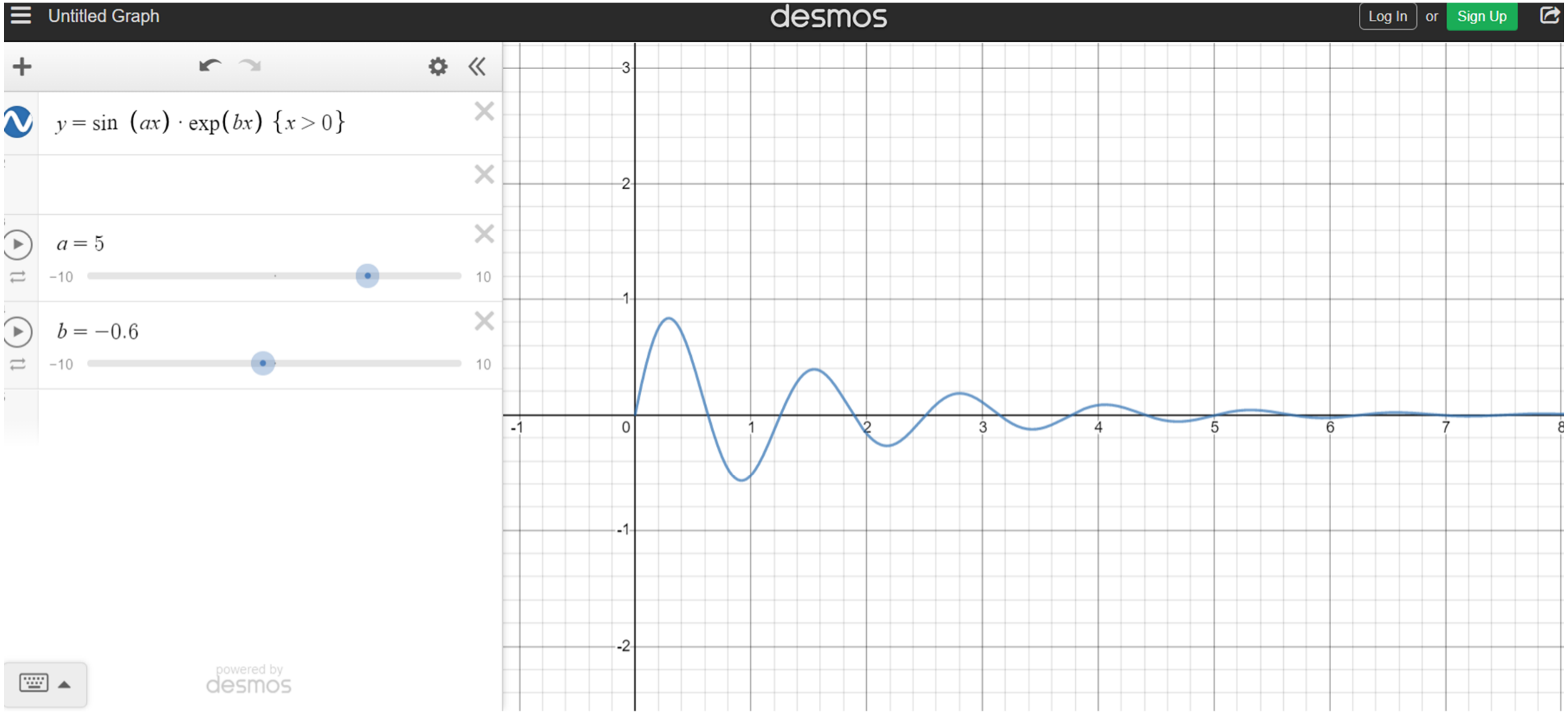
  – i.e. $e^{-st} = e^{-\alpha t}e^{-j\omega t} = e^{-\alpha t}(\sin \omega t + j \cos \omega t)$

- What this does:
  – It turns a periodic function *f(t)* in terms of α and ω into an algebraic function *F(s)* in terms of s
  – Much easier to manipulate
    - And the really good thing is that all the $\int_0^\infty f(t)e^{-st}dt$ has been done for you …

https://www.youtube.com/watch?v=n2y7n6jw5d0

https://www.youtube.com/watch?v=3gjJDuCAEQQ

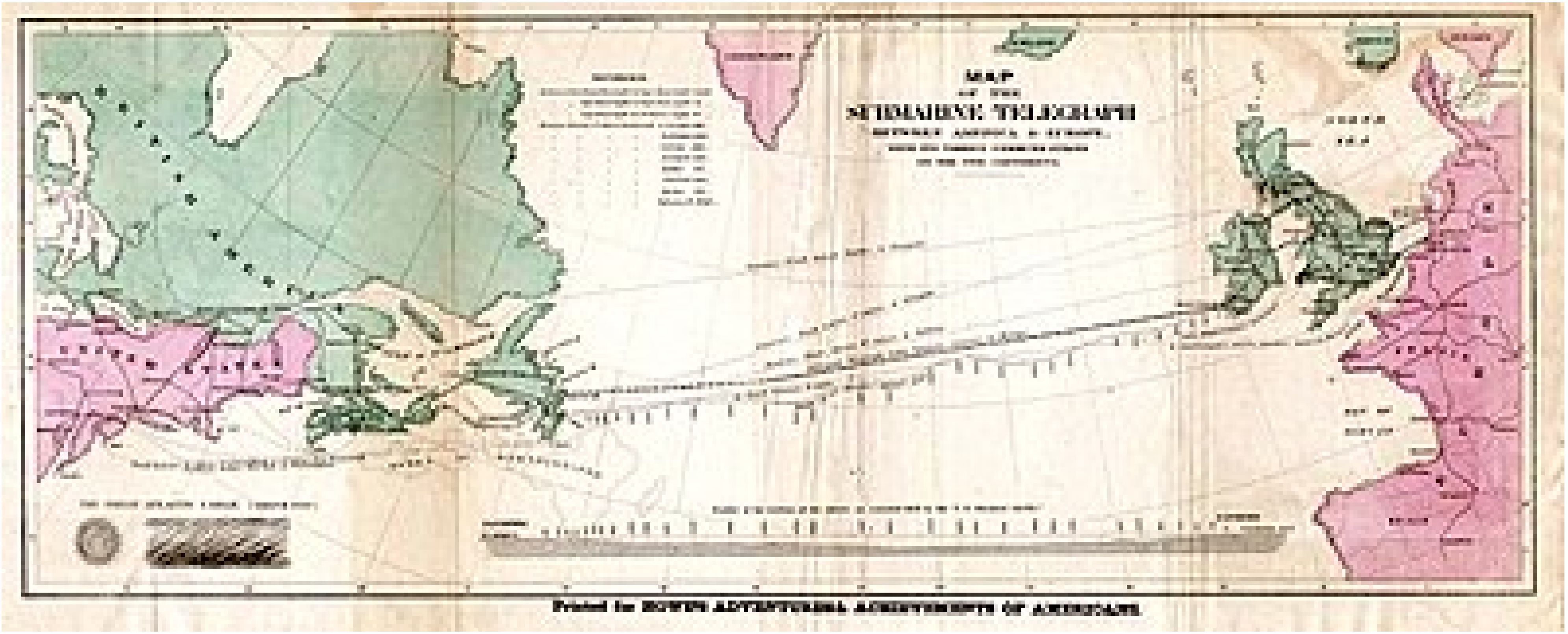
$$e^{-st} = e^{-\alpha t} e^{-j\omega t}$$



Graphic calculator: Desmos, https://www.desmos.com/calculator

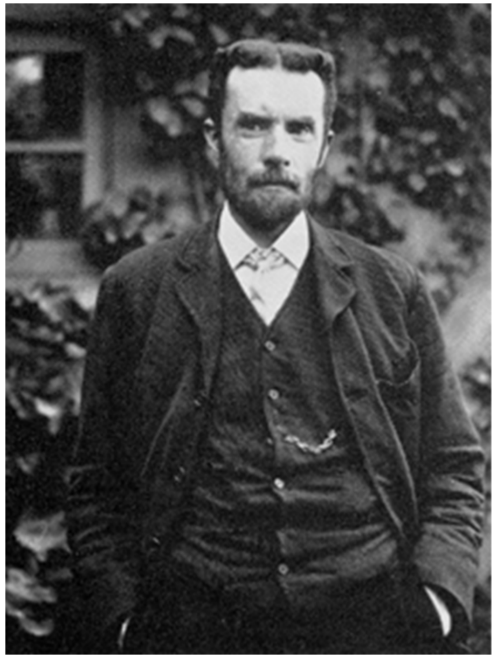
Jump forward to the late 19^{th} Century

- Very long telegraph cables carrying morse signals suffered losses, making signals hard to decipher even with amplification: The first transatlantic cable laid at great expense in 1858 didn’t work as expected (and got fried when they tried running it at 2000 volts)!

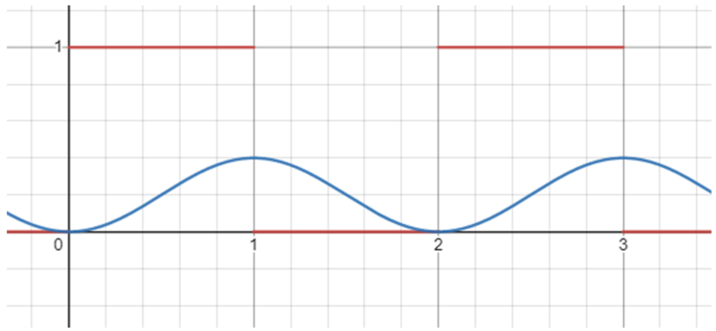


Jump forward to the late 19th Century

- Enter Oliver Heaviside
  - Notice a pattern here: the engineers get called in to solve a problem no-one else had anticipated.
- Heaviside used a method similar to Laplace's to solve practical problems in electromagnetics, particularly the 'telegraph equations':



$$\frac{d^2V}{dx^2} + k^2V = 0 \qquad \frac{d^2I}{dx^2} + k^2I = 0$$



Laplace Transforms

- Use in modelling control systems began in the 1950s, with the development of powered control systems

Revision: Laplace Transforms

The Laplace transform of a function $f(t)$ is $F(s)$ and is defined as:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

where $s = \sigma + j\omega$ is a complex variable and $f(t) = 0$ for $t < 0$.

When solving a differential equation using Laplace transforms the following step-by-step procedure should be followed:

STEP 1 Transform the equation from the time-domain to the Laplace domain (taking account of the initial conditions). This is already done for you in the Laplace Transform tables.

STEP 2 Solve and simplify the resulting equations in the *s*-domain.

STEP 3 Take partial fractions and use tabulated transforms to get the solution in the time domain.

We can deduce system stability and behaviour after step 2: often we don’t need to go as far as step 3.

Useful Results Relating Laplace Transforms

i) Addition and Subtraction

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

ii) Multiplication by a constant

$$\mathcal{L}[Kf(t)] = KF(s)$$

iii) Final Value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

This theorem is only valid if the final value is finite and constant.

iv) Shifting Theorem

$$\text{If } \mathcal{L}[f(t)] = F(s) \text{ then } \mathcal{L}[f(t-\tau)] = e^{-s\tau}F(s)$$

A table of Laplace transform pairs will be provided (also at the exam).

Table of Laplace Transforms

	$f(t)$	$F(S)$
1	$\frac{df(t)}{dt}$	$sF(s) - f(0)$
2	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - \ldots. - f^{n-1}(0)$
3	$\int f(t)dt$	$\frac{1}{s} F(s)$
4	Unit impulse $\delta(t)$ at t=0	1
5	Unit step at t=0	$\frac{1}{s}$
6	Unit ramp $f(t) = t$	$\frac{1}{s^2}$
7	e^{-at}	$\frac{1}{s+a}$
8	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
9	$t - \frac{1}{a}(1 - e^{-at})$	$\frac{a}{s^2(s+a)}$

Table of Laplace Transforms (continued)

	$f(t)$	$F(S)$
10	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
12	$\frac{1}{(\omega^2-p^2)}\left[\sin(pt)-\frac{p}{\omega}\sin(\omega t)\right]$	$\frac{p}{(s^2+p^2)(s^2+\omega^2)}$
13	$\frac{1}{(\omega^2-p^2)}[\cos(pt)-\cos(\omega t)]$	$\frac{s}{(s^2+p^2)(s^2+\omega^2)}$
14	$\frac{\omega}{\sqrt{1-\gamma^2}}e^{-\gamma\omega t}\sin\left(\omega t\sqrt{1-\gamma^2}\right)$	$\frac{\omega^2}{s^2+2\omega\gamma s+\omega^2}$
15	$1-\frac{e^{-\gamma\omega t}}{\sqrt{1-\gamma^2}}\sin\left(\omega t\sqrt{1-\gamma^2}+\phi\right)$	$\frac{\omega^2}{s(s^2+2\omega\gamma s+\omega^2)}$
16	$t-\frac{2\gamma}{\omega}-\frac{e^{-\gamma\omega t}}{\omega\sqrt{1-\gamma^2}}\sin\left(\omega t\sqrt{1-\gamma^2}+\phi\right)$ where $\cos\phi=\gamma$	$\frac{\omega^2}{s^2(s^2+2\omega\gamma s+\omega^2)}$

This Table will be provided in the exam handout.

Examples of the Use of Laplace Transforms

Example 1)

Determine the Laplace transform of $f(t)$ if

$$f(t) = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

and $x = 2$, $\frac{dx}{dt} = 1$ at $t = 0$ (initial conditions).

Solution:

Using Entry 2 of the attached table of Laplace transforms, the Laplace transform of $f(t)$ is

$$F(s) = s^2 X(s) - s x(0) - \dot{x}(0)$$

Substituting the initials conditions into this equation gives

$$\boxed{F(s) = s^2 X(s) - 2s - 1}$$

If the initial conditions are each zero

$$F(s) = s^2 X(s)$$

Example 2)

Use Laplace transforms to determine the solution to

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega_\mathrm{n}^2 x = \cos(pt)$$

with zero initial conditions.

Solution:

STEP 1: Taking Laplace transforms (with zero initial conditions) (Entries 2 & 11)

$$s^2 X(s) + \omega_\mathrm{n}^2 X(s) = \frac{s}{s^2 + p^2}$$

STEP 2: Rearranging gives

$$X(s) = \frac{s}{(s^2 + p^2)(s^2 + \omega_\mathrm{n}^2)}$$

STEP 3: Converting back to the time domain gives (Entry 13)

$$x(t) = \frac{1}{\omega_\mathrm{n}^2 - p^2}\left[\cos(pt) - \cos(\omega_\mathrm{n} t)\right]$$

Or alternatively…

- Example 2, Page 8, by conventional method:

$$\frac{d^2x}{dt^2} + \omega_n^2 x = \cos pt \quad (1)$$

- General solution and particular integral:

$$\frac{d^2x}{dt^2} + \omega_n^2 x = 0 \quad (2)$$

- General solution and particular integral:

GS: $$\frac{d^2x}{dt^2} + \omega_n^2 x = 0$$

$$x = A\cos\omega_n t$$

PI: $$x = B\cos pt$$

$$B(-p^2\cos pt + \omega_n^2\cos pt) = \cos pt$$

$$B = \frac{1}{(\omega_n^2 - p^2)}$$

Example 3)

i) Determine the transfer function of the system

$$\dot{x}_o + a x_o = a x_i$$

where x_o is the output and x_i is the input.

ii) Use the transfer function to determine the output to a unit step input.

Solution:
i) Taking Laplace transforms (with zero initial conditions)

$$sX_o(s) + aX_o(s) = aX_i(s)$$

where $X_o(s)$ and $X_i(s)$ are Laplace transforms of output and input

Re-arranging this equation for the transfer function

$$G(s) = \frac{X_o(s)}{X_i(s)} = \frac{a}{s+a}$$

Example 3)

ii) The output in the Laplace domain can be deduced from

$$X_\mathrm{o}(s) = G(s)X_\mathrm{i}(s)$$

Thus, if the input xi is a unit step, then (from Entry 5 in the table of Laplace transforms)

$$X_\mathrm{i}(s) = \frac{1}{s}$$

and from the transfer function the Laplace transform of the output is

$$X_\mathrm{o}(s) = \frac{a}{s(s+a)}$$

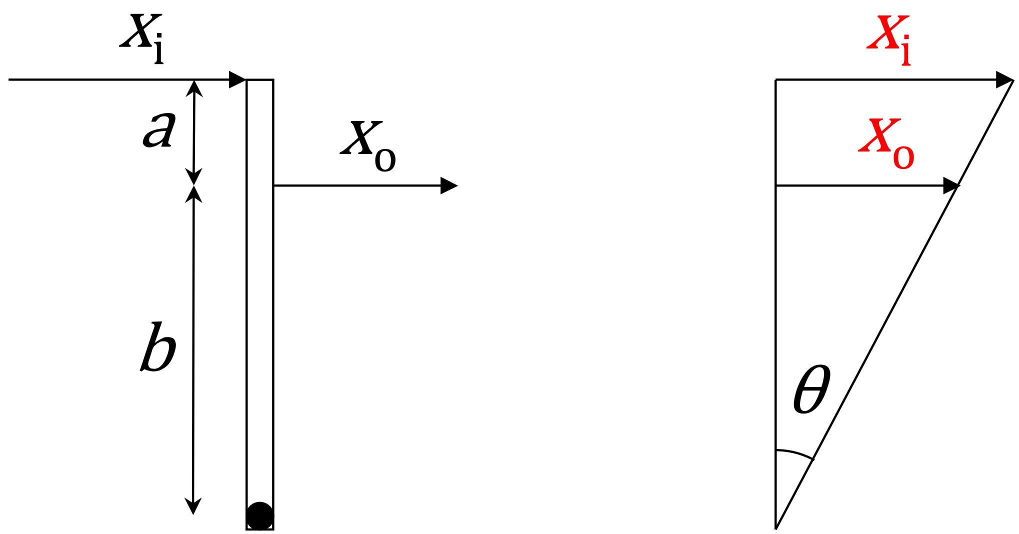
Taking inverse Laplace transforms (using Entry 8 in the table of Laplace transforms) gives

$$x_\mathrm{o}(t) = 1 - \mathrm{e}^{-at}$$

Modelling of Simple Components

- Lever Systems
- Rotor with Viscous Drag
- Mass-Spring-Damper System (Exercise)
- Hydraulic Ram

a) Simple Lever System



Determine the **transfer function** and **block diagram** for the rigid lever system.

Assuming that the displacements are small, then:

$$\tan\theta = \frac{x_i}{a+b} = \frac{x_o}{b}$$

Thus, the relationship between the output and input is

$$\frac{x_o}{x_i} = \frac{b}{a+b}$$

a) Simple Lever System

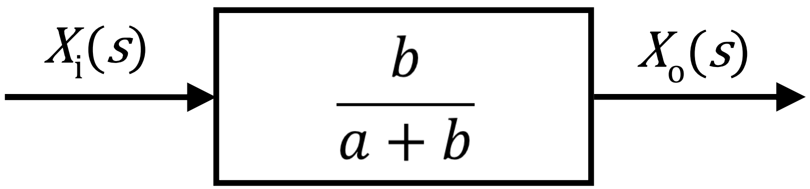
Taking Laplace transforms gives

$$\frac{X_o(s)}{X_i(s)} = \frac{b}{a+b}$$

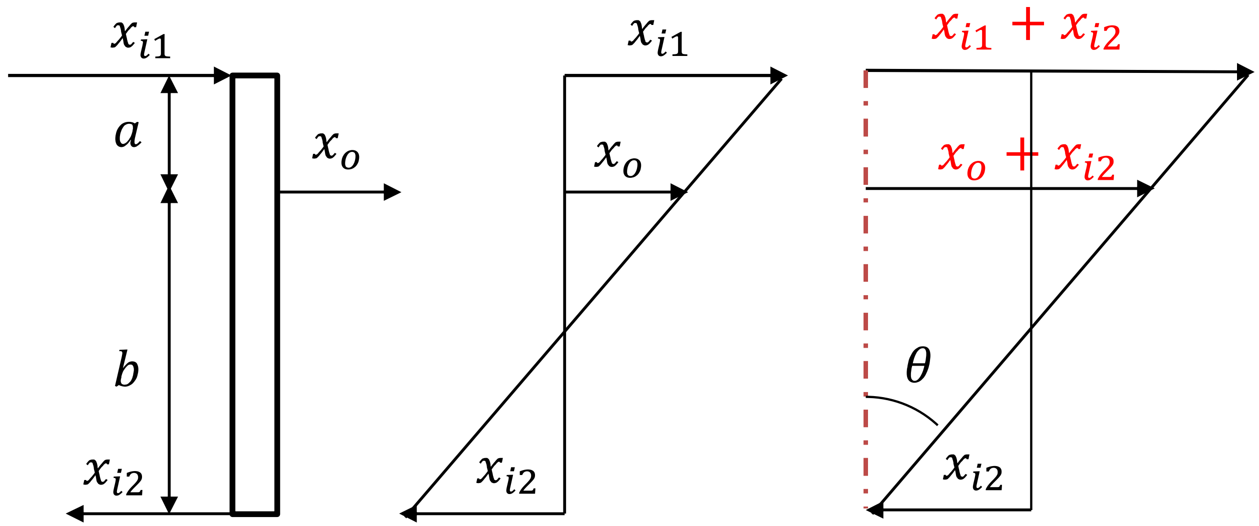
The **transfer function** G(s) is given by

$$G(s) = \frac{b}{a+b}$$

The **block diagram** for the simple lever system is



b) More Complex Lever System



Determine the **transfer function** and **block diagram** for the rigid lever system.

Assuming that the displacements are small, then:

$$\tan\theta = \frac{x_{i1} + x_{i2}}{a+b} = \frac{x_o + x_{i2}}{b}$$

The output can be written as

$$x_o = \frac{b}{a+b} x_{i1} - \frac{a}{a+b} x_{i2}$$

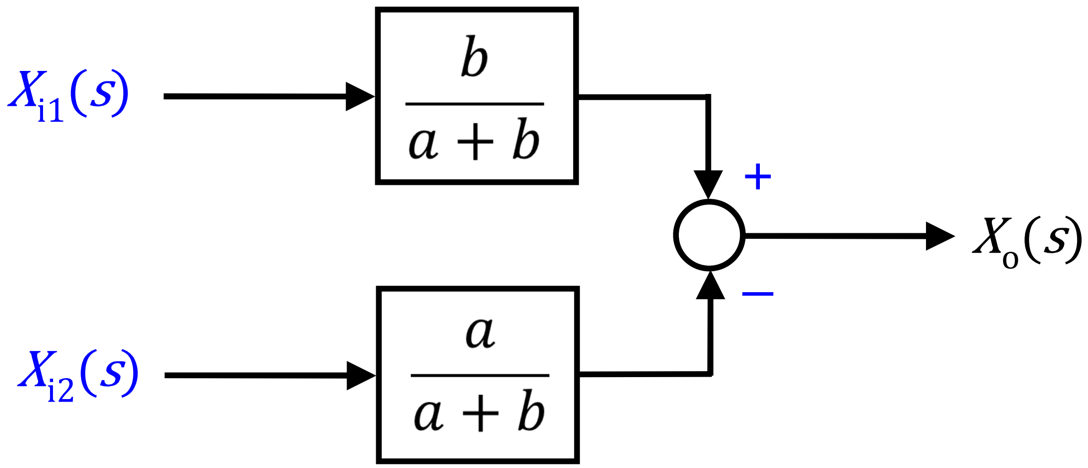
b) More Complex Lever System

Taking Laplace transforms gives

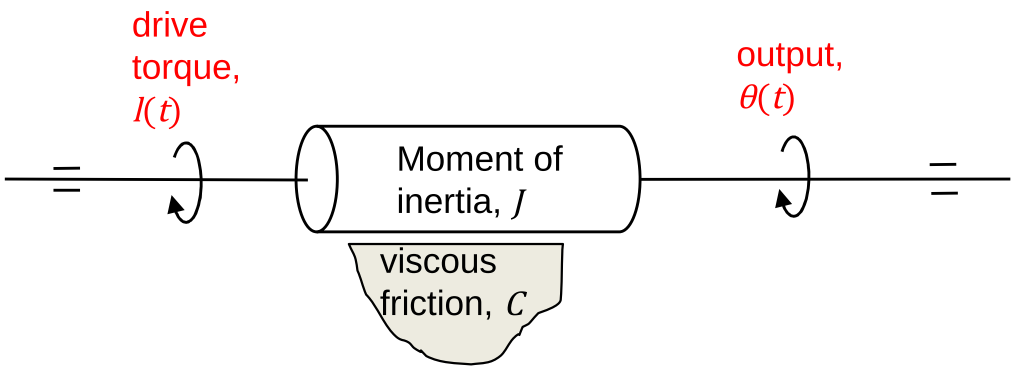
$$X_{\mathrm{o}}(s) = \frac{b}{a+b} X_{\mathrm{i1}}(s) - \frac{a}{a+b} X_{\mathrm{i2}}(s)$$

This is the **transfer function** for the system.

The **block diagram** for this system is



c) Rotor with Viscous Drag



Determine the **transfer function** and **block diagram** for the above system when the input is the drive torque $l(t)$ and the output is the angular displacement θ.

The equation of motion of this system is

$$l(t) - C\dot{\theta}(t) = J\ddot{\theta}(t)$$

Assuming zero initial conditions and taking Laplace transforms gives

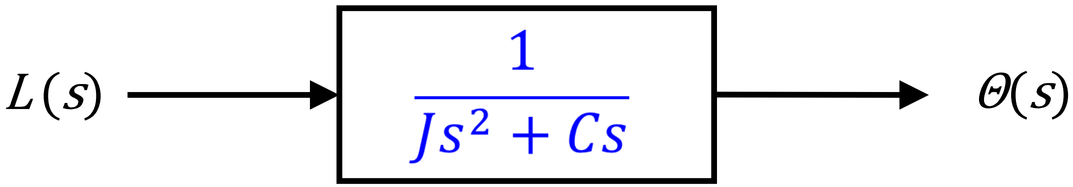
$$Js^2\Theta(s) + Cs\Theta(s) = L(s)$$

c) Rotor with Viscous Drag

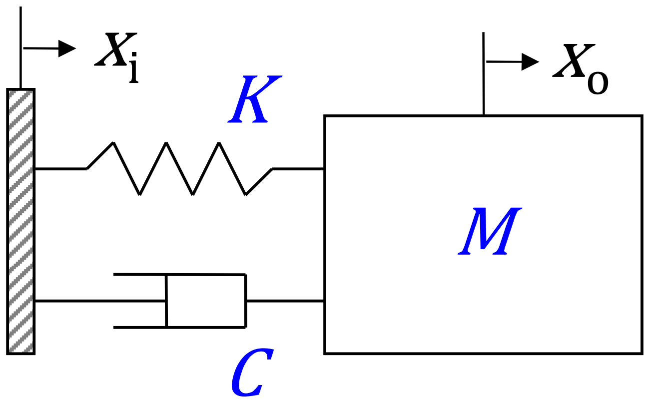
Rearranging, the **transfer function** is given by

$$G(s) = \frac{\Theta(s)}{L(s)} = \frac{1}{Js^2 + Cs}$$

The **block diagram** can be drawn as follows



d) Spring-Mass-Damper System



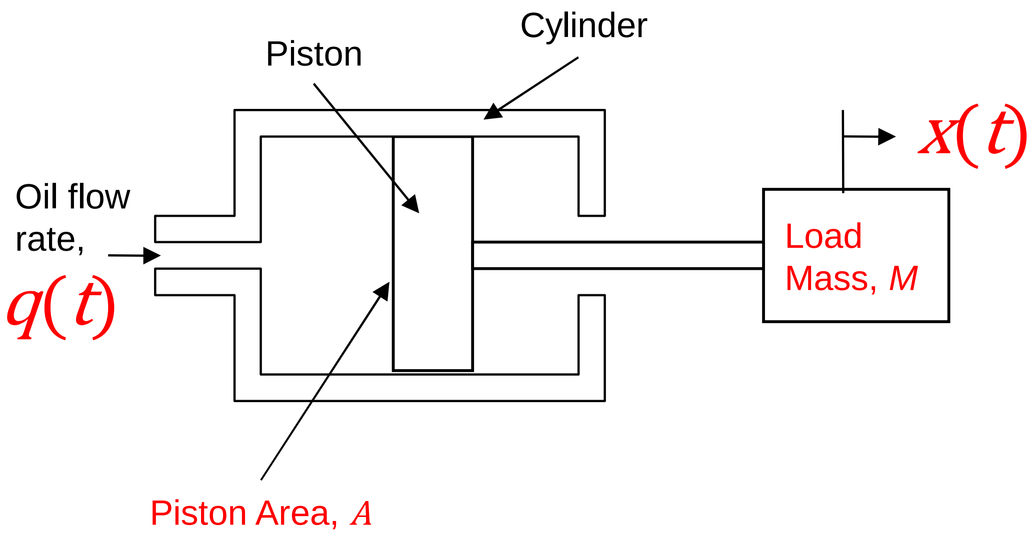
Exercise: Noting that the input to the above system is a displacement, show that the **transfer function** for the system is given by

$$G(s) = \frac{X_o(s)}{X_i(s)} = \frac{Cs + K}{Ms^2 + Cs + K} = \frac{2\gamma\omega_n s + \omega_n^2}{s^2 + 2\gamma\omega_n s + \omega_n^2}$$

where

$$\omega_n^2 = \frac{K}{M} \quad \text{and} \quad \gamma = \frac{C}{2\sqrt{KM}}$$

e) Hydraulic Ram



Determine the **transfer function** between the input $q(t)$ and the output $x(t)$.

Assumptions:

i) Neglect any leakage past the piston

ii) Neglect the compressibility of the oil

$$\frac{\mathrm{d(Vol)}}{\mathrm{d}t} = \mathrm{Area} \times \frac{\mathrm{d}x}{\mathrm{d}t} = q(t)$$

e) Hydraulic Ram

To obtain the transfer function the continuity equation for the oil flow is formed, such that

$$q(t) = q_{\text{piston}} = A\frac{\mathrm{d}x}{\mathrm{d}t}$$

Taking Laplace transforms with zero initial conditions and rearranging, the transfer function is

$$G(s) = \frac{X(s)}{Q(s)} = \frac{1}{As}$$

Note: In this simplified case the load mass *M* does not appear in the transfer function and the ram acts as an "**integrator**" (i.e. $1/s$)

$$q(t) = A\frac{\mathrm{d}x}{\mathrm{d}t} \quad \text{or} \quad x(t) = \frac{1}{A}\int q(t)\mathrm{d}t$$

Seminar – 1/2/2023

- Example sheet 0
  – I'll be going through Question 2
- Example sheet 1
  – Questions 2, 4
- Full solutions for odd numbered problems are posted for you.

What Next?

- Non-Linearity and Linearisation
- Introduction to Transient and Steady-State Responses
- Hydraulic Position Control System