MTHS2007 2021/22 Exam solutions

Q1(b) Auxiliary equation

$$0 = m^2 - 9m + 20$$
$$= (m-4)(m-5)$$
$$\Rightarrow m = 4, 5$$

so the complementary function is

$$y_c(x) = Ae^{4x} + Be^{5x}.$$

For P.I. try

$$y_p(x) = ae^{3x}$$

$$\Rightarrow \quad y_p'' - 9y_p' + 20y_p = (9a - 27a + 20a)e^{3x}$$
$$= 2ae^{3x}$$
$$= 2e^{3x} \text{ if } a = 1$$

Then $y_p(x) = e^{3x}$ and the general solution is

$$y(x) = e^{3x} + Ae^{4x} + Be^{5x}$$

IC's $\quad y(0) = 1 + A + B = 3 \quad \Rightarrow A = B = 1$

$\quad\quad y'(0) = 3 + 4A + 5B = 12$

$$\Rightarrow y(x) = e^{3x} + e^{4x} + e^{5x}$$

(b) Differentiate the first equation w.r.t. x :

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} + \frac{dy}{dt}$$

$$= 2\frac{dx}{dt} - 2x + \sin t$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = \sin t$$

Auxiliary equation: $0 = m^2 - 2m + 2$

$$= (m-1)^2 + 1$$

$$\Rightarrow m = 1 \pm i$$

$$\Rightarrow x_c(t) = e^t(C\cos t + D\sin t)$$

For P.I. try $x_p(t) = a\cos t + b\sin t$

$$x_p'(t) = -a\sin t + b\cos t$$

$$x_p''(t) = -a\cos t - b\sin t$$

$$\Rightarrow x_p'' - 2x_p' + 2x_p = -a\cos t - b\sin t$$
$$-2(-a\sin t + b\cos t)$$
$$+2(a\cos t + b\sin t)$$
$$= (a-2b)\cos t + (b+2a)\sin t$$

$\Rightarrow$ $a - 2b = 0$ $\quad \Rightarrow a = 2b,\ 5b = 1,\ b = \frac{1}{5},\ a = \frac{2}{5}$

$b + 2a = 1$

$$\Rightarrow x(t) = e^t(C\cos t + D\sin t) + \frac{1}{5}(2\cos t + \sin t)$$

Then

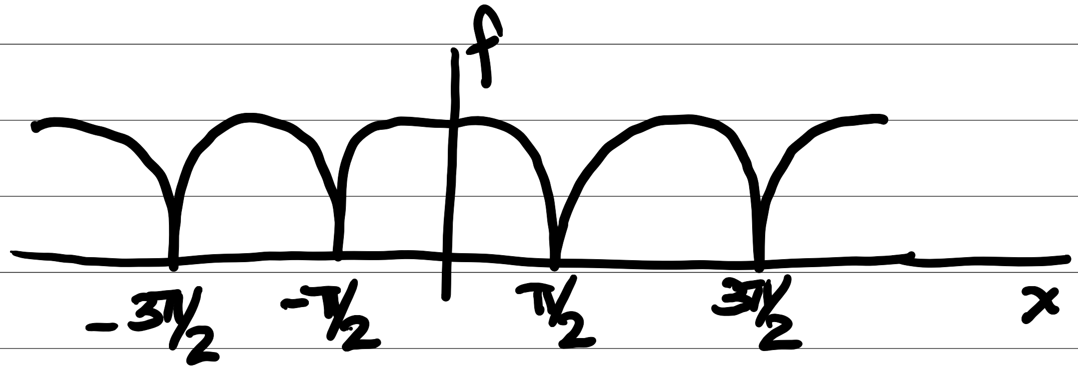
$$y = \frac{dx}{dt} - 2x$$

$$= e^t\big((C+D)\cos t + (D-C)\sin t\big) + \frac{1}{5}(-2\sin t + \cos t)$$

$$+ e^t(-2C\cos t - 2D\sin t) + \frac{1}{5}(-4\cos t - 2\sin t)$$

$$= e^t\big((D-C)\cos t - (C+D)\sin t\big) - \frac{1}{5}(3\cos t + 4\sin t)$$

Q2 (a)



(b) The shortest period is $L = \pi$.

(c) Since f is even and the half period is $L = \pi/2$, F.S. is of the form

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos 2nx$$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos 2nx \cos x \, dx$$

$$= \frac{4}{\pi}\int_{0}^{\pi/2} \cos 2nx \cos x \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2} \left[\cos((2n-1)x) + \cos((2n+1)x)\right] dx$$

$$= \frac{2}{\pi}\left[\frac{\sin((2n-1)x)}{2n-1} + \frac{\sin((2n+1)x)}{2n+1}\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left(\frac{\sin(2n-1)\frac{\pi}{2}}{2n-1} + \frac{\sin(2n+1)\frac{\pi}{2}}{2n+1}\right)$$

$$= \frac{2}{\pi}\left(\frac{(-1)^{n-1}}{2n-1} + \frac{(-1)^n}{2n+1}\right)$$

$$= (-1)^{n-1}\frac{2}{\pi}\,\frac{2}{(2n-1)(2n+1)}$$

Special case $a_0 = \frac{4}{\pi}$ and then

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos 2nx}{(2n-1)(2n+1)}$$

(d) $f(x)$ is continuous everywhere so

(i) F.S. $\to f(0) = 1$

(ii) F.S. $\to f(\pi/2) = 0$.

(e) Evaluate F.S. at $x=0$:

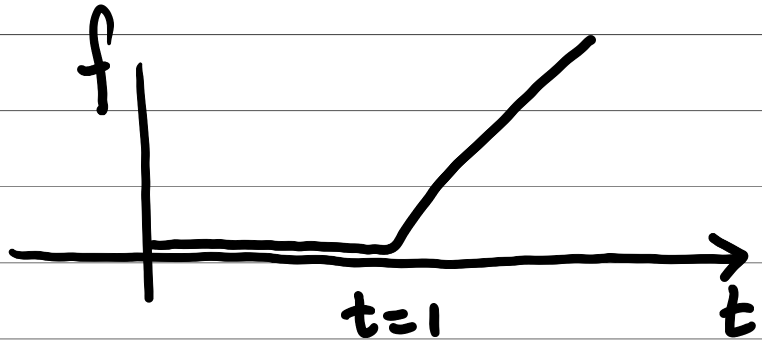
$$f(0) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)(2n+1)}$$

$$= \frac{2}{\pi} + \frac{4}{\pi} S$$

$$= 1$$

$$\Rightarrow \quad S = \frac{\pi}{4} - \frac{1}{2}$$

Q3 (a)



(b) Using the second shifting theorem

$$\mathcal{L}\big((t-1)H(t-1)\big) = e^{-s}\mathcal{L}(t)$$
$$= e^{-s}/s^2$$

(c) $y'' - y' = f \xrightarrow{\text{L.T.}} s^2\bar{y} - sy(0) - y'(0) - s\bar{y} + y(0)$
$$= \bar{f}(s)$$

$$\Rightarrow (s^2 - s)\bar{y}(s) - s + 1 = e^{-s}/s^2$$

$$\Rightarrow \bar{y}(s) = \frac{s-1}{s^2-s} - \frac{e^{-s}}{s^2(s^2-s)}$$
$$= \frac{1}{s} - \frac{e^{-s}}{s^3(s-1)}$$

(d) Use partial fractions to expand

$$\frac{1}{s^3(s-1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s-1}$$

$$= \frac{A(s-1) + Bs(s-1) + Cs^2(s-1) + Ds^3}{s^3(s-1)}$$

$$\Rightarrow A(s-1) + Bs(s-1) + Cs^2(s-1) + Ds^3 = 1$$

$s=1 \Rightarrow D=1$

$s=0 \Rightarrow A=-1$

$$\left.\begin{array}{l} s=-1 \Rightarrow -2A+2B-2C-D=1 \\ s=2 \Rightarrow A+2B+4C+8D=1 \end{array}\right\} \begin{array}{l} 2B-2C=0 \\ 2B+4C=-6 \end{array}$$

$$\Rightarrow B=C=-1$$

Then $\frac{1}{s^3(s-1)} = -\frac{1}{s^3} - \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}$ and

$$\mathcal{L}^{-1}\left(\frac{1}{s^3(s-1)}\right) = -\frac{1}{2}t^2 - t - 1 + e^t$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{e^{-s}}{s^3(s-1)}\right)$$

$$= 1 - H(t-1)\left(e^{t-1} - 1 - (t-1) - \frac{1}{2}(t-1)^2\right)$$

$$= \begin{cases} 1 & t<1 \\ e^{t-1} - (t-1) - \frac{1}{2}(t-1)^2 & t>1 \end{cases}$$

Q4 (a) Substituting $\varphi = X(x)T(t)$ gives

$$X''T - XT'' = XT$$

$$\Rightarrow \frac{X''}{X} = 1 + \frac{T''}{T} = -\lambda = \text{const}$$ by usual SoV argument

Then $X'' + \lambda X = 0$

$$T'' + (1+\lambda)T = 0$$

which is in the form requested with $\lambda' = 1+\lambda$.

(b) If $\lambda > 0$ then $\lambda' = \lambda + 1 > 0$ and general solutions of these equations are

$$X(x) = A\cos kx + B\sin kx \quad k = \sqrt{\lambda}$$

$$T(t) = C\cos\omega t + D\sin\omega t \quad \omega = \sqrt{\lambda'} = \sqrt{\lambda+1} = \sqrt{k^2+1}$$

(c) Given boundary conditions are satisfied if

$$X(0) = 0 = X(L)$$

$X(0) = A = 0 \Rightarrow X(x) = B\sin kx$, then

$X(L) = B\sin kL = 0 \Rightarrow kL = n\pi$, $n = 1, 2, 3 \ldots$
for nontrivial solution with $B \neq 0$.

Then SoV solutions are, setting $B=1$, WLOG:

$$X(x)T(t) = (C\cos\omega_n t + D\sin\omega_n t)\sin\frac{n\pi x}{L}$$

where $\omega_n = \sqrt{1 + (n\pi/L)^2}$, and general soln is

$$\varphi(x,t) = \sum_{n=1}^{\infty} (C_n \cos\omega_n t + D_n \sin\omega_n t)\sin\frac{n\pi x}{L}$$

(d) Impose initial conditions

$$\varphi(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} = 0 \Rightarrow C_n = 0$$

$$\varphi_t(x,0) = \sum_{n=1}^{\infty} \omega_n D_n \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \sin\frac{n\pi x}{L}$$

$$\Rightarrow D_n = \frac{1}{\omega_n}(-1)^n/n^4$$

$$\Rightarrow \varphi(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4\omega_n} \sin\omega_n t \sin\frac{n\pi x}{L}$$

where $\omega_n = \sqrt{1 + n^2\pi^2/L^2}$.

Q5 (a) (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow 0.84 = 0.32 + 0.67 - P(A \cap B)$

$\Rightarrow P(A \cap B) = 0.32 + 0.67 - 0.84 = 0.15$

(ii) $P\left((A \cup B)^c\right) = 1 - P(A \cup B)$

$= 1 - 0.84$

$= 0.16$

(iii) $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

$= \dfrac{0.15}{0.32}$ (from (i))

$= 0.469..$

(b) Use $P_o(\lambda)$ with $\lambda = 3$

(i) $P(X=0) = \dfrac{e^{-\lambda}\lambda^0}{0!} = e^{-3} \simeq 0.0498$

(ii) $P(X \geq 2) = 1 - P(X=0) - P(X=1)$

$= 1 - e^{-\lambda}(1+\lambda)$

$= 1 - e^{-3}(4)$

$\simeq 0.801$

(iii) Here use $P_o(\lambda)$ with $\lambda = 3+3 = 6$

$P(X=0) = e^{-6} \simeq 0.00248$

(c) (i) The height is governed by normal distribution

$$N(20+1.5+20+1.5+20,\ 1^2+0.3^2+1^2+0.3^2+1^2)$$

$$= N(63,\ 3.18)$$

(ii) Let $Z = \frac{X-\mu}{\sigma}$ with $\mu = 63,\ \sigma^2 = 3.18$

$$P(X>65) = P\left(Z > \frac{65-63}{\sqrt{3.18}}\right)$$
$$= P(Z > 1.1215)$$
$$= 1 - F(1.1215)$$
$$\simeq 1 - 0.869 = 0.131$$

(d) Let $Z = \frac{\bar{X}-\mu}{s/\sqrt{100}} = \frac{\bar{X}-621.54}{\sqrt{1.41}/10}$

Then the 95% confidence interval is

$$-1.960 < Z < 1.960 = F(0.975)$$

$$\Rightarrow 621.54 - 1.960\frac{\sqrt{1.41}}{10} < \mu < 621.54 + 1.960\frac{\sqrt{1.41}}{10}$$

$$621.54 - 0.23 < \mu < 621.54 + 0.23$$

$$621.31 < \mu < 621.77$$