MTHS2007 Advanced Mathematics and Statistics for Mechanical Engineers

Chapter 2: differential equations

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Introduction

This part of the module is concerned with solution techniques for linear ordinary differential equations (ODEs).

- These equations arise naturally in the description of physical, chemical and biological systems.
- The aim of the module is to familiarise you with basic solution techniques to these equations to enable you to study problems arising in engineering.
- There is hardly any subject in science or engineering that does not involve differential equations, so these are *extremely important*.

Dependent and independent variables

The full solution to a differential equation, or DE, expresses the dependent variables in terms of the independent variables. So for example in

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 1, \qquad (1)$$

y is the dependent variable, x is the independent variable and the solution

$$y(x) = Ae^{-x} + Bxe^{-x} + 1$$

expresses y (the dependent variable) in terms of x (the independent variable).

If a DE has more than one independent variable, it is called a *partial differential equation* (PDE).

An example of a PDE is

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

where φ is the dependent variable, and x and y the independent variables. The solution $\varphi(x, y)$ is a function of two variables.

- If a DE has only one independent variable, for example (1), it is called an *ordinary differential equation* (ODE).
- We will study PDEs later in the module: In this section, we solve ODEs.

Order

The order of a differential equation (DE) is that of the highest order derivative appearing in the equation. So for example

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} - \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \sin x \tag{2}$$

is a fourth order ODE.

In general, an n^{th} order ODE can be written as

$$F(x, y(x), y'(x), \cdots, y^{(n)}(x)) = 0. \tag{3}$$

Linear differential equations

In a *linear* DE, there are no non-linear functions of the dependent variable or its derivatives, and derivatives are not multiplied together with other derivatives or the dependent variable, *etc.*.

The most general second order linear ODE is thus

$$a(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}y}{\mathrm{d}x} + c(x)y = r(x). \tag{4}$$

Any equation for which this is not true, for example

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{q}{L}\sin\theta,$$

is called *nonlinear* ($\sin\theta$ is not linear in the dependent variable θ).

Homogeneous ODEs

A linear ODE is in *standard form* if all terms containing the dependent variable are on the left hand side, and all terms not containing the dependent variable are on the right hand side.

A linear ODE is called a *homogeneous* ODE, if in its standard form, the right hand side is zero. Hence,

$$a(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}y}{\mathrm{d}x} + c(x)y = 0 \tag{5}$$

is the most general second-order homogeneous ODE.

An ODE that is not homogeneous is called *inhomogeneous*. For example, equation (4) is inhomogeneous if $r(x) \neq 0$ for some x.

2.2 Properties of linear ODEs

Linear ODEs often appear in Engineering problems, and many useful non-linear ODEs can be approximated by linear ODEs for certain values of the independent variable. This is extremely convenient, as linear ODEs can be solved analytically.

Nonlinear ODEs are much harder to deal with, and many of them can only be solved numerically, ie on a computer. Methods for this are considered in module MTHS3001/HG3MCE.

The superposition principle

The **superposition principle** states that if $y_1(x)$ and $y_2(x)$ are two solutions of a **linear homogeneous** ODE, *eg.* equation (5), then

$$y(x) = Ay_1(x) + By_2(x)$$

is also a solution of the same ODE for any constants A and B.

If the functions $y_1(x)$ and $y_2(x)$ are solutions of (5), so

$$\begin{aligned} a(x)\frac{\mathrm{d}^2 y_1}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}y_1}{\mathrm{d}x} + c(x)y_1 &= 0, \qquad (6) \\ a(x)\frac{\mathrm{d}^2 y_2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}y_2}{\mathrm{d}x} + c(x)y_2 &= 0. \qquad (7) \end{aligned}$$

then adding (6) multiplied by A to (7) multiplied by B gives

$$a\frac{\mathrm{d}^2}{\mathrm{d}x^2}(Ay_1 + By_2) + b\frac{\mathrm{d}}{\mathrm{d}x}(Ay_1 + By_2) + c(Ay_1 + By_2) = 0,$$

so $y = Ay_1 + By_2$ is also a solution of (5).

The function $y(x) = Ay_1(x) + By_2(x)$ contains two arbitrary constants A and B. This is a general rule.

If we have two *linearly independent* solutions $y_1(x)$ and $y_2(x)$ of a homogeneous, linear, 2^{nd}-order ODE (*ie*. solutions that are not just multiples of each other), then *every* solution of that ODE can be written as

$$y(x) = Ay_1(x) + By_2(x)$$

with some constants A and B. This is called the *general solution* of the ODE.

A similar rule holds for ODEs of higher orders. The general solution of an n^{th} order, homogeneous, linear ODE is a linear combination of n linearly independent solutions: it contains n arbitrary constants.

Example

Consider the ODE

$$\frac{d^2y}{dx^2} - y = 0.$$

You can easily check that

$$y_1(x) = e^x \qquad \text{and} \qquad y_2(x) = e^{-x}$$

are solutions of the ODE. Thus the general solution is

$$y(x) = Ae^x + Be^{-x}.$$

The question is, how do we find two fundamental solutions y_1 and y_2 in general? This is tackled in the Section 2.3.

Boundary and initial conditions for ODEs

We have seen that a stand-alone differential equation such as

$$\frac{d^2y}{dx^2} - y = 0$$

has many solutions. Each different value of the constants in

$$y(x) = Ae^x + Be^{-x}$$

gives a different function of x.

In a properly modelled engineering problem we would expect to get a *unique* solution. To achieve this we need more information, usually given in the form of **initial conditions** or **boundary conditions**.

Initial Conditions

With initial conditions, we are given information about the solution at a single value of the independent variable.

For example, for second order equations, initial conditions are typically of the following form.

Initial conditions: Find a solution such that, when $x = x_0$, y and $\dfrac{dy}{dx}$ are given numbers:

$$y(x_0) = c_1 \quad \text{and} \quad y'(x_0) = c_2.$$

Typically here, x_0 represents time.

Since there are two arbitrary constants for a second-order equation, we need two independent bits of information to get a unique solution.

Example Along with

$$\frac{d^2y}{dx^2} - y = 0,$$

we are told that

$$y(0) = 1 \quad \text{and} \quad y'(0) = 0.$$

Substituting these initial conditions into the general solution $y(x) = Ae^x + Be^{-x}$ gives

$$\begin{aligned} y(0) = A + B &= 1 \\ y'(0) = A - B &= 0 \end{aligned} \quad \Rightarrow \quad A = B = \frac{1}{2}$$

so the *explicit* solution is

$$y(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}.$$

Boundary Conditions

With boundary conditions, we are given information about the solution at multiple values of the independent variable.

An example of boundary conditions for second order equations:

Boundary conditions: Find a solution such that conditions are satisfied when $x = x_0$ and $x = x_1$. For example

$$y'(x_0) = c_3 \quad \text{and} \quad y(x_1) = c_4.$$

Typically here, x is a spatial coordinate.

As with initial conditions, we need two independent bits of information to get a unique solution for a second-order equation.

Example Along with

$$\frac{d^2y}{dx^2} - y = 0,$$

we are told that

$$y'(0) = 0 \quad \text{and} \quad y(1) = 1.$$

Substituting these boundary conditions into the general solution $y(x) = Ae^x + Be^{-x}$ gives

$$\left.\begin{array}{rcl} y'(0) & = & A - B = 0 \\ y(1) & = & Ae + Be^{-1} = 1 \end{array}\right. \Rightarrow \quad A = B = \frac{1}{e + e^{-1}}$$

so the *explicit* solution is

$$y(x) = \frac{e^x + e^{-x}}{e + e^{-1}}.$$

Boundary and initial conditions for ODEs

In general, n conditions are required to obtain a unique solution for an n^{th} order ODE.

For example, the solution of the given by n-th order ODE (3) can be made unique by imposing the initial conditions

$$y(x_0) = Y_0, \quad y'(x_0) = Y_1, \quad \cdots, \quad y^{(n-1)}(x_0) = Y_{n-1},$$

specified at the same point $x = x_0$.

Example Imposing the initial conditions

$$y(0) = 2, \quad y'(0) = 3/2, \quad y''(0) = 1, \quad y'''(0) = 1/2,$$

on the fourth order equation (2) leads to the unique solution

$$y(x) = 1 + \frac{\sin x}{2} + e^x.$$

Check by substitution that this works.

Example Imposing the boundary conditions

$$y(0) = 1, \quad y'(0) = 1/2, \quad y(\pi) = 1, \quad y'(\pi) = -1/2$$

on (2) leads to the unique solution

$$y(x) = 1 + \frac{\sin x}{2}.$$

2.3 Second order homogeneous linear ODEs with constant coefficients

We consider homogeneous linear ODEs, where the coefficients a, b and c are *constant*, ie do not depend on x,

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0,$$

where a, b, c are constants.

We want to find the general solution. This is done by "guessing" solutions of the type $y = \mathrm{e}^{mx}$, where m is a constant. Then

$$y = \mathrm{e}^{mx}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = m\mathrm{e}^{mx}, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = m^2\mathrm{e}^{mx}.$$

Substituting this into the ODE, we obtain

$$(am^2 + bm + c)\mathrm{e}^{mx} = 0.$$

As e^{mx} cannot be zero, we must have

$$am^2 + bm + c = 0.$$

This quadratic equation for m is called the *auxiliary equation*.

In general, it has two solutions,

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

So we find the following two solutions to the ODE,

$$y_1(x) = e^{m_1 x} \quad \text{and} \quad y_2(x) = e^{m_2 x}.$$

The general solution to the ODE is a combination of these,

$$y(x) = Ae^{m_1 x} + Be^{m_2 x},$$

with arbitrary (real or complex) constants A and B.

Example

The auxiliary equation for the ODE

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4y = 0$$

is

$$m^2 - 4 = 0 \quad \Rightarrow \quad m = \pm 2.$$

The general solution is then

$$y(x) = A\mathrm{e}^{2x} + B\mathrm{e}^{-2x},$$

with arbitrary constants A and B.

Note The same method can be used for an n-th order homogeneous linear ODE: In this case the we have to find the zeroes of a polynomial of degree n.

Example Solve

$$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0.$$

Solution: The auxiliary equation

$$3m^2 - 5m - 2 = 0$$

can be factorised as

$$\begin{aligned} 3m^2 - 5m - 2 &= (3m+1)(m-2) \\ \Rightarrow &\quad 3m+1 = 0 \quad \text{or} \quad m-2=0 \\ \Rightarrow &\quad m = -\frac{1}{3} \quad \text{or} \quad m = 2. \end{aligned}$$

Then

$$y_1(x) = \mathrm{e}^{-x/3} \quad \text{and} \quad y_2(x) = \mathrm{e}^{2x}$$

both satisfy the differential equation and the general solution is

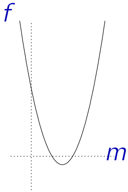
$$y(x) = A\mathrm{e}^{-x/3} + B\mathrm{e}^{2x}.$$

The auxiliary equation is a quadratic equation, and the roots of a quadratic may be

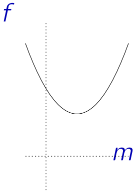
1. distinct and real (as in the examples above),
2. distinct and complex conjugates of each other,
3. real and equal.

This can be seen from the graph of

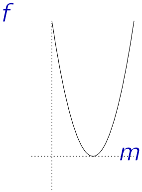
$$f(m) = am^2 + bm + c.$$



Case 1



Case 2



Case 3

Case 1 Real distinct roots, $m_1 \neq m_2$.

The general solution is as previously written

$$y(x) = Ae^{m_1 x} + Be^{m_2 x},$$

with arbitrary constants A and B. Job done!

Case 2 Complex conjugate roots,

$$m_1 = \alpha + \mathrm{i}\beta \quad \text{and} \quad m_2 = \alpha - \mathrm{i}\beta$$

The general solution is

$$y(x) = Ae^{m_1 x} + Be^{m_2 x} = Ae^{(\alpha+\mathrm{i}\beta)x} + Be^{(\alpha-\mathrm{i}\beta)x},$$

with arbitrary constants A and B.

However, $y_1(x) = e^{m_1 x} = e^{(\alpha+\mathrm{i}\beta)x}$ and $y_2(x) = e^{m_2 x} = e^{(\alpha-\mathrm{i}\beta)x}$ are complex-valued and not particularly convenient for representing physical quantities.

It is better to use Euler's formula

$$e^{i\beta x} = \cos(\beta x) + i\sin(\beta x)$$

to rewrite the solution in terms of trig functions as follows:

$$\begin{aligned}
Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} &= Ae^{\alpha x}e^{i\beta x} + Be^{\alpha x}e^{-i\beta x} \\
&= e^{\alpha x}[Ae^{i\beta x} + Be^{-i\beta x}] \\
&= e^{\alpha x}[A(\cos\beta x + i\sin\beta x) + B(\cos\beta x - i\sin\beta x)] \\
&= e^{\alpha x}[(A+B)\cos\beta x + i(A-B)\sin\beta x] \\
&= e^{\alpha x}[C\cos\beta x + D\sin\beta x]
\end{aligned}$$

where

$$C = A + B \quad \text{and} \quad D = i(A - B)$$

are alternative integration constants. The general solution can be written

$$y(x) = e^{\alpha x}(C\cos\beta x + D\sin\beta x).$$

Note: C and D can be **real** if A and B are **complex**.

Example: Solve

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0.$$

Solution: The auxiliary equation

$$m^2 + 2m + 2 = 0$$

has the roots

$$m_1 = -1 + i \quad \text{and} \quad m_2 = -1 - i,$$

so we get the general solution

$$\begin{aligned} y(x) &= Ae^{(-1+i)x} + Be^{(-1-i)x} \\ &= e^{-x}[Ae^{ix} + Be^{-ix}] \\ &\vdots \\ &= e^{-x}[C\cos x + D\sin x], \end{aligned}$$

where C and D are arbitrary constants (and actually, we *don't care* how they're related to A and B since all the constants are arbitrary anyway).

Case 3 Equal roots, $m_1 = m_2$.

We have found only one fundamental solution solution,

$$y_1(x) = e^{m_1 x}.$$

We need another! In this special case it turns out that

$$y_2(x) = x e^{m_1 x}$$

is also a solution. You can check by substituting into the ODE.

In this case the general solution is

$$\begin{aligned} y(x) &= A y_1(x) + B y_2(x) \\ &= (A + Bx) e^{m_1 x} \end{aligned}$$

with arbitrary constants A and B.

Example: Solve

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$$

Solution: The auxiliary equation is

$$m^2 + 2m + 1 = (m + 1)^2 = 0$$

and $m = -1$ is a double root. Then we know

$$y_1(x) = e^{-x}$$

is one root and have been told

$$y_2(x) = xe^{-x}$$

is another.

Check this:

$$\begin{aligned}\frac{\mathrm{d}y_2}{\mathrm{d}x} &= \mathrm{e}^{-x} - x\mathrm{e}^{-x} = (1-x)\mathrm{e}^{-x} \\ \frac{\mathrm{d}^2y_2}{\mathrm{d}x^2} &= -\mathrm{e}^{-x} - (1-x)\mathrm{e}^{-x} = (x-2)\mathrm{e}^{-x}\end{aligned}$$

and therefore

$$\frac{\mathrm{d}^2y_2}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y_2}{\mathrm{d}x} + y_2 = [((x-2) + 2(1-x) + x]\mathrm{e}^{-x} = 0.$$

The general solution is then

$$y(x) = Ay_1(x) + By_2(x) = (A + Bx)\mathrm{e}^{-x}.$$

Example: Solve

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Solution: The auxiliary equation is

$$\begin{aligned} m^2 + 3m + 2 &= 0 \\ \Rightarrow \quad (m+1)(m+2) &= 0, \end{aligned}$$

so the roots are $m_1 = -1$, $m_2 = -2$. We have real distinct roots, so the general solution is

$$y(x) = Ae^{-x} + Be^{-2x}.$$

The imposed initial conditions give

$$\begin{aligned} y(0) &= A + B = 1 \\ y'(0) &= -A - 2B = 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= 2 \\ B &= -1 \end{aligned}$$

The explicit solution is therefore $y(x) = 2e^{-x} - e^{-2x}$.

Example: Solve

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$$

subject to the boundary conditions $y(0) = 1$ and $y(\pi/4) = 0$.

Solution: The auxiliary equation is

$$m^2 + 2m + 5 = 0,$$

with complex roots, $m = -1 \pm 2i$, so the general solution is

$$y(x) = e^{-x}\left[C\cos(2x) + D\sin(2x)\right].$$

The imposed boundary conditions give

$$\begin{aligned} y(0) &= C = 1 \\ y(\pi/4) &= e^{-\pi/4}\left[C\cos\frac{\pi}{2} + D\sin\frac{\pi}{2}\right] = e^{-\pi/4}D = 0 \end{aligned}$$

The explicit solution is therefore $y(x) = e^{-x}\cos(2x)$.

Example:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0.$$

subject to the initial conditions $y(0) = 0$ and $y'(0) = 1$.

Solution: The auxiliary equation is

$$\begin{aligned} m^2 + 4m + 4 &= 0 \\ \Rightarrow\ (m+2)(m+2) &= 0, \end{aligned}$$

with equal roots $m = -2$ and -2. The general solution is then

$$y = (Ax + B)\mathrm{e}^{-2x}.$$

The imposed initial conditions give

$$y(0) = B = 0 \quad \Rightarrow \quad y(x) = Ax\mathrm{e}^{-2x} \Rightarrow \quad y'(x) = A(1-2x)\mathrm{e}^{-2x}$$

$$\Rightarrow \quad y'(0) = A = 1$$

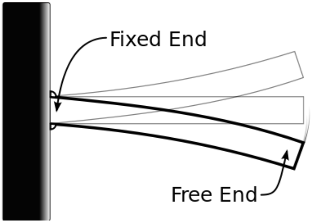
The explicit solution is therefore $y(x) = x\mathrm{e}^{-2x}$.

Example: This is a more challenging example. It involves a higher-order equation and some quantities are expressed as general parameters rather than simple numbers.

A common model for the shape of a vibrating beam is

$$\frac{d^4 y}{dx^4} - k^4 y = 0,$$

where $y(x)$ describes how far the beam deflects at position coordinate x.

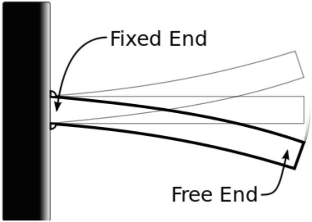


- The parameter k depends on the material parameters of the beam, its geometry and the frequency of vibration.
- This is a *fourth-order* differential equation, so we need *four* independent bits of information to get an explicit solution - provided by the boundary conditions overleaf.

Boundary conditions for a vibrating cantilever

The problem in the diagram shows a *cantilever*, fixed to a wall at one end and free at the other.

Let the fixed and free ends correspond to $x = 0$ and $x = L$ respectively.



- The boundary conditions

$$y(0) = 1 \quad \text{and} \quad y'(0) = 0.$$

model the situation where the cantilever is mounted horizontally and the point of attachment oscillates up and down, with amplitude equal to one unit of length.

- But a fourth-order equation needs two more conditions! If no forces/moments are applied at the free end, it can be shown that

$$y''(L) = 0 \quad \text{and} \quad y'''(L) = 0.$$

General solution of the vibrating beam equation

If we substitute the usual guess $y(x) = \mathrm{e}^{mx}$ in the vibrating beam equation we get the auxiliary equation

$$m^4 = k^4.$$

Here k is a fixed parameter and we are to find m. There are *four* solutions:

$$m = k, \qquad m = -k, \qquad m = \mathrm{i}k \quad \text{and} \quad m = -\mathrm{i}k.$$

The general solution of the equation then has *four* independent constants, which can be expressed in the following alternative ways:

$$\begin{aligned} y(x) &= E\mathrm{e}^{kx} + F\mathrm{e}^{-kx} + A\mathrm{e}^{\mathrm{i}kx} + B\mathrm{e}^{-\mathrm{i}kx} \\ &= E\mathrm{e}^{kx} + F\mathrm{e}^{-kx} + C\cos kx + D\sin kx, \end{aligned}$$

where $C = A + B$ and $D = \mathrm{i}(A - B)$.

We will make one more change to the form

$$y(x) = C\cos kx + D\sin kx + E\mathrm{e}^{kx} + F\mathrm{e}^{-kx},$$

of the general solution given on the previous slide. Recall the *hyperbolic functions*, defined

$$\cosh\theta = \frac{1}{2}\left(\mathrm{e}^{\theta} + \mathrm{e}^{-\theta}\right) \quad \text{and} \quad \sinh\theta = \frac{1}{2}\left(\mathrm{e}^{\theta} - \mathrm{e}^{-\theta}\right)$$

and satisfying the identities

$$\cosh\theta + \sinh\theta = \mathrm{e}^{\theta} \quad \text{and} \quad \cosh\theta - \sinh\theta = \mathrm{e}^{-\theta}.$$

Then

$$E\mathrm{e}^{kx} + F\mathrm{e}^{-kx} = G\cosh kx + H\sinh kx$$

where $G = E + F$ and $H = E - F$. The general solution can then be alternatively written

$$y(x) = C\cos kx + D\sin kx + G\cosh kx + H\sinh kx,$$

This form makes applying boundary conditions ~~easier~~ less messy!

Explicit solution of the vibrating cantilever

Let's start from the general solution expressed in the form

$$y(x) = C\cos kx + D\sin kx + G\cosh kx + H\sinh kx,$$

where C, D, G and H are four independent constants to be determined from the four boundary conditions.

The boundary conditions at $x = 0$ give

$$\begin{aligned} y(0) &= C + G = 1 \quad \Rightarrow \quad G = 1 - C \\ y'(0) &= k\,(D + H) = 0 \quad \Rightarrow \quad H = -D, \end{aligned}$$

so now we know the solution is of the form

$$y(x) = \cosh kx + C\,(\cos kx - \cosh kx) + D\,(\sin kx - \sinh kx)$$

There are still *two* undetermined constants, C and D, which are fixed by the boundary conditions at the other end.

The boundary conditions at $x = L$ give

$$0 = \frac{y''(L)}{k^2} = \cosh kL - C\left(\cos kL + \cosh kL\right) - D\left(\sin kL + \sinh kL\right)$$

$$0 = \frac{y'''(L)}{k^3} = \sinh kL + C\left(\sin kL - \sinh kL\right) - D\left(\cos kL + \cosh kL\right)$$

which can be rearranged as two simultaneous equations

$$\begin{aligned} \left(\cos kL + \cosh kL\right) C + \left(\sin kL + \sinh kL\right) D &= \cosh kL \\ \left(-\sin kL + \sinh kL\right) C + \left(\cos kL + \cosh kL\right) D &= \sinh kL \end{aligned}$$

or in matrix form

$$\begin{pmatrix} \cos kL + \cosh kL & \sin kL + \sinh kL \\ -\sin kL + \sinh kL & \cos kL + \cosh kL \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} \cosh kL \\ \sinh kL \end{pmatrix}$$

The equations are still quite complicated, so we won't attempt to solve them here, but they can be solved easily numerically. In an engineering sense this problem is *done*.

2.4 Inhomogeneous linear ODEs with constant coefficients

The linear ODE

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = r(x),$$

where the RHS involves x, but not y, is an *inhomogeneous ODE*.

This equation is solved in two stages:

1. Solve the homogeneous equation obtained by setting $r(x) = 0$:

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0,$$

   using the method from the previous subsection. The general solution, denoted $y_c(x) = Ay_1(x) + By_2(x)$, is called the *complementary function*.

2 Find a particular solution or *particular integral* $y_{\mathrm{p}}(x)$ of the inhomogeneous equation

Method of undetermined parameters: Depending on the form of $r(x)$, substitute an informed guess for $y_p(x)$ that has undetermined parameters in it. Plug in the differential equation to get equations for the parameters and solve them.

Once we have found the complementary function and a particular integral, the *general* solution of the inhomogeneous equation is

$$y(x) = y_{\mathrm{c}}(x) + y_{\mathrm{p}}(x) = y_{\mathrm{p}}(x) + Ay_1(x) + By_2(x).$$

The main challenge here is to find particular integrals - which we do by example.

Example: Solve

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \mathrm{e}^{-x}.$$

Solution:

- First solve the associated homogeneous equation. The auxiliary equation is

$$m^2 - 3m + 2 = 0,$$

with solutions $m = 1$ and $m = 2$. Hence the *complementary function* is

$$y_{\mathrm{c}}(x) = A\mathrm{e}^{x} + B\mathrm{e}^{2x}.$$

- Now find a *particular integral* corresponding to $r(x) = \mathrm{e}^{-x}$. What sort of function could give $r(x) = \mathrm{e}^{-x}$ when differentiated and added? *Guess* a solution of the form

$$y_{\mathrm{p}}(x) = a\mathrm{e}^{-x},$$

where a is a constant.

Differentiating $y_{\mathrm{p}}(x) = a\mathrm{e}^{-x}$ gives

$$\frac{\mathrm{d}y_{\mathrm{p}}}{\mathrm{d}x} = -a\mathrm{e}^{-x} \qquad \text{and} \qquad \frac{\mathrm{d}^2 y_{\mathrm{p}}}{\mathrm{d}x^2} = a\mathrm{e}^{-x}.$$

Substitute y_{p} into the ODE:

$$\begin{aligned} y_{\mathrm{p}}''(x) - 3y_{\mathrm{p}}'(x) + 2y_{\mathrm{p}}(x) &= a\mathrm{e}^{-x} - 3\left(-a\mathrm{e}^{-x}\right) + 2a\mathrm{e}^{-x} \\ &= 6a\mathrm{e}^{-x} \\ &= r(x) = \mathrm{e}^{-x}. \end{aligned}$$

So we need

$$6a\mathrm{e}^{-x} = \mathrm{e}^{-x}.$$

This works if we choose we $a = 1/6$. Then the particular integral is

$$y_{\mathrm{p}}(x) = \frac{1}{6}\mathrm{e}^{-x}$$

and the general solution of the inhomogeneous ODE is

$$y(x) = A\mathrm{e}^{x} + B\mathrm{e}^{2x} + \frac{1}{6}\mathrm{e}^{-x}.$$

Example: Solve

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 3 - 2x^2$$

Solution:

- Since the LHS is the same as in the previous example, we have the same complementary function:

$$y_c(x) = Ae^x + Be^{2x}.$$

- Now let's find the particular integral: what function might give $r(x) = 3 - 2x^2$ after repeated differentiation and then adding the results? Try

$$y_p(x) = ax^2 + bx + c,$$

where a, b and c are parameters to be determined.

Differentiating $y_\mathrm{p}(x) = ax^2 + bx + c$ a couple of times gives

$$\frac{\mathrm{d}y_\mathrm{p}}{\mathrm{d}x} = 2ax + b, \qquad \frac{\mathrm{d}^2 y_\mathrm{p}}{\mathrm{d}x^2} = 2a.$$

Substitute into the ODE:

$$\begin{aligned} y_\mathrm{p}''(x) - 3y_\mathrm{p}'(x) + 2y_\mathrm{p}(x) &= 2a - 3(2ax + b) + 2(ax^2 + bx + c) \\ &= 2ax^2 + (-6a + 2b)x + (2a - 3b + 2c) \\ &= r(x) = 3 - 2x^2. \end{aligned}$$

Now find a, b and c by comparing coefficients,

$$\begin{array}{lrcl} x^2: & 2a = -2 & \Rightarrow & a = -1, \\ x^1: & -6a + 2b = 0 & \Rightarrow & b = -3, \\ x^0: & 2a - 3b + 2c = 3 & \Rightarrow & c = -2. \end{array}$$

So the complete solution to the ODE is

$$y(x) = A\mathrm{e}^x + B\mathrm{e}^{2x} - x^2 - 3x - 2.$$

Example: Solve

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin x$$

Solution:

- Again, the complementary function is the same as before:

$$y_c(x) = Ae^x + Be^{2x}.$$

- Try to find a particular integral: what kind of function might give $r(x) = \sin x$ after substitution in the differential equation? The guess

$$y_p(x) = a\sin x,$$

is tempting but doesn't work:

$$y_p''(x) - 3y_p'(x) + 2y_p(x) = -a\sin x - 3a\cos x + 2a\sin x = \sin x.$$

There is no value of a that satisfies both

$$a\sin x = \sin x \quad \text{and} \quad -3a\cos x = 0.$$

So we try a particular integral of the more general form

$$y_p(x) = a\sin x + b\cos x.$$

Substitute into the ODE:

$$\begin{aligned} y_p''(x) - 3y_p'(x) + 2y_p(x) &= -a\sin x - b\cos x \\ &\qquad -3a\cos x + 3b\sin x \\ &\qquad\qquad +2a\sin x + 2b\cos x \\ &= (a+3b)\sin x + (b-3a)\cos x \\ &= r(x) = \sin x. \end{aligned}$$

Now find a and b by comparing coefficients:

$$\begin{aligned} \cos x:&\quad b - 3a = 0 \;\Rightarrow\; b = 3a. \\ \sin x:&\quad a + 3b = 1 \;\Rightarrow\; a + 9a = 1. \end{aligned} \quad\Rightarrow\quad \begin{aligned} a &= \frac{1}{10} \\ b &= \frac{3}{10}. \end{aligned}$$

This works! The solution to the ODE is then

$$y(x) = Ae^{x} + Be^{2x} + \frac{1}{10}\sin x + \frac{3}{10}\cos x.$$

Let us summarise the lessons so far about how to choose a particular integral.

How to choose the particular integral

- If the RHS is a polynomial of degree n, choose the most general polynomial of the same degree.
- If the RHS is $e^{\alpha x}$, choose

$$y_p(x) = ae^{\alpha x}.$$

- If the RHS is $\cos(\alpha x)$ or $\sin(\alpha x)$ or some linear combination of them, choose

$$y_p(x) = a\cos(\alpha x) + b\sin(\alpha x).$$

2.5 Right hand sides needing more complicated guesses

There are cases where the strategies on the previous slide fail or need to be extended. Let's look at some examples.

Example Solve

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \mathrm{e}^x$$

Solution This is an example of an *exceptional case:* the RHS $r(x) = \mathrm{e}^x$ is a solution of the homogeneous equation, ie it appears in the complementary function, $y_\mathrm{c}(x) = A\mathrm{e}^x + B\mathrm{e}^{2x}$.

- In this case the natural first choice for the particular integral,

  $$y_\mathrm{p}(x) = a\mathrm{e}^x,$$

  fails, as it gives

  $$y_\mathrm{p}''(x) - 3y_\mathrm{p}'(x) + 2y_\mathrm{p}(x) = a\mathrm{e}^x - 3a\mathrm{e}^x + 2a\mathrm{e}^x = 0.$$

  No matter what we choose for a, this can't match the desired RHS $r(x) = \mathrm{e}^x$.

The solution is to try a particular integral of the form

$$y_p(x) = axe^x.$$

Then

$$\frac{dy_p}{dx} = axe^x + ae^x, \qquad \frac{d^2y_p}{dx^2} = axe^x + 2ae^x.$$

Substitute into the ODE:

$$\begin{aligned} {y_p}''(x) - 3{y_p}'(x) + 2y_p(x) &= axe^x + 2ae^x - 3(axe^x + ae^x) + 2axe^x \\ &= -ae^x \\ &= r(x) = e^x \end{aligned}$$

(all the terms containing xe^x have cancelled). So the guess works if we choose $a = -1$. The solution to the ODE is

$$y = Ae^x + Be^{2x} - xe^x.$$

A similar modification of the standard guess is needed in all cases where the RHS is a solution of the homogeneous equation.

Example Solve

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \mathrm{e}^x$$

Subject to the initial conditions

$$y(0) = 1 \quad \text{and} \quad y'(0) = 0.$$

Solution The general solution of the ODE,

$$y(x) = A\mathrm{e}^x + B\mathrm{e}^{2x} - x\mathrm{e}^x,$$

was found in the previous example. At $x = 0$,

$$y(0) = A + B = 1 \quad \text{and} \quad y'(0) = A + 2B - 1 = 0.$$

Thus $A = 1$ and $B = 0$, and the solution to the ODE with the given initial conditions is

$$y = \mathrm{e}^x - x\mathrm{e}^x.$$

The following examples show cases where the RHS is more complicated than the simple functions encountered so far.

Example: Find the general solution of

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - y = \mathrm{e}^{-2x} + 3.$$

Solution The auxiliary equation is $m^2 - 1 = 0$ and the complementary function is

$$y_\mathrm{c}(x) = A\mathrm{e}^{x} + B\mathrm{e}^{-x}.$$

Now we look for the particular integral. The right hand side is a sum of an exponential and a constant, so we should try something of that type: an exponential with the same exponent plus a constant:

$$y_\mathrm{p}(x) = a\mathrm{e}^{-2x} + b.$$

Substituting $y_{\mathrm{p}}(x) = a\mathrm{e}^{-2x} + b$ in the differential equation gives

$$y_{\mathrm{p}}'(x) = -2a\mathrm{e}^{-2x}, \qquad y_{\mathrm{p}}''(x) = 4a\mathrm{e}^{-2x},$$

which we can substitute into the inhomogeneous equation to get

$$y_{\mathrm{p}}''(x) - y_{\mathrm{p}}(x) = 3a\mathrm{e}^{-2x} - b = \mathrm{e}^{-2x} + 3.$$

Thus we have $a = 1/3$ and $b = -3$, so the particular integral is

$$y_{\mathrm{p}}(x) = \frac{1}{3}\mathrm{e}^{-2x} - 3$$

and the general solution is

$$y(x) = A\mathrm{e}^{x} + B\mathrm{e}^{-x} + \frac{1}{3}\mathrm{e}^{-2x} - 3.$$

Example: Find the general solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - y = \mathrm{e}^{-2x}\cos x.$$

Solution The complementary function is the same as in the previous example. For the particular integral, let us try a product of our usual guess for e^{-2x} and our usual guess for $\cos x$:

$$y_{\mathrm{p}}(x) = \mathrm{e}^{-2x}\left(a\cos x + b\sin x\right).$$

Then

$$\begin{aligned} y_{\mathrm{p}}'(x) &= \mathrm{e}^{-2x}\left(-a\sin x + b\cos x\right) - 2\mathrm{e}^{-2x}\left(a\cos x + b\sin x\right) \\ &= \mathrm{e}^{-2x}\left((b-2a)\cos x - (a+2b)\sin x\right) \end{aligned}$$

and

$$\begin{aligned} y_{\mathrm{p}}''(x) &= \mathrm{e}^{-2x}\left(-(b-2a)\sin x - (a+2b)\cos x\right) \\ &\qquad -2\mathrm{e}^{-2x}\left((b-2a)\cos x - (a+2b)\sin x\right) \\ &= \mathrm{e}^{-2x}\left((3a-4b)\cos x + (4a+3b)\sin x\right). \end{aligned}$$

If we substitute this into the inhomogeneous equation, we get

$$\begin{aligned} y_{\mathrm{p}}''(x) - y_{\mathrm{p}}(x) &= \mathrm{e}^{-2x}\left((3a-4b)\cos x + (4a+3b)\sin x\right) \\ &\quad -\mathrm{e}^{-2x}\left(a\cos x + b\sin x\right) \\ &= \mathrm{e}^{-2x}\left((2a-4b)\cos x + (4a+2b)\sin x\right) \\ &= r(x) = \mathrm{e}^{-2x}\cos x. \end{aligned}$$

This works if

$$\begin{aligned} 2a - 4b &= 1 \\ 4a + 2b &= 0, \end{aligned}$$

which can be solved to give $a = 1/10$, $b = -1/5$. Therfore the particular integral is

$$y_{\mathrm{p}}(x) = \frac{1}{10}\mathrm{e}^{-2x}\left(\cos x - 2\sin x\right)$$

and the general solution is

$$y(x) = A\mathrm{e}^{x} + B\mathrm{e}^{-x} + \frac{1}{10}\mathrm{e}^{-2x}\left(\cos x - 2\sin x\right).$$

Example Find the general solution of

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - y = x\mathrm{e}^{2x}.$$

Solution Again, we know the complementary function from the previous example. For the particular integral, try

$$y_{\mathrm{p}}(x) = (ax + b)\mathrm{e}^{2x}.$$

Then

$$\begin{aligned} y_{\mathrm{p}}'(x) &= 2(ax+b)\mathrm{e}^{2x} + a\mathrm{e}^{2x} \\ &= (2ax + a + 2b)\mathrm{e}^{2x} \end{aligned}$$

and

$$\begin{aligned} y_{\mathrm{p}}''(x) &= 2(2ax + a + 2b)\mathrm{e}^{2x} + 2a\mathrm{e}^{2x} \\ &= (4ax + 4a + 4b)\mathrm{e}^{2x}. \end{aligned}$$

Substituting this into the inhomogeneous equation, we get

$$\begin{aligned} {y_\mathrm{p}}''(x) - y_\mathrm{p}(x) &= (4ax + 4a + 4b)\mathrm{e}^{2x} - (ax + b)\mathrm{e}^{2x} \\ &= (3ax + 4a + 3b)\mathrm{e}^{2x} \\ &= r(x) = x\mathrm{e}^{2x}. \end{aligned}$$

Comparing coefficients then reveals that

$$3a = 1 \qquad \Rightarrow \qquad a = \frac{1}{3}$$

and

$$4a + 3b = \frac{4}{3} + 3b = 0 \qquad \Rightarrow \qquad b = -\frac{4}{9},$$

which gives us the particular integral

$$y_\mathrm{p}(x) = \frac{1}{9}(3x - 4)\mathrm{e}^{2x}$$

and the general solution

$$y(x) = A\mathrm{e}^{x} + B\mathrm{e}^{-x} + \frac{1}{9}(3x - 4)\mathrm{e}^{2x}.$$

2.6 Systems of ODEs

We can apply what we have learned about solving second-order equations to solving *systems* of equations.

For example, the system of coupled, first-order equations

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= z\\ \frac{\mathrm{d}z}{\mathrm{d}x} &= x - 2y - 3z\end{aligned}$$

is equivalent to the second order equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x.$$

Proof If starting from the system

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= z \\ \frac{\mathrm{d}z}{\mathrm{d}x} &= x - 2y - 3z,\end{aligned}$$

simply substitute $z = \mathrm{d}y/\mathrm{d}x$ from the first equation into the second to get

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x - 2y - 3\frac{\mathrm{d}y}{\mathrm{d}x},$$

which can be rearranged to give the second order equation on the previous slide.

Alternatively, if starting form the second-order equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x,$$

then *define* $z = \mathrm{d}y/\mathrm{d}x$ and then

$$\frac{\mathrm{d}z}{\mathrm{d}x} + 3z + 2y = x,$$

which can be rearranged to give the system on the previous slide.

Example Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} + u = 3$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} + y = 2.$$

Solution First differentiate the first equation to get

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}u}{\mathrm{d}x} = 0.$$

Then use the second equation to eliminate $\mathrm{d}u/\mathrm{d}x = 2 - y$, giving

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - y + 2 = 0.$$

This is called the *method of elimination*. This second-order equation can now be solved by the methods we already know. The solution is

$$y(x) = A\mathrm{e}^{x} + B\mathrm{e}^{-x} + 2$$

(check this!).

The solution for u can be found directly from the first equation in the original system:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + u = 3 \quad \Rightarrow \quad u(x) = 3 - \frac{\mathrm{d}y}{\mathrm{d}x} = 3 - Ae^{x} + Be^{-x}.$$

The constants A and B can be found from initial or boundary conditions. For example, if the initial conditions

$$y(0) = 1 \qquad \text{and} \qquad u(0) = 0$$

are imposed, then we find

$$\begin{array}{c} A + B + 2 = 1 \\ 3 - A + B = 0 \end{array} \qquad \Rightarrow \qquad \begin{array}{l} A = 1 \\ B = -2. \end{array}$$

The explicit solution is then

$$\begin{aligned} y(x) &= 2 + e^{x} - 2e^{-x} \\ u(x) &= 3 - e^{x} - 2e^{-x}. \end{aligned}$$

Example Solve

$$\begin{aligned}\frac{dx}{dt} + 2x + y &= 0\\ \frac{dy}{dt} - 3x - 2y &= e^{-2t}\end{aligned}$$

subject to the initial conditions $x(0) = 0$ and $y(0) = 0$.

Solution Differentiate the first equation to get

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \frac{dy}{dt} = 0,$$

and then use the second to substitute

$$\frac{dy}{dt} = e^{-2t} + 3x + 2y,$$

giving

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x + 2y + e^{-2t} = 0.$$

This still involves y, so we're not done yet!

Use the first equation to get

$$y = -\frac{\mathrm{d}x}{\mathrm{d}t} - 2x$$

so substitute to give

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - x = -\mathrm{e}^{-2t}.$$

(ii) Solve the auxiliary equation $m^2 - 1 = 0$. This gives $m = \pm 1$, so the complementary function is $x_\mathrm{c}(t) = A\mathrm{e}^{t} + B\mathrm{e}^{-t}$.
For the particular integral, try $x_\mathrm{p}(t) = C\mathrm{e}^{-2t}$, and find $C = -\frac{1}{3}$.
The general solution is

$$x(t) = A\mathrm{e}^{t} + B\mathrm{e}^{-t} - \frac{1}{3}\mathrm{e}^{-2t}$$

and

$$y(t) = -\frac{\mathrm{d}x}{\mathrm{d}t} - 2x = -3A\mathrm{e}^{t} - B\mathrm{e}^{-t}.$$

(iii) Apply initial conditions

$$\begin{aligned} x(0) = 0 &= A + B - \frac{1}{3} \\ y(0) = 0 &= -3A - B \end{aligned}$$

so $A = -1/6$ and $B = 1/2$.

Therefore

$$\begin{aligned} x(t) &= -\frac{1}{6}\mathrm{e}^{t} + \frac{1}{2}\mathrm{e}^{-t} - \frac{1}{3}\mathrm{e}^{-2t} \\ y(t) &= \frac{1}{2}\mathrm{e}^{t} - \frac{1}{2}\mathrm{e}^{-t} \end{aligned}$$