University of Nottingham
Department of Mechanical Engineering

MM2MS3 Mechanics of Solids 3
Exercise Sheet - Finite Element Method

Part 1: Matrix Method

1. Two dissimilar rods are connected together and loaded as shown Figure 1. Using the stiffness matrix approach, calculate the displacement at the interface and the forces at the supports. E_{steel} = 200 GPa, $E_{aluminium}$ = 70 GPa.

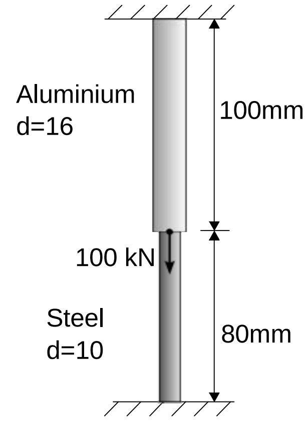

Figure 1

2. For the pin jointed structure shown in Figure 2. Determine the vertical and horizontal displacements at the loading point. The value of AE for each member is 200MN.

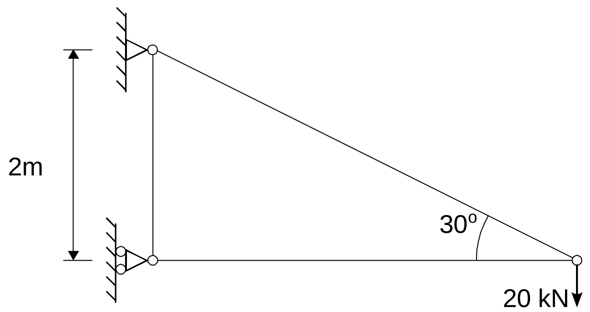

Figure 2

The stiffness matrix of a truss element is:

$$[k_e] = \left(\frac{AE}{L}\right)\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

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Part 2: Practical FE Problems

For the following cases, describe briefly the approach taken to model the problem in FE to obtain a good solution for the stresses. Sketch the geometry of the model (any symmetry?) including applied loads and boundary conditions and consider what mesh (element distribution; element type e.g. plane stress, plane strain, axisymmetric etc.) would be appropriate. Also consider any special features of the analysis.

1. A square plate of side length *L*, width *b* and thickness *t* with a central circular hole of radius *r*, is subjected to a uniaxial stress σ_0 as shown in Figure 3. You wish to determine the stress distribution around the hole. ($t << L$)

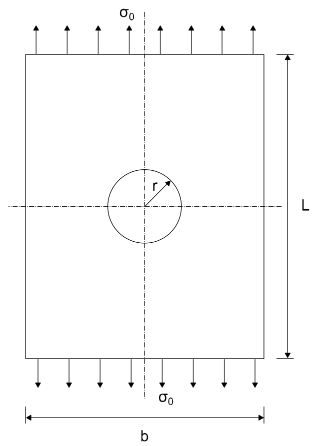


Figure 3

2. A steel cylindrical roller of radius *r* and length *L* is pressed on a flat block of regular cross-section of the same material of depth *h*, width *w* and length *L*, by a vertical line load of magnitude *F* per unit length as shown in Figure 4. You wish to determine the stress distribution in the block under the cylinder. (*L* is long and the loading is in-plane)

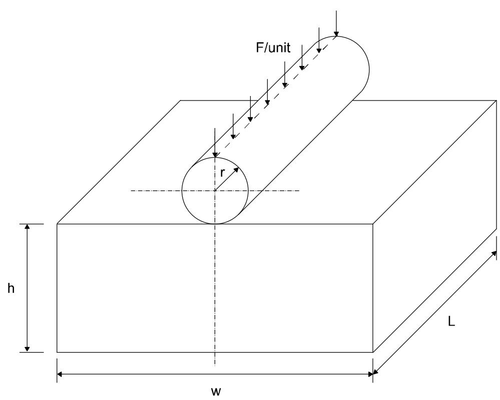
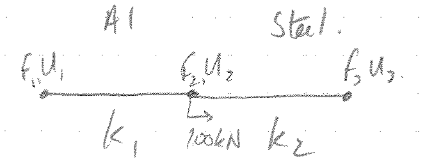


Figure 4

3. Consider a similar situation to Q2. However, in this case the cylinder is replaced by a sphere.

Part 1:

Q1. May be represented as two bar/spring elements



$\rightarrow$ +ve

$E_{Al} = 70 GPa$

$E_{St} = 200 GPa$

$A_1 = \pi \times (8 \times 10^{-3})^2 = 2.0 \times 10^{-4} m^2$ $\qquad L_1 = 100 \times 10^{-3} m$

$A_2 = \pi \times (5 \times 10^{-3})^2 = 7.9 \times 10^{-5} m^2$ $\qquad L_2 = 80 \times 10^{-3} m$

$$k_1 = \frac{A_1 E_1}{L_1} = \frac{2.0 \times 10^{-4} \times 70 \times 10^9}{0.1} = 1.4 \times 10^8 \ N/m$$

$$k_2 = \frac{A_2 E_2}{L_2} = \frac{7.9 \times 10^{-5} \times 200 \times 10^9}{0.08} = 1.98 \times 10^8 \ N/m.$$

Element stiffness matrices:

El 1 $$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} 1.4 \times 10^8 & -1.4 \times 10^8 \\ -1.4 \times 10^8 & 1.4 \times 10^8 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

El 2 $$\begin{Bmatrix} f_2 \\ f_3 \end{Bmatrix} = \begin{bmatrix} 1.98 \times 10^8 & -1.98 \times 10^8 \\ -1.98 \times 10^8 & 1.98 \times 10^8 \end{bmatrix}$$

Global Stiffness Matrix; $\qquad$ BCs $u_1 = u_3 = 0$

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{bmatrix} 1.4 \times 10^8 & -1.4 \times 10^8 & 0 \\ -1.4 \times 10^8 & (1.4 \times 10^8 + 1.98 \times 10^8) & -1.98 \times 10^8 \\ 0 & -1.98 \times 10^8 & 1.98 \times 10^8 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

gives $f_2 = (1.4 \times 10^8 + 1.98 \times 10^8) u_2$ $\qquad (f_2 = 100000 \ N)$

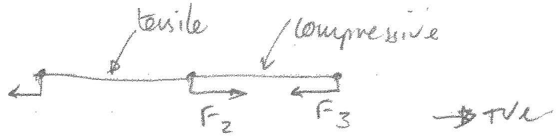
displacement $\Rightarrow u_2 = \dfrac{100\,000}{3.38 \times 10^8} = \underline{\underline{2.96 \times 10^{-4} m = 0.296 mm}}$

Q1 Reaction forces given by: $f_1 = -1.4\times10^8 \times U_2$

$$f_3 = -1.98\times10^8 \times U_2$$

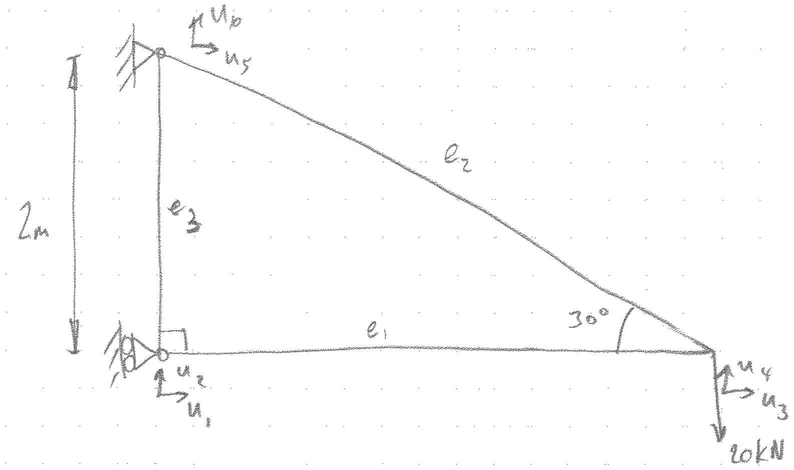
$$\Rightarrow f_1 = -41400N$$

$$f_3 \quad -58600N$$



As expected.

Q2



AE = 200MN.

$$L_{e_1} = \frac{2}{\tan 30} = 3.46\,m.$$

$$L_{e_2} = \frac{2}{\sin 30} = 4\,m.$$

$$L_{e_3} = 2\,m.$$

$$\left(\frac{AE}{L}\right)_{e_1} = \frac{200\times10^6}{3.46} = 5.78\times10^7\ N/m$$

$$\left(\frac{AE}{L}\right)_{e_2} = \frac{200\times10^6}{4} = 5\times10^7\ N/m.$$

$$\left(\frac{AE}{L}\right)_{e_3} = \frac{200\times10^6}{2} = 1\times10^8\ N/m.$$

$$\therefore [K_{e_1}] = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 5.78\times10^7 & 0 & -5.78\times10^7 & 0 \\ 0 & 0 & 0 & 0 \\ -5.78\times10^7 & 0 & 5.78\times10^7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\angle = 0°$

Q2. 2

$\angle = 150^\circ$

$$[K_{e_2}] = \begin{array}{cc} & \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 3.75\times10^7 & -2.16\times10^7 & -3.75\times10^7 & 2.16\times10^7 \\ -2.16\times10^7 & 1.25\times10^7 & 2.16\times10^7 & -1.25\times10^7 \\ -3.75\times10^7 & 2.16\times10^7 & 3.75\times10^7 & -2.16\times10^7 \\ 2.16\times10^7 & -1.25\times10^7 & -2.16\times10^7 & 1.25\times10^7 \end{bmatrix} \end{array}$$

$\angle = 270^\circ$

$$[K_{e_3}] = \begin{array}{cc} & \begin{array}{cccc} 5 & 6 & 1 & 2 \end{array} \\ \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1\times10^8 & 0 & -1\times10^8 \\ 0 & 0 & 0 & 0 \\ 0 & -1\times10^8 & 0 & 1\times10^8 \end{bmatrix} \end{array}$$

Assemble Stiffness Matrix.

$$[K] = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 5.78\times10^7 & 0 & -5.78\times10^7 & 0 & 0 & 0 \\ 0 & 1\times10^8 & 0 & 0 & 0 & -1\times10^8 \\ -5.78\times10^7 & 0 & 5.78\times10^7 + 3.75\times10^7 & -2.16\times10^7 & -3.75\times10^7 & -2.16\times10^7 \\ 0 & 0 & -2.16\times10^7 & 1.25\times10^7 & -2.16\times10^7 & -1.25\times10^7 \\ 0 & 0 & -3.75\times10^7 & -2.16\times10^7 & 3.75\times10^7 & -2.16\times10^7 \\ 0 & -1\times10^8 & -2.16\times10^7 & -1.25\times10^7 & -2.16\times10^7 & 1.25\times10^7 + 1\times10^8 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 5.78 & 0 & -5.78 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & -10 \\ -5.78 & 0 & 9.53 & -2.16 & -3.75 & -2.16 \\ 0 & 0 & -2.16 & 1.25 & -2.16 & -1.25 \\ 0 & 0 & -3.75 & -2.16 & 3.75 & -2.16 \\ 0 & -10 & -2.16 & -1.25 & -2.16 & 11.25 \end{bmatrix} \times 10^7$$

Q2 $\{F\} = [K]\{u\}$. BCs are $u_1 = u_5 = u_6 = 0$

so,

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 9.53 & -2.16 \\ 0 & -2.16 & 1.25 \end{bmatrix} \times 10^7 \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -20000 \end{Bmatrix}$$

$$9.53\times10^7 u_3 + -2.16\times10^7 u_4 = 0 \qquad (1)$$

$$-2.16\times10^7 u_3 + 1.25\times10^7 u_4 = -20000. \qquad (2)$$

From (1) $9.53\times10^7 u_3 = -2.16\times10^7 u_4$.

$$= u_4 = \frac{9.53\times10^7}{2.16\times10^7} u_3 = 4.41 \quad u_3.$$

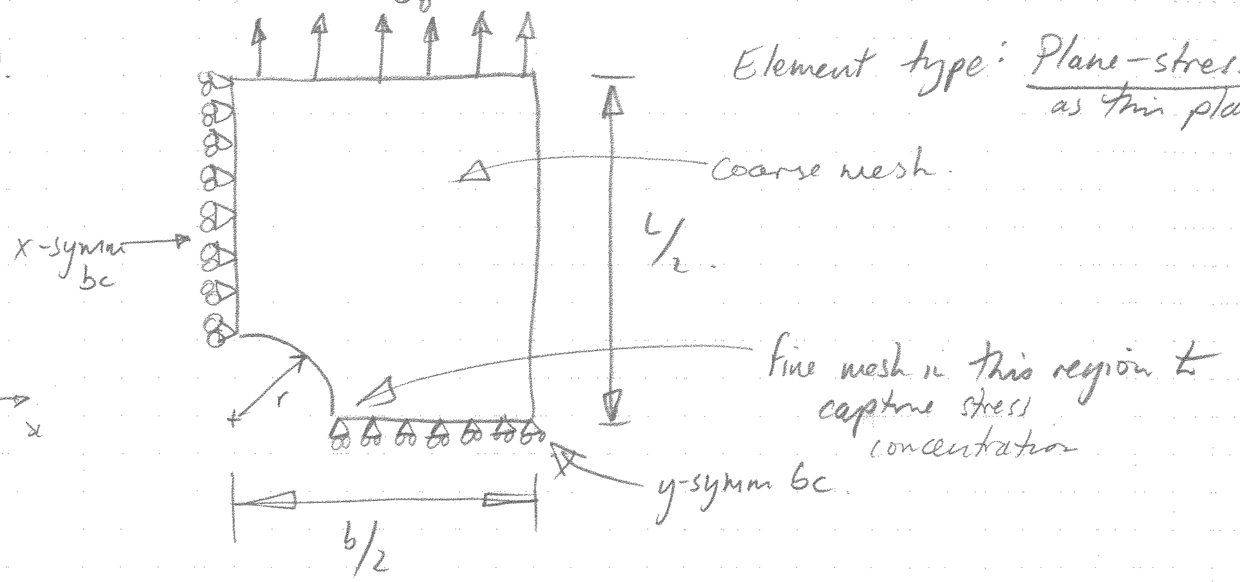
Subs into (2) gives $3.35\times10^7 u_3 = -20000$

$$\Rightarrow u_3 = \underline{6\times10^{-4}\,m} = \underline{-0.6\,mm}.$$

$$u_4 = \underline{-2.6\times10^{-3}\,m} = \underline{-2.6\,mm}$$

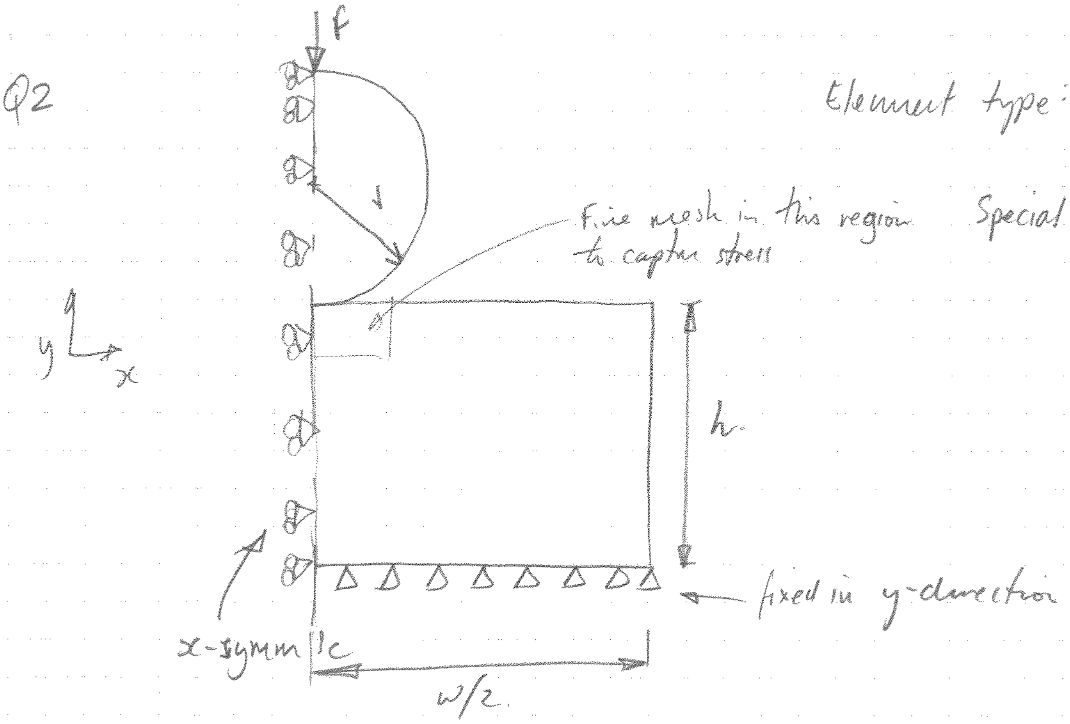
Part 2:

Q1.



Element type: Plane-stress as thin plate

Q2



Element type: Plane-strain as thick $\varepsilon_z \approx 0$

Special features: contact

Q3. As above but use Axisymmetric elements + no need for x-symmetry bc on left edge.