Solutions for Exercise Sheet 4: VIBRATION

Free Vibration

① Checking the damping ratio $\xi = \frac{c}{2m\omega_n}$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{500}{2}} = 15.81 \text{ rad/s}$$

$$\xi = \frac{2}{2 \times 2 \times 15.81} = 0.031 \quad (\text{light damping})$$

Damped free vibration: $x(t) = C.e^{-\xi\omega_n t}\sin(\omega_d t + \phi)$

Initial conditions: i) $x(0) = 0 = C.\sin\phi \quad \rightarrow \phi = 0$ since $C \neq 0$.

ii) $\dot{x}(0) = v$

$$\dot{x}(t) = -\xi\omega_n C.e^{-\xi\omega_n t}\sin(\omega_d t) + \omega_d C.e^{-\xi\omega_n t}\cos(\omega_d t)$$

$$\dot{x}(0) = v = \omega_d . C \quad \rightarrow C = \frac{v}{\omega_d}$$

Thus, $x(t) = \frac{v}{\omega_d}\cdot e^{-\xi\omega_n t}\sin(\omega_d t)$ where $\omega_d = \omega_n\sqrt{1-\xi^2}$

② The equivalent spring stiffness $k_e = k + k = 500$ N/m.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{500}{2}} = 15.81 \text{ rad/s}$$

$$\xi \text{ (damping ratio)} = \frac{63.24}{2 \times 2 \times 15.81} \approx 1 \quad (\text{critically damped})$$

$$x(t) = (C + Dt)e^{-\omega_n t}$$

Initial conditions;

i) $x(0) = C = 0.02$

ii) $\dot{x}(t) = D.e^{-\omega_n t} + (C + Dt)e^{-\omega_n t} \times (-\omega_n)$

$\Leftrightarrow \dot{x}(0) = D + C\times(-\omega_n) = 0$

$D = C\cdot\omega_n = 0.02\times 15.81 = 0.316$.

So $x(t) = (0.02 + 0.316t)\,e^{-15.81t}$ [m]

$\dot{x}(t) = 0.316e^{-15.81t} - (0.316 + 4.995t)e^{-15.81t}$ [m/s].

$\Leftrightarrow \dot{x}(t) = -4.995t\cdot e^{-15.81t}$ [m/s]

③ To ensure the system is lightly damped:

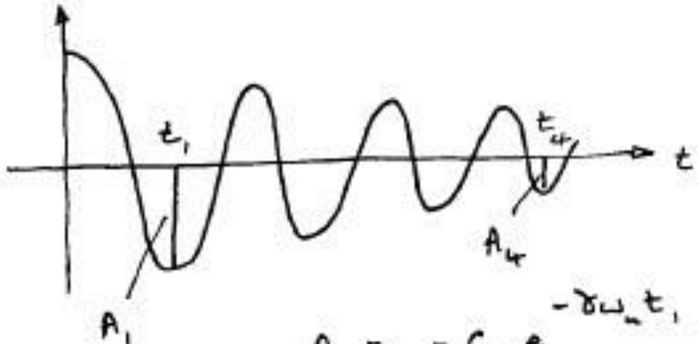
$\zeta < 1$

$\Leftrightarrow \frac{c}{2\sqrt{km}} < 1$ where $c = 63.24$ Ns/m

$\Leftrightarrow c^2 < 4km \longrightarrow km > \frac{c^2}{4}$ or $km > 9.998\times10^2 \left[\frac{N\,kg}{m}\right]$

Since the damper is not changed, either k or m can be increased to satisfy the above criterion.

4.



$$A_1 = -C_0 e^{-\gamma\omega_n t_1} \qquad A_2 = -C_0 e^{-\gamma\omega_n t_4}$$

$$\frac{A_1}{A_4} = 4 = e^{\gamma\omega_n(t_4 - t_1)} \qquad (1)$$

But $t_4 - t_1 = 3T_n$ (where T_n = period of free vibration)

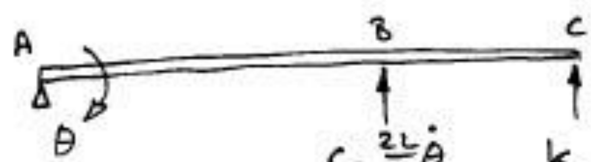
If $\gamma << 1$, $T_n \simeq \frac{2\pi}{\omega_n} = 0.281\,s$

From (1) $\quad \log_e 4 = \gamma\omega_n . 3T$

$\therefore \gamma = 0.0735$ (which is $<< 1$)

Damping coefficient, $c = \gamma . c_{CRIT} = \gamma . 2\sqrt{km}$

5.



$AC = L$

$$A \circlearrowright \ddot{\theta} \qquad -\left(c\frac{2L}{3}\dot{\theta}\right).\frac{2L}{3} - (kL\theta).L = I_A\ddot{\theta}$$

or $\quad I_A\ddot{\theta} + c\frac{4L^2}{9}\dot{\theta} + kL^2\theta = 0$

or $\quad 10\ddot{\theta} + 200\dot{\theta} + 56{,}250\,\theta = 0$

c.f. $\quad M\ddot{z} + C\dot{z} + Kz = 0$

$$\omega_n = \sqrt{\frac{K}{M}} = 75 \text{ rad/s} \qquad \gamma = \frac{C}{2\sqrt{KM}} = 0.133$$

Damped natural frequency, $\Omega_d = \omega_n\sqrt{1-\gamma^2} = 74.3$ rad/s

General solution is $\theta(t) = e^{-\gamma\omega_n t}\left(A\cos\Omega_d t + B\sin\Omega_d t\right)$

Initial conditions: $\theta = \Theta_0$, $\dot{\theta} = 0$ at $t = 0$

Hence $A = \Theta_0$ and $B = \dfrac{\gamma\omega_n}{\Omega_d}\Theta_0$

Here, $\Theta_0 = \dfrac{0.01}{1.5} = 6.67$ mrad

Period of <u>damped</u> vibration $= \dfrac{2\pi}{\Omega_d} = 0.0845$ s

Maximum <u>upward</u> displacement occurs $\frac{1}{2}$ cycle after the start — that is, after 0.423 s

Hence, $\theta = -4.369$ m rad and tip displacement = 6.55mm

6. This is an old examination question.

The equation of motion is

$$\left(m + \frac{I}{r^2}\right)\ddot{x} + 2c\dot{x} + \left(2k + \frac{K}{r^2}\right)x = 0$$

where m = mass of rack
I = moment of inertia of pinion
c = damping coefficient between rack and ground
k = spring stiffness " " " "
K = torsional stiffness of pinion shaft
x = rack displacement.

This gives $\quad 2.5\ddot{x} + 100\dot{x} + 58{,}000\,x = 0$

* If you have difficulty, go to the next Subject Tutorial

General solution is

$$x(t) = e^{-\gamma \omega_n t}\left(A \cos \Omega_d t + B \sin \Omega_d t\right) \quad (1)$$

Initial conditions: $x = 0$, $\dot{x} = 1$ m/s at $t = 0$

Hence, $A = 0$ and $B = 6.62$ mm

The maximum displacement occurs approximately $\frac{1}{4}$ cycle after the start

i.e., at $t = \frac{1}{4} \cdot \frac{2\pi}{\Omega_d} = 0.01040$ s

This gives $x_{MAX} = 5.38$ mm

A more accurate value can be found by differentiating (1) to get $\dot{x}$ and then equating to zero.

This gives $t = 0.00953$ s and $x_{MAX} = 5.43$ mm