Elastic Instability (Buckling)

Lecture 1 – Introduction and Ideal Struts

Department of Mechanical, Materials & Manufacturing Engineering

MMME2053 – Mechanics of Solids

University of Nottingham
UK | CHINA | MALAYSIA

Elastic Instability (Buckling)

Introduction

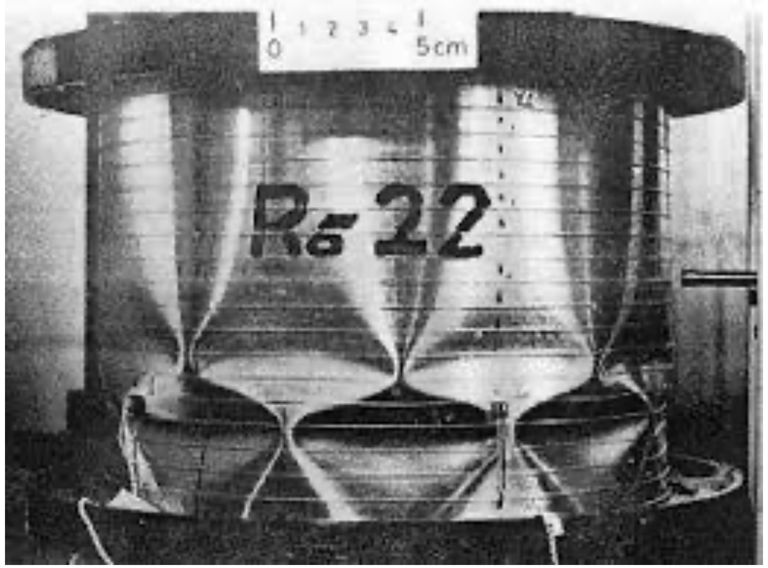
For many structural problems, it is reasonable to assume that the system is in stable equilibrium. However, not all structural arrangements are stable. For example, consider a one-meter long stick with the cross-sectional area of a pencil. If this stick were stood on its end, the axial stress would be small, but the stick could easily topple over sideways. This simple example demonstrates that in some configurations, stability considerations can be primary.

This section is concerned with the stability of struts. Struts are compression members with cross-sectional dimensions which are small compared to the length, i.e., they are slender. If a circular rod of, say, 5mm diameter, which has its ends machined flat and perpendicular to the axis, were made 10mm long to act as a column, there would not be a problem of instability and it could carry considerable force. However, if the same rod were made a meter long, the rod would become laterally unstable at a much smaller applied force and could collapse.

Buckling also occurs in many other situations with compressive forces. Examples include thin sheets which have no problem carrying tensile loads and vacuum tanks, as well as submarine hulls. Thin-walled tubes can wrinkle like paper when subjected to torque.

Buckling

Examples



Elastic Instability (Buckling) Methods

Learning Outcomes

1. Know the meanings of and the differences between stable, unstable and neutral equilibria (knowledge);

2. Be able to apply Macaulay's method for determining beam deflection in situations with axial loading (application);

3. Be able to determine the buckling loads for ideal struts (application);

4. Be able to include the interaction of yield behaviour with buckling and how to represent this interaction graphically (knowledge/application).

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The Stability of Equilibrium

Some examples were given in the introduction slide:

- One-meter long stick with the cross-sectional area of a pencil. If this stick were stood on its end, the axial stress would be small, but the stick could easily topple over sideways.
- Circular rod of, say, 5 mm diameter, which has its ends machined flat and perpendicular to the axis, were made 10mm long to act as a column, there would not be a problem of instability and it could carry considerable force. However, if the same rod were made a meter long, the rod would become laterally unstable at a much smaller applied force and could collapse.

Assume this pin is frictionless

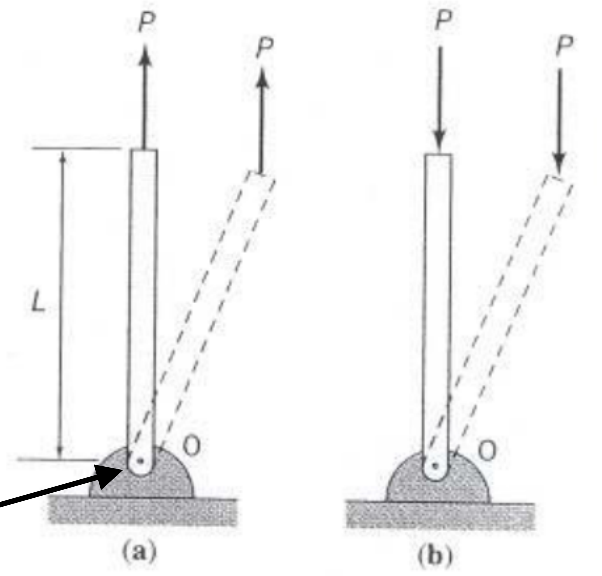

Fig 3.6.2 Examples of stable and unstable equilibrium

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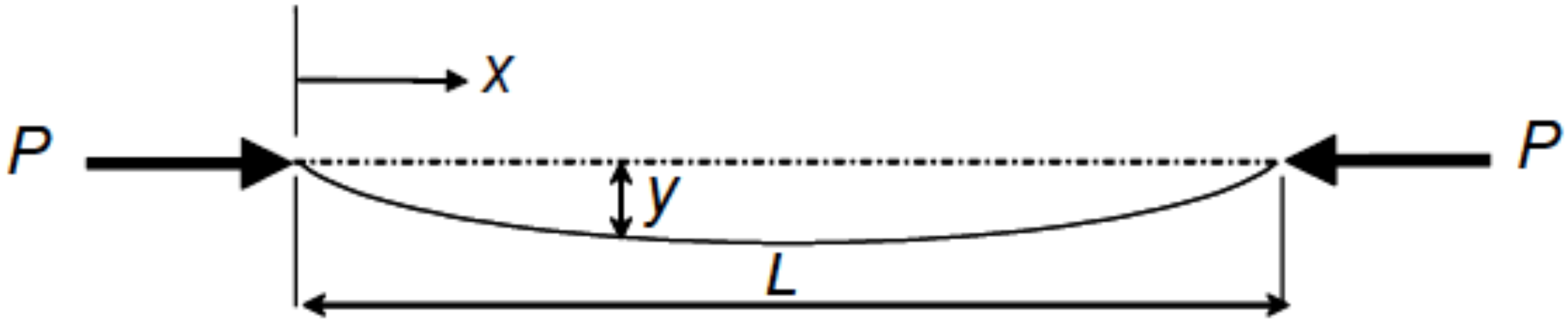
4. Be able to include the interaction of yield behaviour with buckling and how to represent this interaction graphically (knowledge/application).

Struts

Hinged-Hinged End Conditions

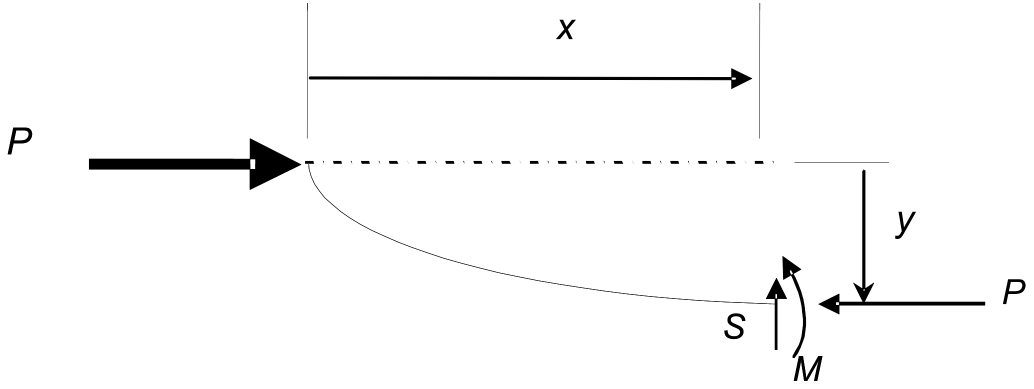
Consider an initially straight strut with its ends free to rotate around frictionless pins. The strut is now considered to be perturbed, from its initially straight position.

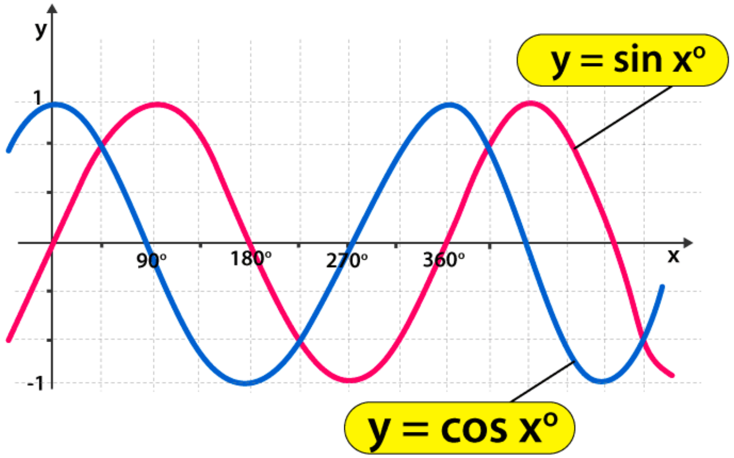


Struts

Hinged-Hinged End Conditions

FBD





Taking moments about the section position in order to determine an expression for the bending moment, M:

$$M = Py$$

Substituting this into the 2nd order differential equation of the elastic line (see Deflection of Beams notes) :

$$EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Py = 0$$

$$\therefore \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{P}{EI}y = 0$$

Let: $\dfrac{P}{EI} = \alpha^2$

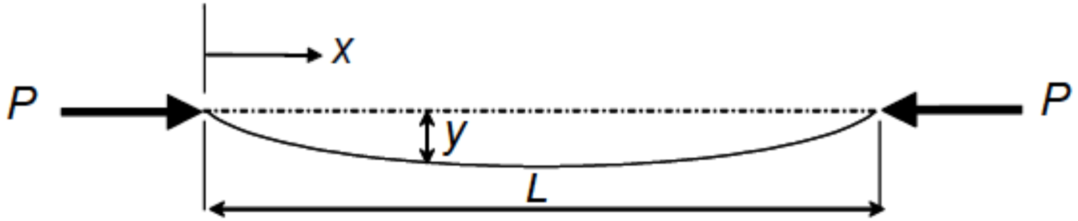
$$\therefore \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \alpha^2 y = 0$$

$$y = A\sin\alpha x + B\cos\alpha x$$

Struts

Hinged-Hinged End Conditions

$$y = A\sin\alpha x + B\cos\alpha x$$



In order to determine values for A and B we need 2 boundary conditions as follows:

$$x = 0, y = 0$$

$$x = L, y = 0$$

$$\therefore B = 0 \quad \text{and} \quad A\sin(\alpha L) = 0$$

Trivial solution:

$$A = 0$$

(for an undeflected strut)

Non-trivial solution:

$$\sin(\alpha L) = 0$$

$$\alpha L = n\pi$$

Where n = 1, 2, …

$$\therefore \alpha^2 L^2 = n^2\pi^2$$

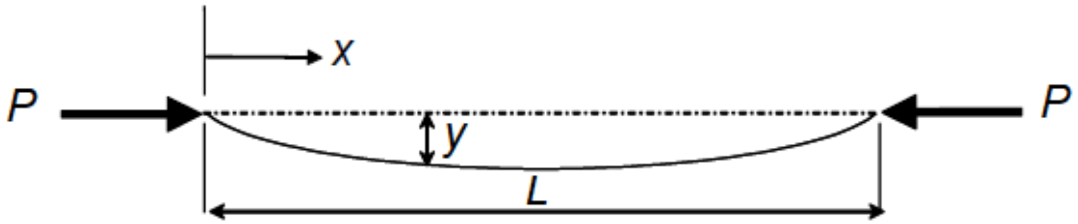
Recalling that: $\frac{P}{EI} = \alpha^2$

$$\therefore \frac{P}{EI} L^2 = n^2\pi^2$$

$$\therefore P = \frac{n^2\pi^2 EI}{L^2}$$

Struts

Hinged-Hinged End Conditions



$$\therefore P = \frac{n^2\pi^2 EI}{L^2}$$

n = 0 gives the trivial solution:

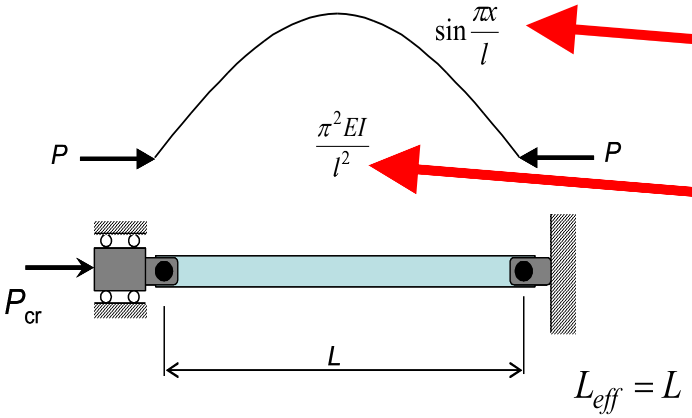
$$\therefore P = 0$$

n = 1 gives the Euler Buckling load MODE 1

$$\therefore P_c = \frac{\pi^2 EI}{L^2}$$

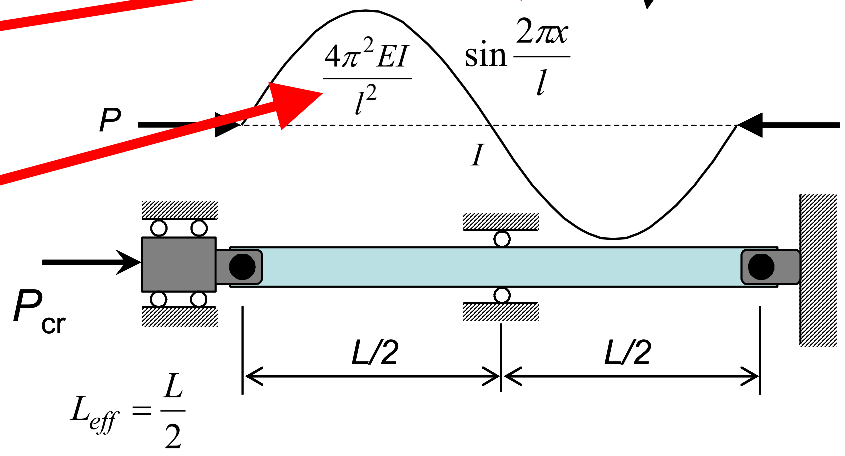
n = 2 MODE 2

$$\therefore P = \frac{4\pi^2 EI}{L^2}$$



Eigenvector (mode shape)

Eigenvalue (buckling load)



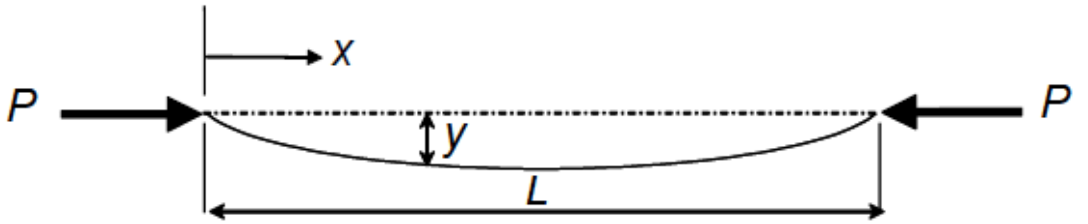
Struts

Hinged-Hinged End Conditions

$$\therefore P = \frac{n^2 \pi^2 EI}{L^2} \longrightarrow$$

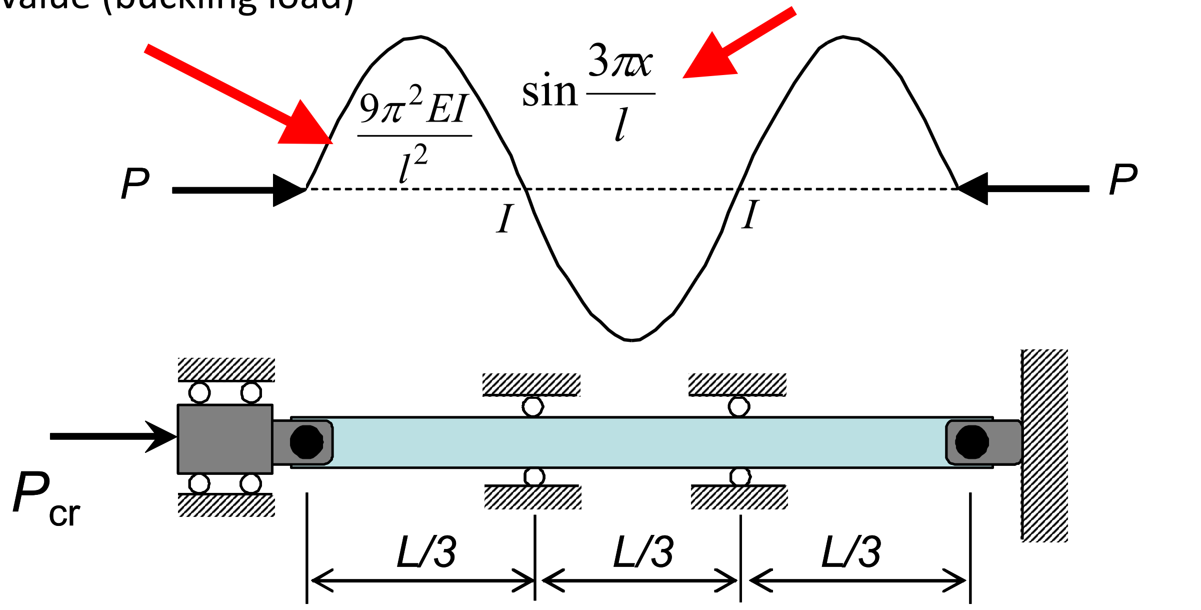
n = 3

MODE 3



Eigenvalue (buckling load)

Eigenvector (mode shape)

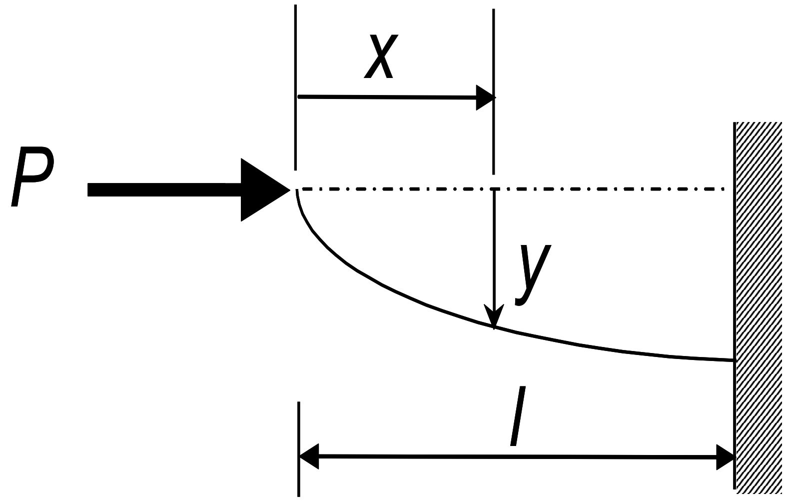


$$L_{eff} = \frac{L}{3}$$

Struts

Fixed-Free End Conditions

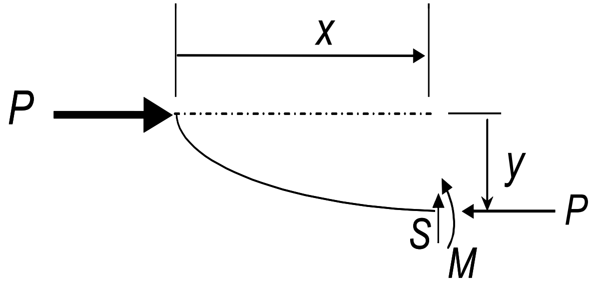
Consider an initially straight strut with one end built-in (fixed) and the other end free (unconstrained). The strut is now considered to be perturbed, from its initially straight position.

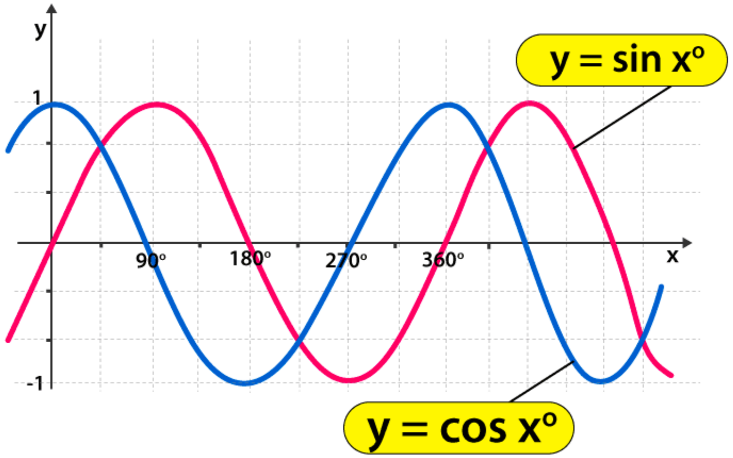


Struts

Fixed-Free End Conditions

FBD





Taking moments about the section position in order to determine an expression for the bending moment, M:

$$M = Py$$

Substituting this into the 2nd order differential equation of the elastic line (see Deflection of Beams notes) :

$$EI\frac{d^2y}{dx^2} + Py = 0$$

$$\therefore \frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

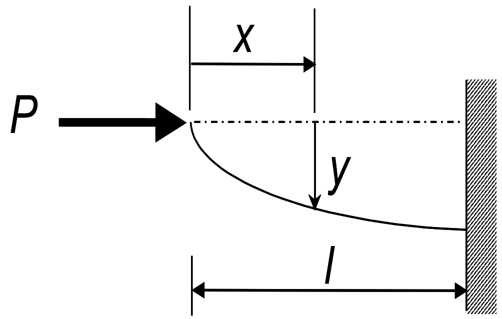
Let: $\frac{P}{EI} = \alpha^2$

$$\therefore \frac{d^2y}{dx^2} + \alpha^2 y = 0$$

$$y = A\sin\alpha x + B\cos\alpha x$$

Struts

Fixed-Free End Conditions



$$y = A\sin\alpha x + B\cos\alpha x \qquad \frac{dy}{dx} = A\alpha\cos\alpha x - B\alpha\cos\alpha x$$

In order to determine values for A and B we need 2 boundary conditions as follows:

$$x = 0, y = 0$$

$$x = L, \frac{dy}{dx} = 0$$

$$\therefore B = 0 \quad \text{and} \quad A\alpha\cos(\alpha L) = 0$$

Trivial solution:

$$A = 0$$

(for an undeflected strut)

Non-trivial solution:

$$\cos(\alpha L) = 0$$

$$\alpha L = \frac{n\pi}{2}$$

Where n = 1, 2, …

$$\therefore \alpha^2 L^2 = \frac{n^2\pi^2}{4}$$

Recalling that: $\frac{P}{EI} = \alpha^2$

$$\therefore \frac{P}{EI} L^2 = \frac{n^2\pi^2}{4}$$

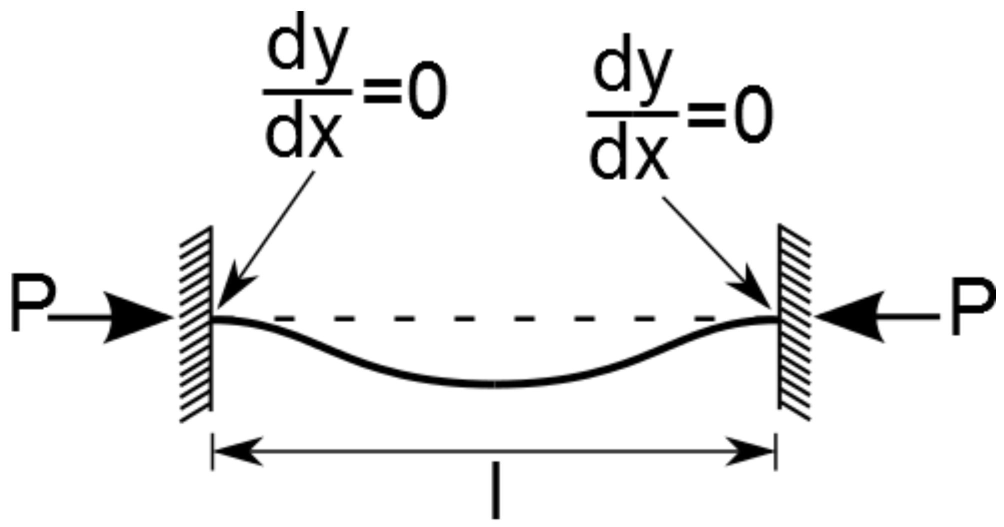
The smallest, non-trivial, value of P occurs when n = 1:

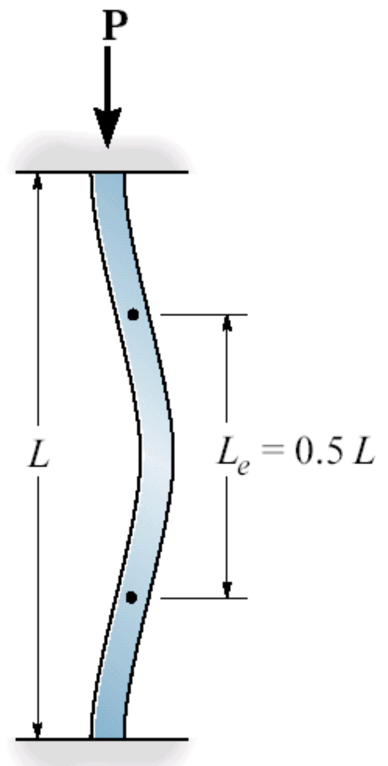
$$\therefore P_c = \frac{\pi^2 EI}{4\mathrm{L}^2}$$

Struts

Fixed-Fixed End Conditions

Consider an initially straight strut that is built-in (fixed) at both ends. The strut is now considered to be perturbed, from its initially straight position.



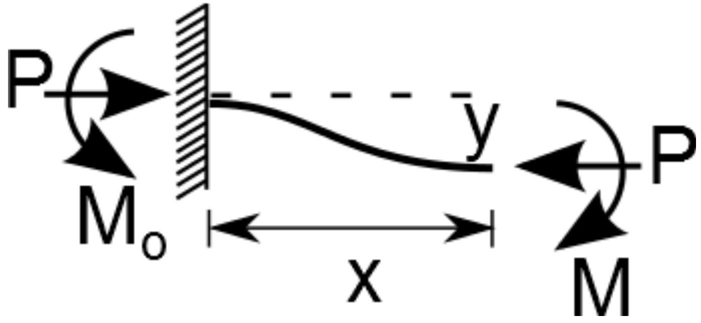


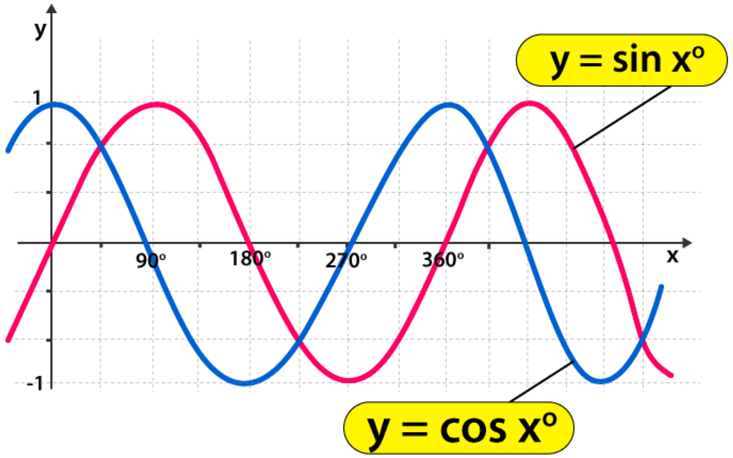
Fixed ends

Struts

Fixed-Fixed End Conditions

FBD





Taking moments about the section position in order to determine an expression for the bending moment, M:

$$M = M_o - Py$$

Substituting this into the 2nd order differential equation of the elastic line (see Deflection of Beams notes) :

$$EI\frac{d^2y}{dx^2} - Py = -M_o$$

$$\therefore \frac{d^2y}{dx^2} - \frac{P}{EI}y = -\frac{M_o}{EI}$$

Let: $\frac{P}{EI} = \alpha^2$

$$\therefore \frac{d^2y}{dx^2} - \alpha^2 y = -\frac{M_o}{EI}$$

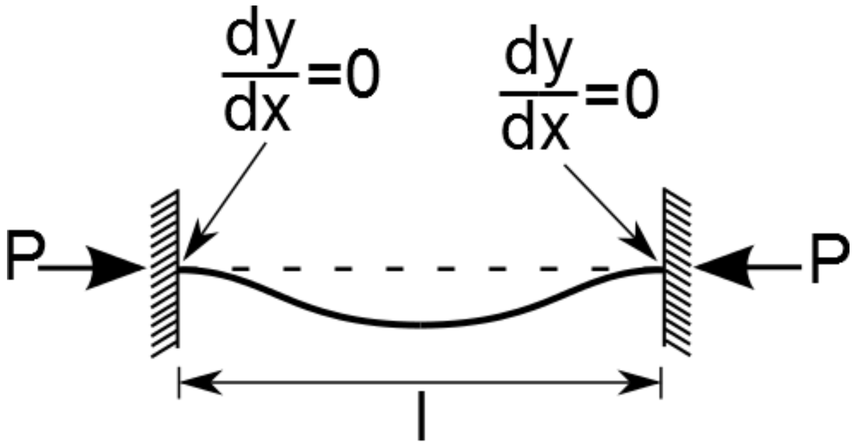
$$y = A\sin\alpha x + B\cos\alpha x - \frac{M_o}{EI}\frac{1}{\alpha^2}$$

Struts

Fixed-Fixed End Conditions

$$y = A\sin\alpha x + B\cos\alpha x - \frac{M_o}{EI}\frac{1}{\alpha^2}$$

$$\frac{dy}{dx} = A\alpha\cos(\alpha x) - B\alpha\sin(\alpha x)$$



In this case, we have 4 boundary conditions:

$x = 0, y = 0$ $\quad\therefore B = \frac{M_o}{EI}\frac{1}{\alpha^2}$

$x = 0, \frac{dy}{dx} = 0$ $\quad\therefore A = 0$

$x = L, y = 0$ $\quad\therefore \cos(\alpha L) = 1$

$x = L, \frac{dy}{dx} = 0$ $\quad\therefore \sin(\alpha L) = 0$

$$\alpha L = \text{n}\pi$$

where n is even

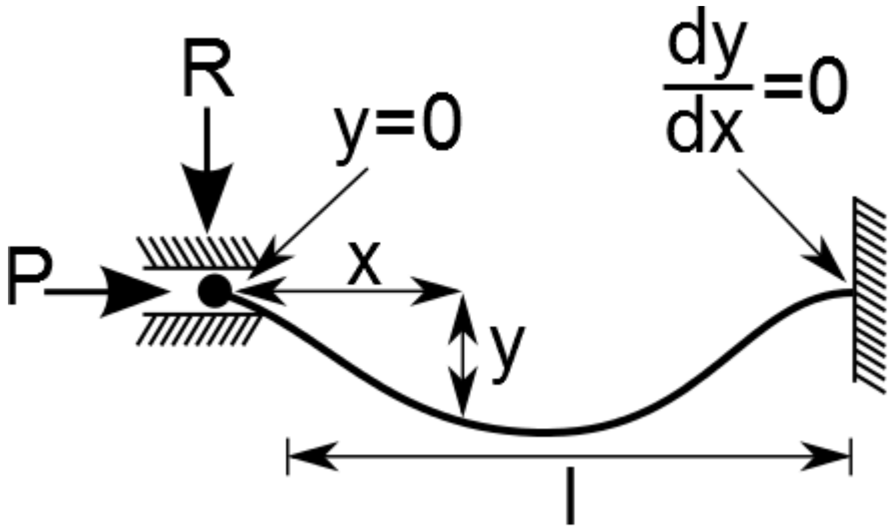
where n is odd or even

$$\therefore y = B\left(\cos\frac{2\pi}{L}x - 1\right) \quad \text{and} \quad P_c = \frac{4\pi^2 EI}{L^2}$$

Struts

Hinged-Fixed End Conditions

Consider an initially straight strut with one end built-in (fixed) and pinned at the other end free (free to rotate around frictionless pins). The strut is now considered to be perturbed, from its initially straight position.

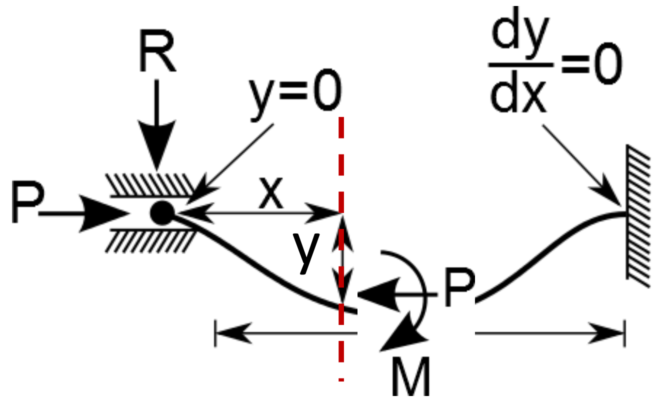


This case differs from the previous examples in that a transverse force, R, shown in the figure, is necessary to create this mode of deformation.

Struts

Hinged-Fixed End Conditions

FBD



Taking moments about the section position in order to determine an expression for the bending moment, M:

$$M = Rx - Py$$

Substituting this into the 2nd order differential equation of the elastic line (see Deflection of Beams notes) :

$$EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Py = Rx$$

$$\therefore \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{P}{EI}y = \frac{Rx}{EI}$$

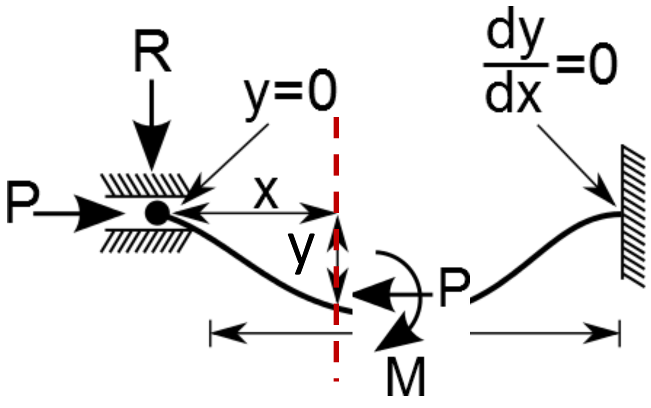
Let: $\frac{P}{EI} = \alpha^2$

$$\therefore \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \alpha^2 y = \frac{Rx}{EI}$$

Struts

Hinged-Fixed End Conditions

FBD



$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{Rx}{EI}$$

Solving this case requires this use of a particular integral (as described in the notes) to yield:

$$y = A\sin\alpha x + B\cos\alpha x + \frac{R}{P}x$$

and the application of the following boundary conditions:

$$x = 0, y = 0$$

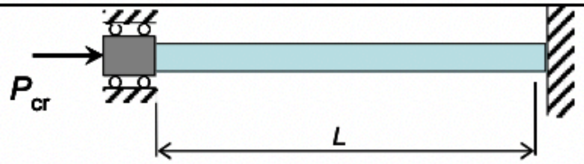
$$x = L, y = 0$$

$$x = L, \frac{dy}{dx} = 0$$

gives: $$P_c = \frac{2.045\pi^2 EI}{L^2}$$

Struts

Summary

Description	Schematic	Critical buckling load, P_c	Effective length, L_{eff}
Free-fixed		$P_c = \frac{\pi^2 EI}{4l^2}$	2*l*
Hinged-hinged		$P_c = \frac{\pi^2 EI}{l^2}$	l
Fixed-hinged		$P_c = \frac{2.045\pi^2 EI}{l^2}$	$0.7l$
Fixed-fixed		$P_c = \frac{4\pi^2 EI}{l^2}$	$\frac{l}{2}$

General Formula:

$$\therefore P_c = \frac{\pi^2 EI}{L_{eff}^2}$$

Elastic Instability (Buckling) Methods

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