The University of Nottingham

SCHOOL OF MATHEMATICAL SCIENCES

A LEVEL 2 MODULE, AUTUMN SEMESTER SEMESTER 2019-2020

ADVANCED MATHEMATICS AND STATISTICS FOR MECHANICAL ENGINEERS

Time allowed TWO Hours

Candidates may complete the front cover of their answer book and sign their desk card but must NOT write anything else until the start of the examination period is announced.

Answer ALL FIVE questions.

Only a calculator from approved list A (or one functionally equivalent) may be used in this examination.

List A

Basic Models	***Scientific Calculators***
Aurora HC133	Aurora AX-582
Casio HS-5D	Casio FX82 family
Deli – DL1654	Casio FX83 family
Sharp EL-233	Casio FX85 family
	Casio FX350 family
	Casio FX570 family
	Casio FX 991 family
	Sharp EL-531 family
	Texas Instruments TI-30 family
	Texas BA II+ family

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

DO NOT turn examination paper over until instructed to do so

ADDITIONAL MATERIAL: Formula Sheet, Table of Laplace Transfron, Table of Normal Distribution.

1. (a) Find the general solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 3x^2 - 2x - 3.$$

[10 marks]

(b) Find the solution of the system of ordinary differential equations

$$\frac{dx}{dt} = 2x - y, \quad \frac{dy}{dt} = x + 2y,$$

[10 marks]

subject to the initial conditions $x(0) = 1$, $y(0) = 0$.

2. Consider the periodic function $f(x)$ defined by

$f(x) = x^2$ for $-1 \leq x \leq 1$, with $f(x+2) = f(x)$ for all x.

(a) Sketch the graph of $f(x)$ for $-2 \leq x \leq 2$. [2 marks]

(b) Is $f(x)$ odd, even or neither odd nor even? [2 marks]

(c) Find the Fourier series of $f(x)$. [12 marks]

(d) By evaluating the Fourier series of $f(x)$ at $x = 1$, evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

[4 marks]

3. (a) Show, using the definition

$$\bar{f}(s) = \int_0^{\infty} f(t)e^{-st}dt,$$

and repeated integration by parts, that the Laplace transform of $\frac{d^3f}{dt^3}$ is

$$s^3\bar{f}(s) - s^2f(0) - sf'(0) - f''(0),$$

where a prime denotes d/dt. [5 marks]

(b) Show that the Laplace transform of the solution of the ordinary differential equation

$$\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - \frac{dy}{dt} + 2y = \delta(t-1),$$

subject to the initial conditions

$$y(0) = \frac{dy}{dt}(0) = 0, \quad \frac{d^2y}{dt^2}(0) = 1,$$

is

$$\bar{y}(s) = \frac{1 + e^{-s}}{(s-1)(s+1)(s-2)},$$

and hence determine $y(t)$. [15 marks]

4. The behaviour of the small displacement, $\phi(x,t)$, of a stretched string of length L that lies in $0 \leq x \leq L$ is governed by the wave equation,

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2},$$

where c is a positive constant and t is time, subject to the boundary conditions

$$\phi(0,t) = \phi(L,t) = 0.$$

A separable solution of this partial differential equation takes the form $\phi(x,t) = X(x)T(t)$.

(a) Determine the ordinary differential equations satisfied by $X(x)$ and $T(t)$. [4 marks]

(b) Solve these equations and hence write down the general solution for $\phi(x,t)$, giving careful consideration to the sign of the separation constant. [10 marks]

(c) If

$$\frac{\partial \phi}{\partial t}(x,0) = 0 \quad \text{and} \quad \phi(x,0) = f(x),$$

[4 marks]

determine an expression for any constants in the general solution for $\phi(x,t)$ in terms of $f(x)$.

(d) If $f(x) = \sin(3\pi x/L)$, determine $\phi(x,t)$. [2 marks]

5. (a) For events A and B it is known that $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{5}$ and $P(A \cap B) = \frac{1}{6}$.

i) Determine the probability that A occurs, given that B occurs.

ii) Determine the probability that A occurs, given that B does not occur.

[4 marks]

(b) Suppose that the number of faulty switches in each batch produced by Switch-a-lot follows a Poisson distribution with mean $3/2$. What is the probability that a given batch contains more than 1 faulty switch? [3 marks]

(c) Suppose that door frames in use on a building site have width that follows a Normal distribution with mean $605\,\text{mm}$ and standard deviation $2\,\text{mm}$; and that the widths of doors being used also follow a Normal distribution, but with mean $602\,\text{mm}$ and standard deviation $3/2\,\text{mm}$.

i) Find the probability that a randomly chosen door is at least $600\,\text{mm}$ wide.

ii) Find the probability that the width of a randomly chosen door is less than that of a randomly chosen door frame.

[7 marks]

(d) Testing of the fuel efficiency of a new design of diesel electricity generator takes place in 50 different scenarios (representative of industrial use) and the observed efficiencies from those scenarios have sample mean $15.72\,\text{litres/hour}$ and sample variance $2.1\,(\text{litres/hour})^2$.

Construct a 95% confidence interval for the mean fuel efficiency of the new generator; and comment on how the data suggests the efficiency of the new generators compares to that of the existing design which achieves an average efficiency of $16.2\,\text{litres/hour}$.

[6 marks]