Minilecture 3E Examples of Fourier series

Recall, $2L$-periodic function $f(x+2L) = f(x)$ can be represented

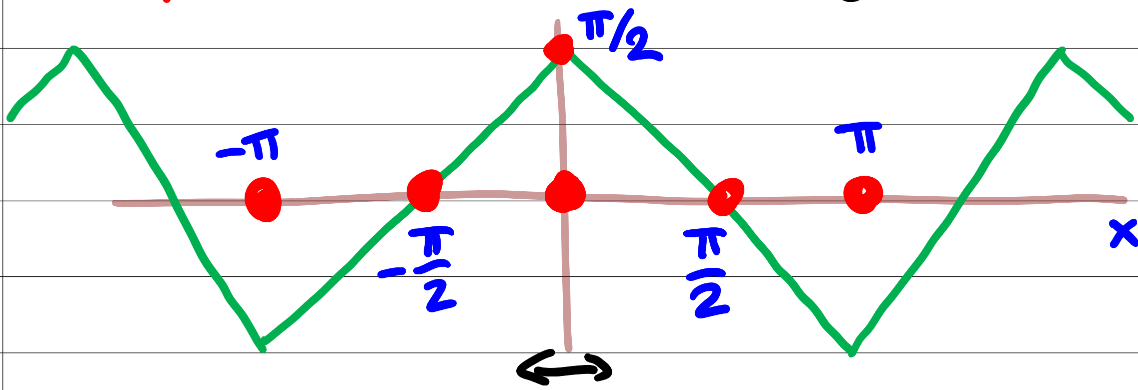
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L} f(x)\,dx \qquad n=0,1,2\ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \sin\frac{n\pi x}{L} f(x)\,dx \qquad n=1,2,3\ldots$$

Example A 2π-periodic triangle wave

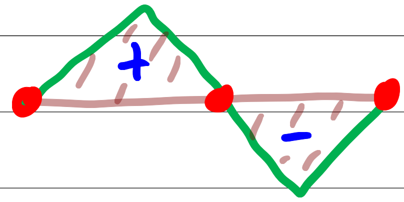


can be defined by the conditions

$$f(x) = \begin{cases} x + \pi/2 & -\pi \le x < 0 \\ \pi/2 - x & 0 \le x < \pi \end{cases} \quad \underline{\text{and}} \quad f(x+2\pi) = f(x)$$

This has period $L = 2\pi$ half-period $L=\pi$ and is an <u>even</u> fn of x.

- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

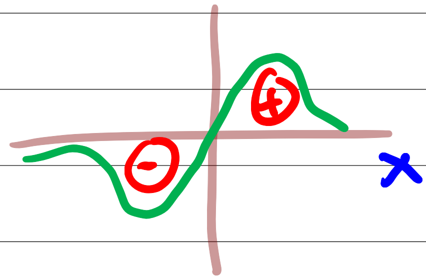


$= 0$ over a full cycle
same area above as below → cancel!

[$\sin\frac{n\pi x}{L} = \sin nx$ if $L=\pi$]

- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx\, f(x)\,dx$

$f(x) \rightarrow$ even fn
$\sin nx \rightarrow$ odd fn
$\sin nx\, f(x) \rightarrow$ odd fn



In general

$$\int_{-a}^{a} (\text{odd function})\,dx = 0$$

$$\Rightarrow b_n = 0$$

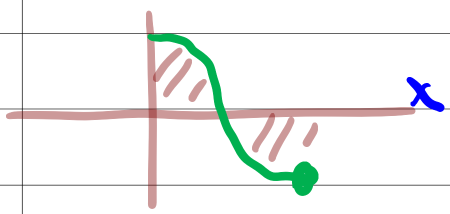
[FS of an even fn involves only constant and cosines → no sines]

- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\cos nx\, f(x)}_{\text{even fn of } x}\,dx$ needs work!

$$= \frac{2}{\pi}\int_{0}^{\pi} \cos nx\, f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^\pi \left(\frac{\pi}{2}-x\right)\cos nx\,dx$$

$$= \underbrace{\int_0^\pi \cos nx\,dx}_{\text{vanishes because}} - \frac{2}{\pi}\int_0^\pi x\cos nx\,dx$$



Alternatively

$$\int_0^\pi \cos nx\,dx = \left[\frac{1}{n}\sin nx\right]_0^\pi$$

$$= \frac{1}{n}\sin n\pi - 0 = 0$$

integration by parts

Then

$$a_n = -\frac{2}{\pi}\int_0^\pi x\cos nx\,dx$$

$$= -\frac{2}{n\pi}\int_0^\pi x\,\frac{d}{dx}(\sin nx)\,dx$$

$$= -\frac{2}{n\pi}\cancel{\left[x\sin nx\right]_0^\pi} + \frac{2}{n\pi}\int_0^\pi \sin nx\,dx$$

0

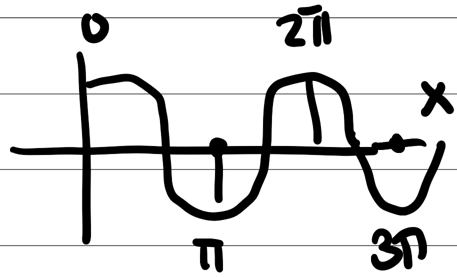
$\sin n\pi = 0$

$$= -\frac{2}{n^2\pi}\left[\cos nx\right]_0^\pi$$

$$= \frac{2}{n^2\pi}\left[1-\cos n\pi\right]$$

$$= \frac{2}{n^2\pi}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0 & n \text{ even} \\ 4/n^2\pi & n \text{ odd} \end{cases}$$



$$f(x) = \sum_{\text{odd } n} \frac{4}{n^2\pi} \cos nx$$

$$= \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots \right)$$

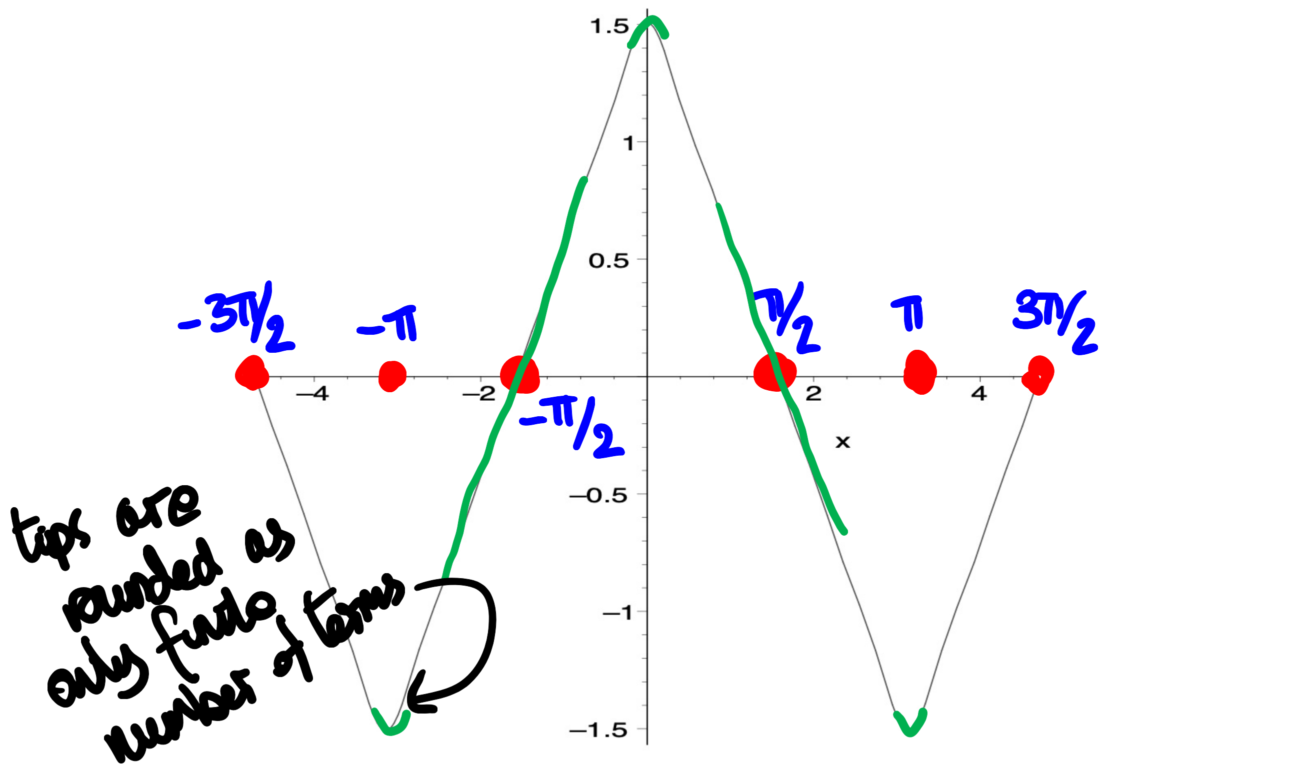
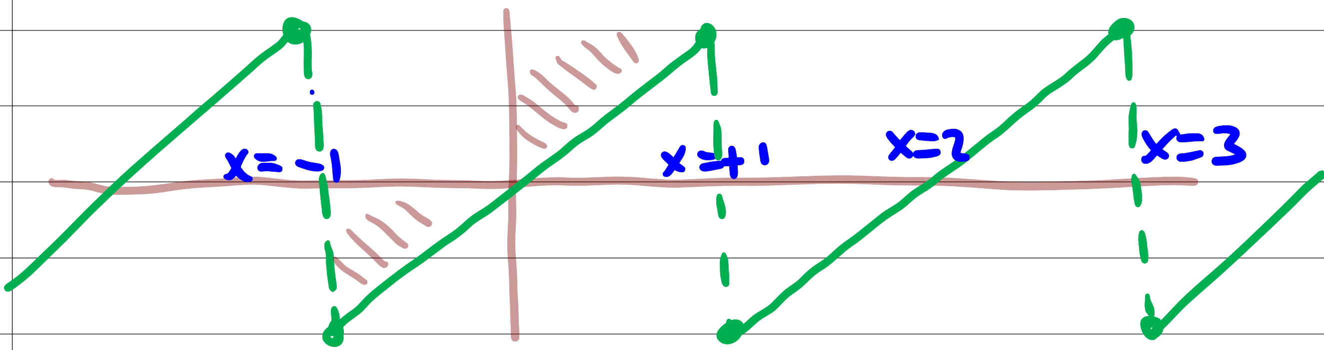


Figure 1: The first 10 terms in the Fourier series of f.

Example $g(x)$ is the sawtooth function defined by the conditions

$$g(x) = x, \quad -1 \leq x < 1, \quad g(x+2) = g(x)$$

Here $2L = 2$, $L = 1$



This is an odd function of x - FS contains only sines!

$$a_0 = \int_{-1}^{1} g(x)\,dx = 0$$

$\left[\cos\frac{n\pi x}{L},\ L = 1 = \cos n\pi x\right]$

$$a_n = \int_{-1}^{1} \underbrace{\cos n\pi x\, g(x)}_{\text{odd}}\,dx = 0$$

Without doing further calculation we know

$$g(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \int_{-1}^{+1} x \sin n\pi x \, dx$$

$$= \int_{-1}^{1} x \frac{d}{dx}\left(-\frac{1}{n\pi}\cos n\pi x\right) dx$$

$$= \left[-\frac{1}{n\pi} x \cos n\pi x\right]_{-1}^{1} + \cancel{\frac{1}{n\pi}\int_{-1}^{+1} \cos n\pi x \, dx}$$

0

complete cycle of trig

$$= -\frac{2}{n\pi}\cos n\pi$$

$$= -\frac{2}{n\pi}(-1)^n$$

$$g(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin n\pi x$$

$$= \frac{2}{\pi}\left(\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \dots\right)$$

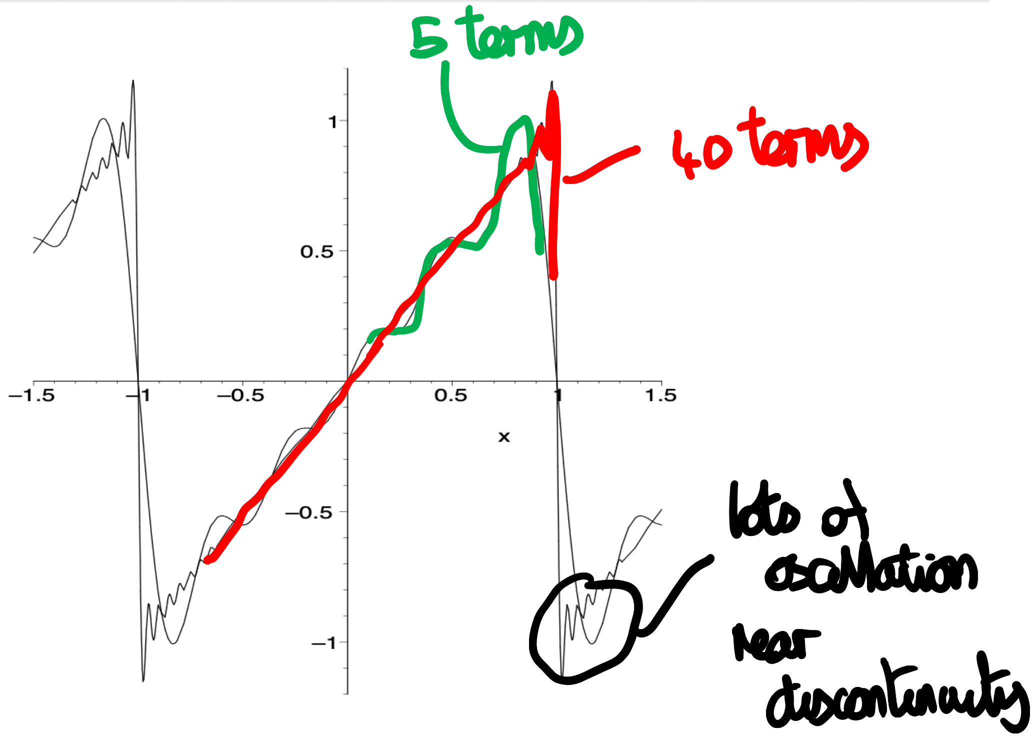


Figure 2: The first 5 terms and the first 40 terms in the Fourier series of g.

"Gibb's phenomenon"