University of Nottingham
UK | CHINA | MALAYSIA

LECTURE 3A

Circuits Revision & AC Power

Electromechanical Devices
MMME2051

Module Convenor – Surojit Sen

Learning Outcomes

- Revision of circuits
  - **Sinusoidal** waveform – phase **angle** v **time**?
  - **Phasors**
  - Series/Parallel **clarification**
  - **Impedance**
  - **General Rule to solve circuits**
  - **Potential Divider Rule**
- Power in AC circuits
  - **Active** v **Reactive** v **Apparent** Power
  - **Power Factor**
  - **Resonance**

Sinusoid

Sinusoid is a mathematical curve defined in terms of the **sine trigonometric function**

Sine and **Cosine** are both examples of sinusoid

Cosine function is simply the Sine function, but **90° advanced**

We will use the **Cosine function to represent variables**

$$y(t) = A\cos(\omega t + \phi)$$

Frequency

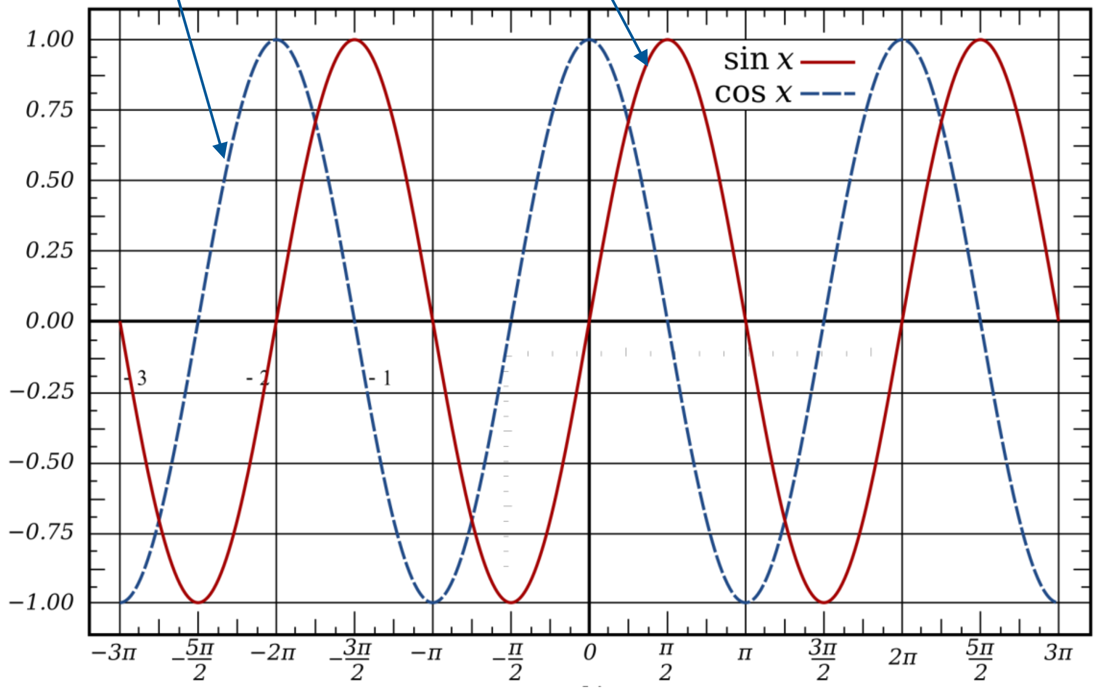
Phase Angle

Variable as function of time

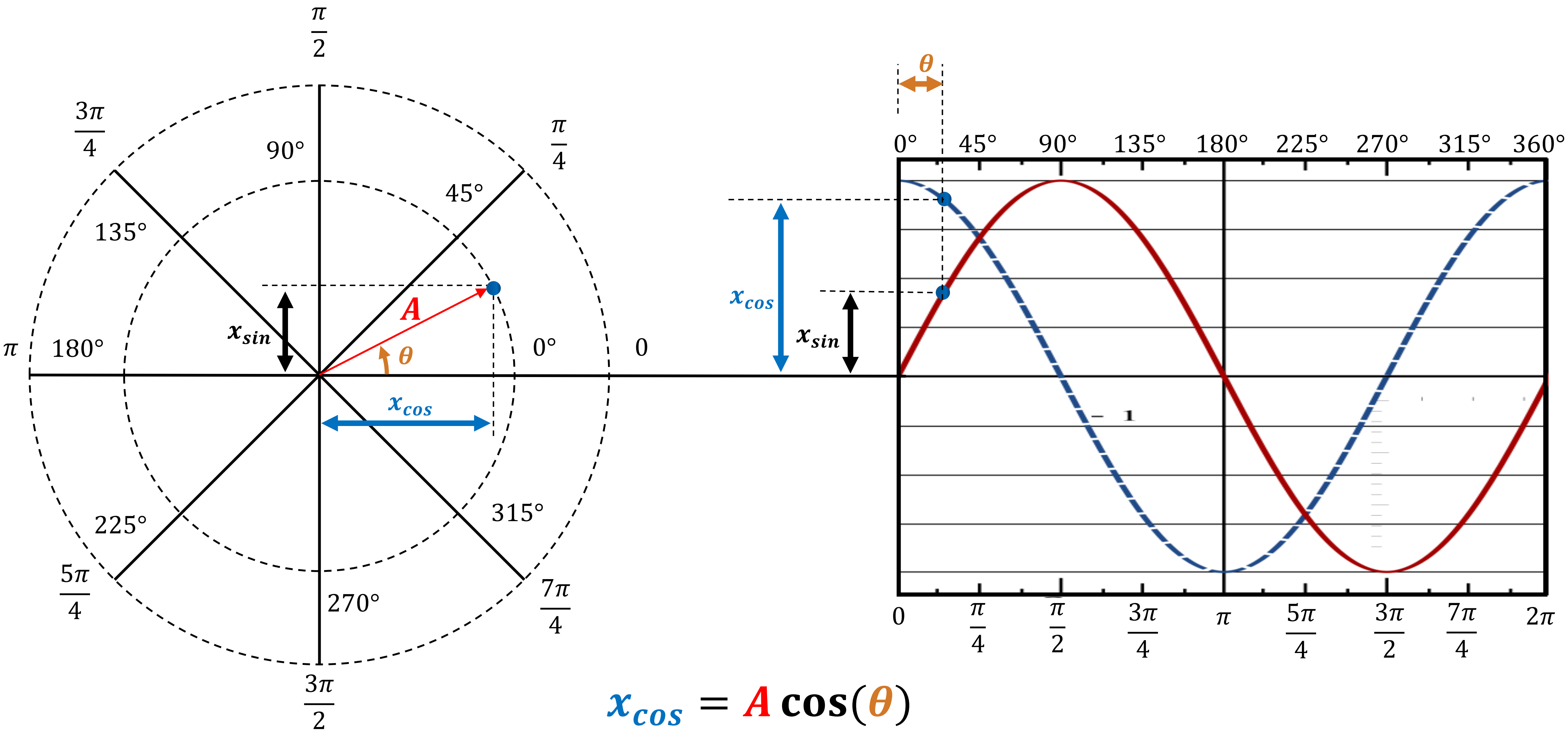
Amplitude

Phase offset

$x_1(t) = \cos t$ $x_2(t) = \sin t$



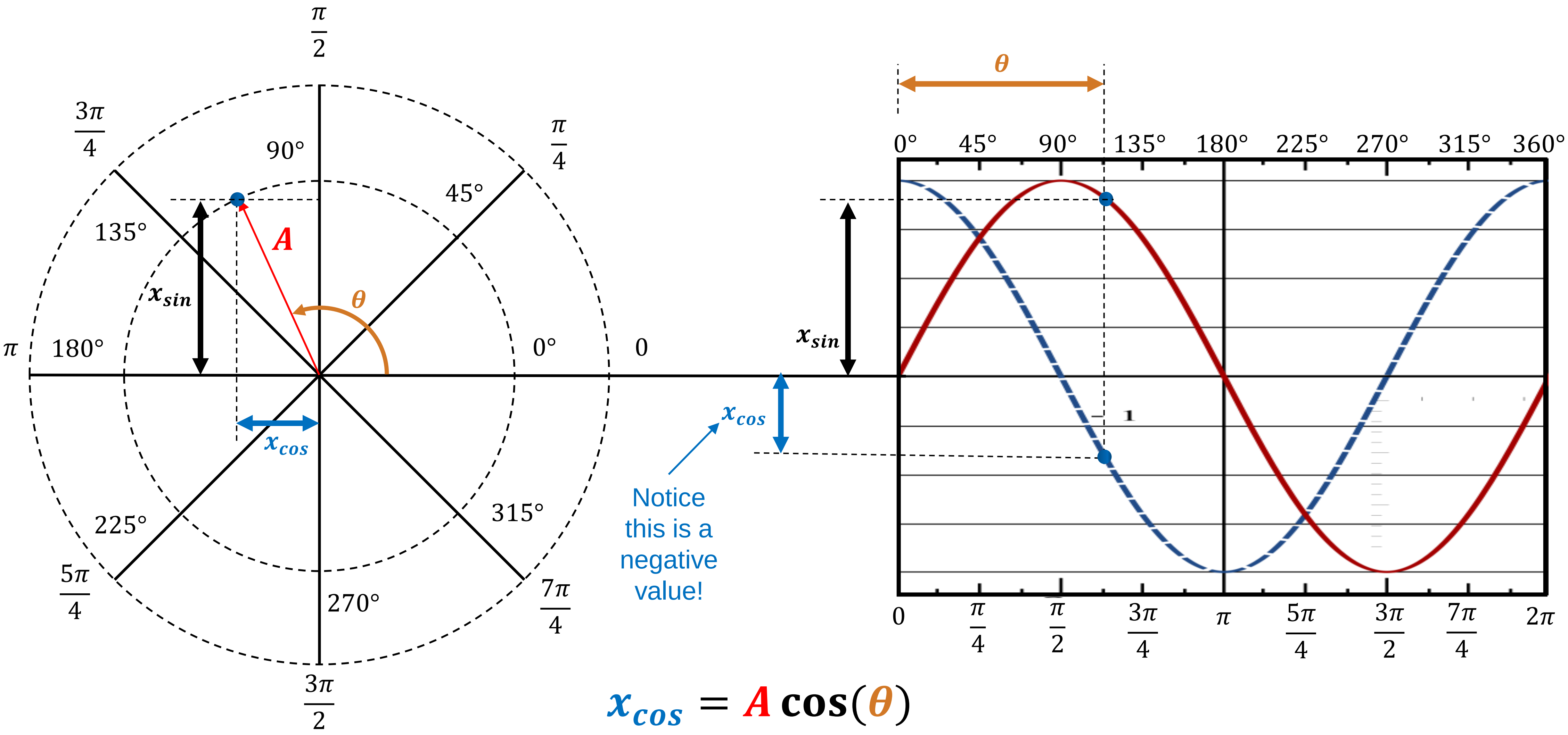
Sinusoid – Phase Angle



$$x_{cos} = A\cos(\theta)$$

$$x_{sin} = A\sin(\theta)$$

Sinusoid – Phase Angle

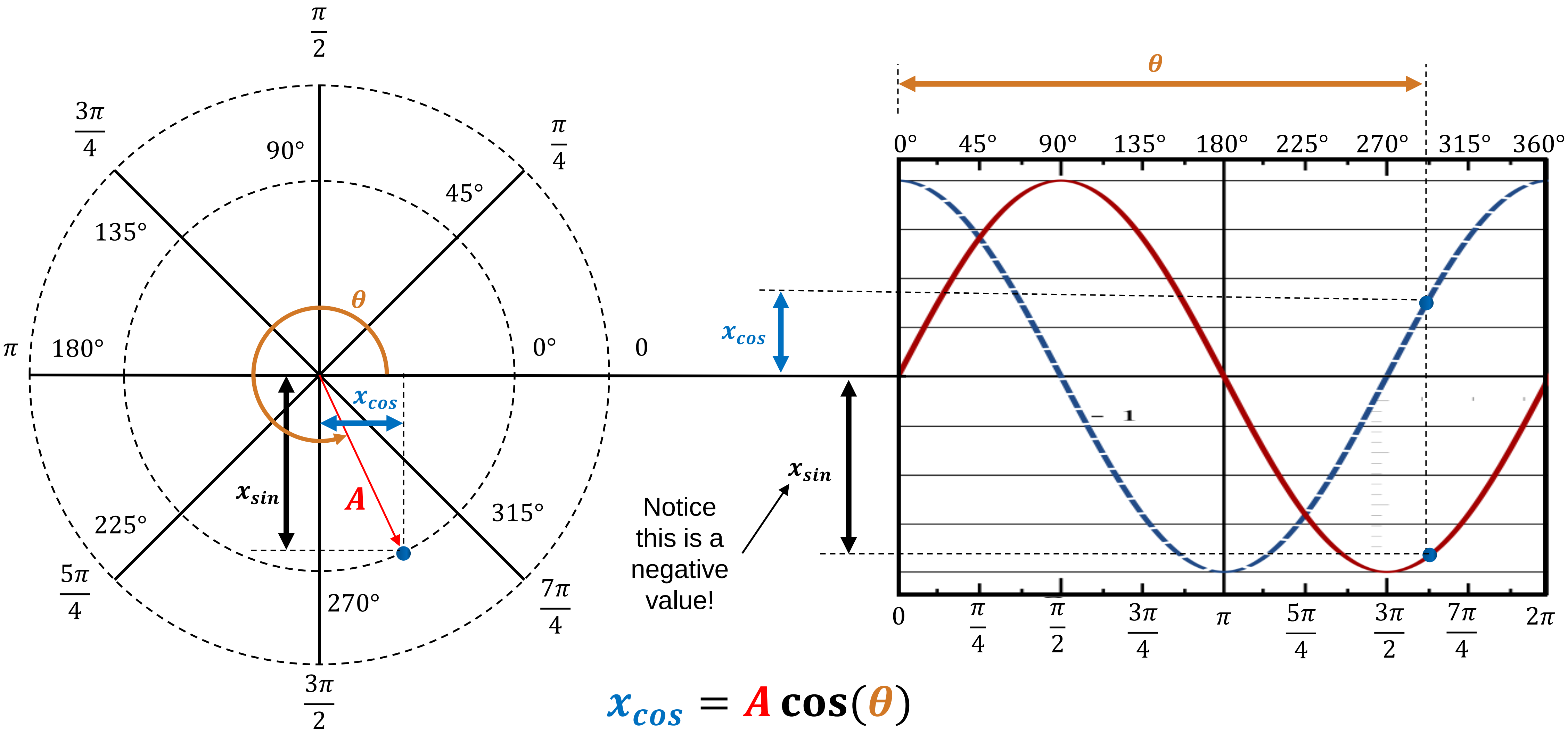


Notice this is a negative value!

$$x_{cos} = A\cos(\theta)$$

$$x_{sin} = A\sin(\theta)$$

Sinusoid – Phase Angle



Notice this is a negative value!

$$x_{cos} = A\cos(\theta)$$

$$x_{sin} = A\sin(\theta)$$

Sinusoid – Phase Angle v Time?

Why are we denoting points on the x-axis with angle values?

Shouldn't x-axis have values in seconds?

$$y(t) = A\cos(\boldsymbol{\omega t} + \boldsymbol{\phi})$$

$$y(t) = A\cos\boldsymbol{\theta}$$

$$\boldsymbol{\omega t} + \boldsymbol{\phi} = \boldsymbol{\theta}$$

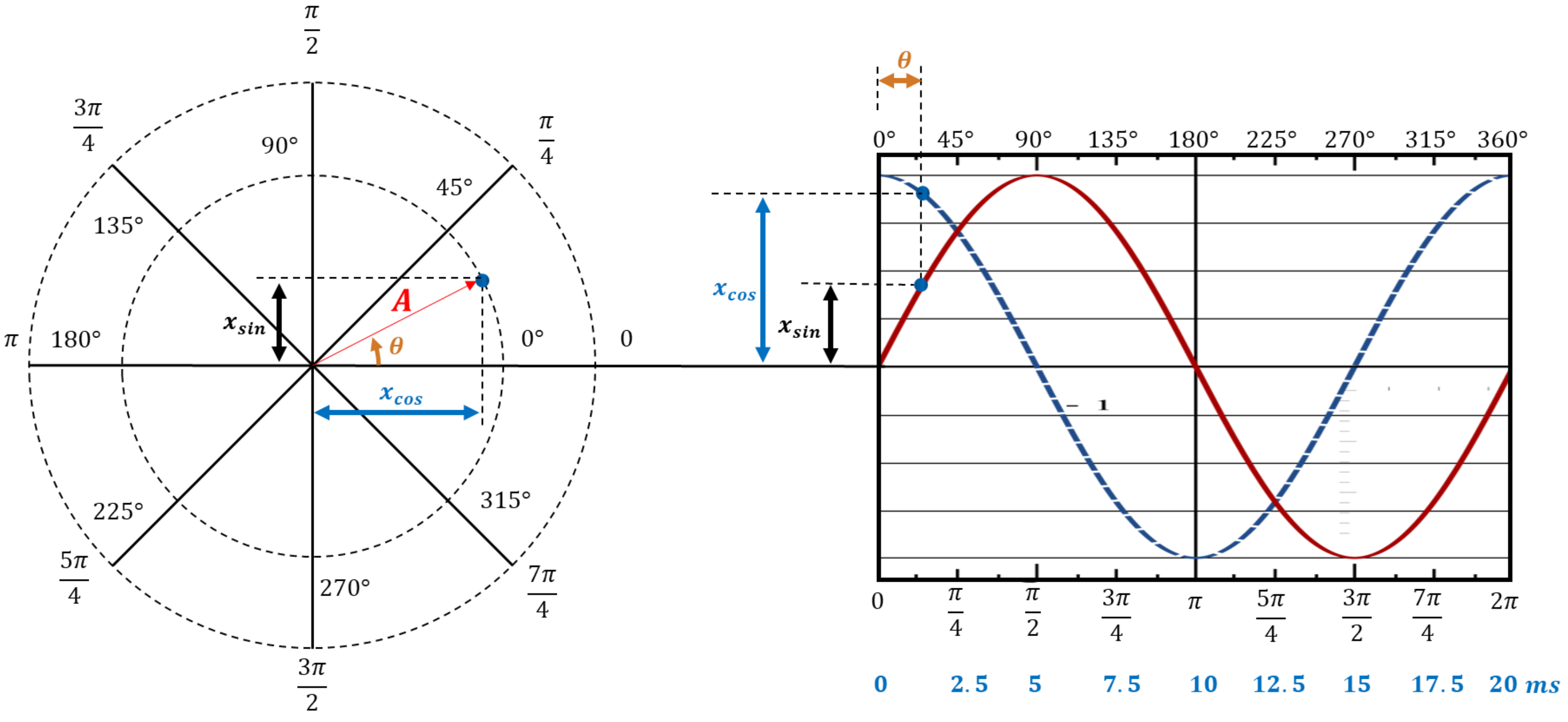
So let us say we are talking about a $50\ Hz$ signal with $0°$ phase offset

$$\boldsymbol{2\pi f t + 0 = \theta}$$

$$\boldsymbol{t = \frac{\theta}{2\pi \times 50} = \frac{\theta}{100\pi}}$$

$$t = 0 \text{ when } \theta = 0$$

$$t = 20ms \text{ when } \theta = 2\pi$$



Sinusoid – Phase Angle v Time?

Let us try another example

$$y(t) = A\cos(\boldsymbol{\omega t} + \boldsymbol{\phi})$$

$$y(t) = A\cos\boldsymbol{\theta}$$

$$\boldsymbol{\omega t + \phi = \theta}$$

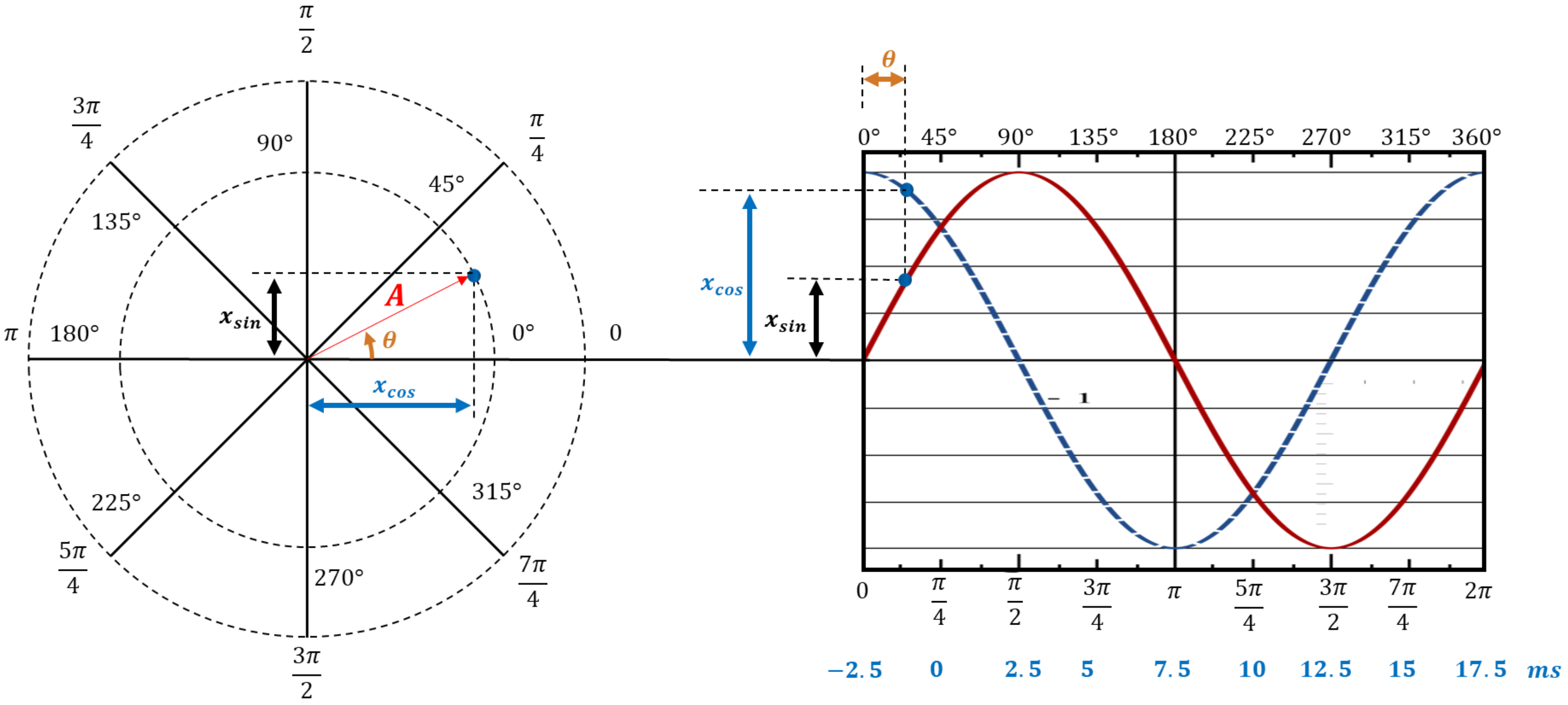
So let us say we are talking about a $50\ Hz$ signal with $\frac{\pi}{4}$ phase offset

$$\boldsymbol{2\pi f t + \frac{\pi}{4} = \theta}$$

$$\boldsymbol{t = \frac{\theta - \frac{\pi}{4}}{2\pi \times 50} = \frac{\theta - \frac{\pi}{4}}{100\pi}}$$

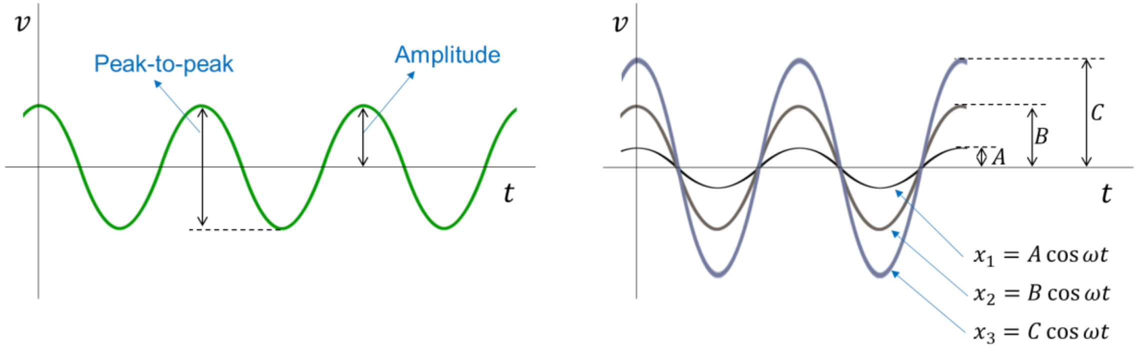
$$t = -2.5ms \text{ when } \theta = 0$$

$$t = 17.5ms \text{ when } \theta = 2\pi$$

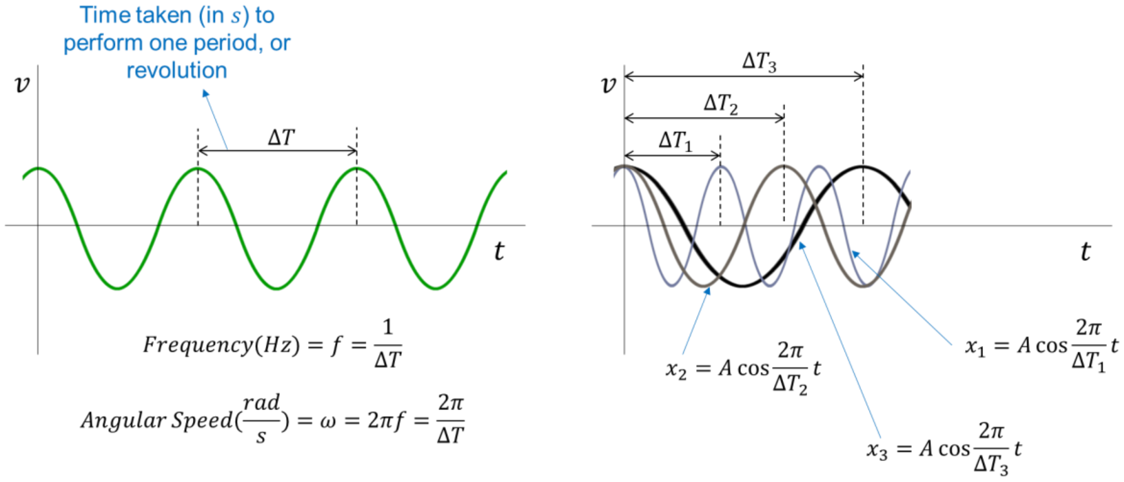


Sinusoid – Amplitude/Frequency/Phase Offset

Maximum magnitude of the variable



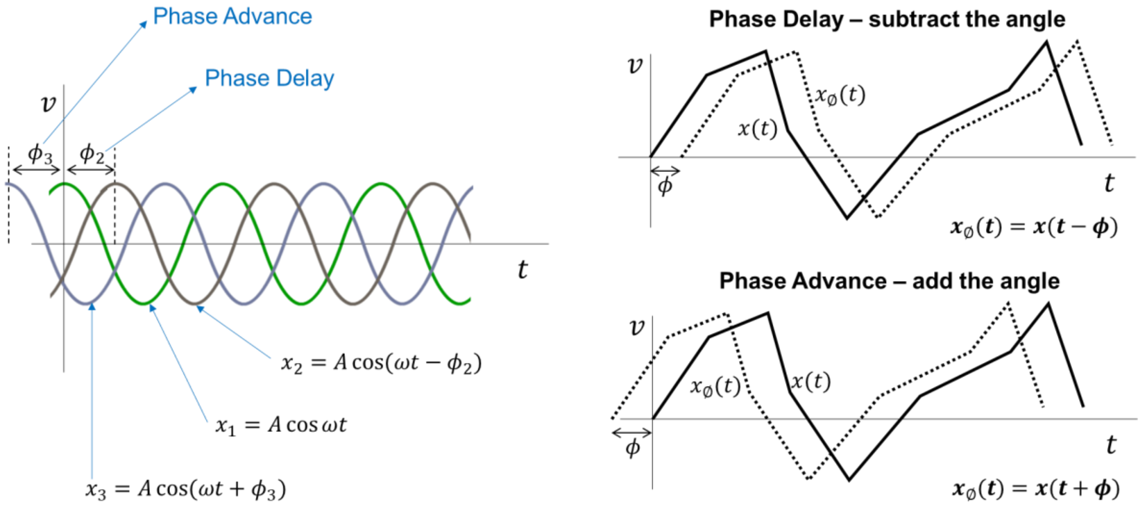
Indicates how fast is the variable changing



$$Frequency(Hz) = f = \frac{1}{\Delta T}$$

$$Angular\ Speed(\frac{rad}{s}) = \omega = 2\pi f = \frac{2\pi}{\Delta T}$$

Phase angle at $t = 0$

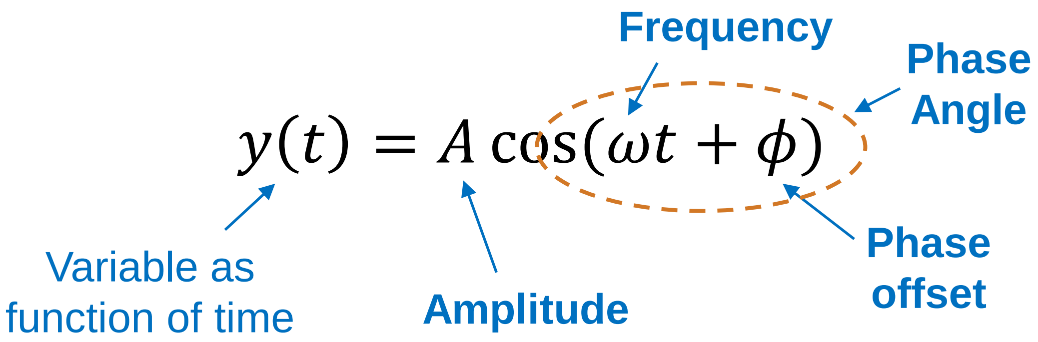


Phase Delay – subtract the angle

$$\boldsymbol{x_\emptyset(t) = x(t - \phi)}$$

Phase Advance – add the angle

$$\boldsymbol{x_\emptyset(t) = x(t + \phi)}$$



$$y(t) = A\cos(\omega t + \phi)$$

Complex Number

Cartesian Form – Use the x & y coordinates to represent the complex number

$$\boldsymbol{4 + j3}$$

$\boldsymbol{x + jy}$ (general form)

Polar Form – Use the magnitude & angle to represent the complex number

$$\boldsymbol{5\angle 37°}$$

$\boldsymbol{|V|\angle\theta}$ (general form)

Exponential Form – Variation of Polar Form

$$\boldsymbol{Re[5e^{j37°}]}$$

$\boldsymbol{Re[|V|e^{j\theta}]}$(general form)

Cartesian to Polar Conversion

$$\boldsymbol{|V| = \sqrt{x^2 + y^2}}$$

$$\boldsymbol{\theta = \tan^{-1}\frac{y}{x}}$$

Polar to Cartesian Conversion

$$\boldsymbol{x = |V|cos\theta}$$

$$\boldsymbol{y = |V|sin\theta}$$

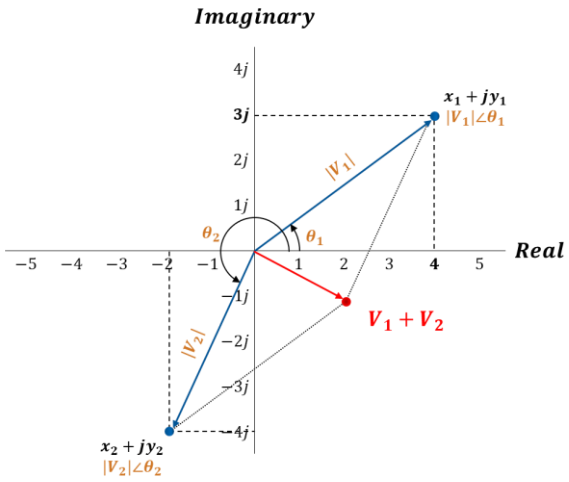
Complex Number

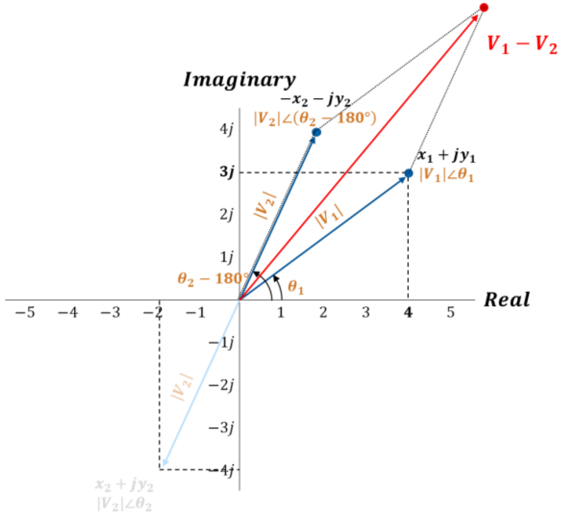
Addition/Subtraction

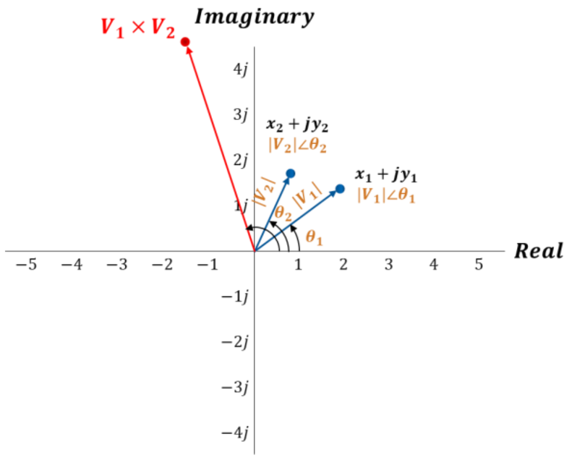
- Convert to cartesian form
- Add/subtract the real terms
- Add/subtract the imaginary terms

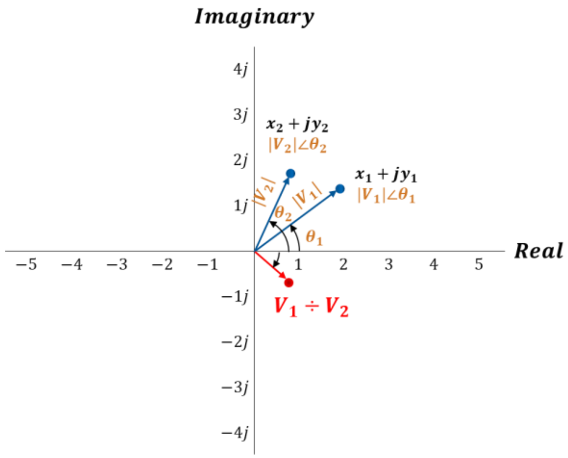
Multiplication/Division

- Convert to polar form
- Multiply/divide the magnitudes
- Add/subtract the angles

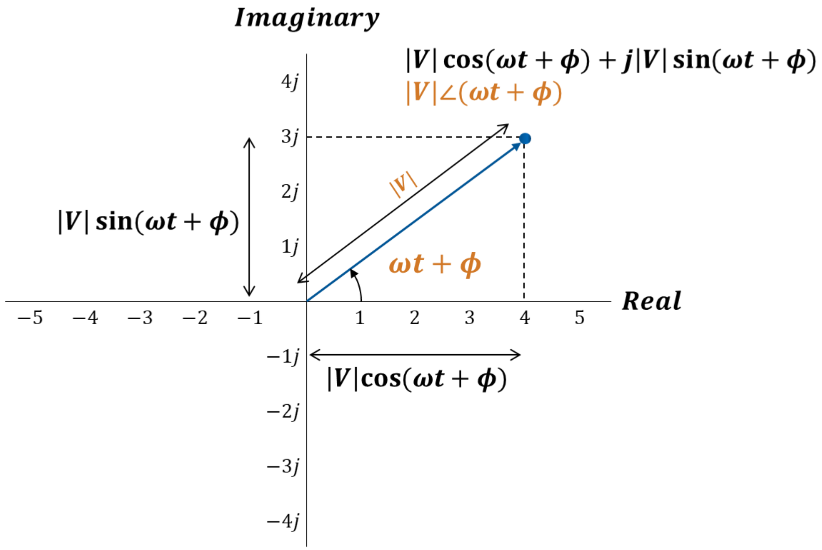








Phasor



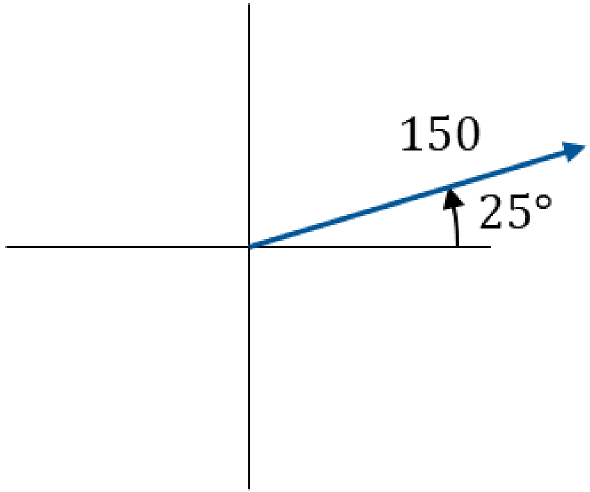
Say we have a voltage variable $\boldsymbol{v} = |\boldsymbol{V}|\boldsymbol{cos}(\boldsymbol{\omega t}+\boldsymbol{\phi})$

We may represent it with a "phasor" which is nothing but a complex number that represents the initial position of the rotating vector (i.e., at $t = 0$), and say the "projection on positive real axis" is the value of the physical variable

For example, voltage $\boldsymbol{v} = \boldsymbol{150\,cos(50t+25°)}$ may be represented in the phasor form as follows:

Numeric Form – $\boldsymbol{150\angle 25°}$

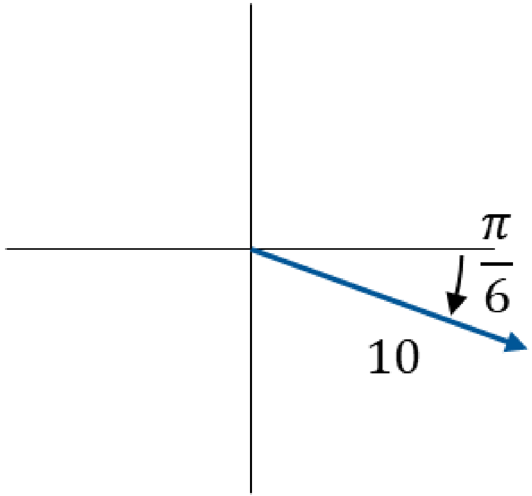
Visual Form –



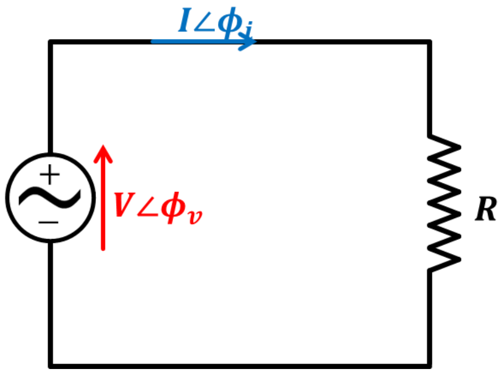
For example, current $\boldsymbol{i} = \boldsymbol{10\,cos(50t-\frac{\pi}{6})}$ may be represented in the phasor form as follows:

Numeric Form – $\boldsymbol{10\angle\frac{\pi}{6}}$

Visual Form –

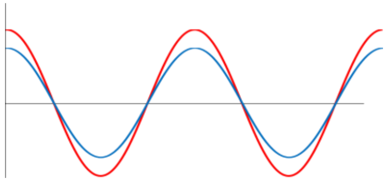


Impedance

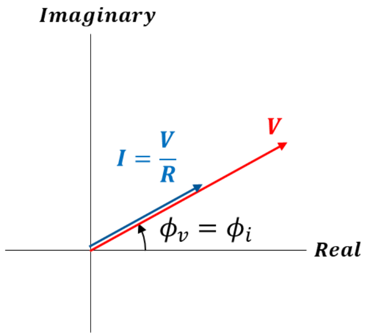


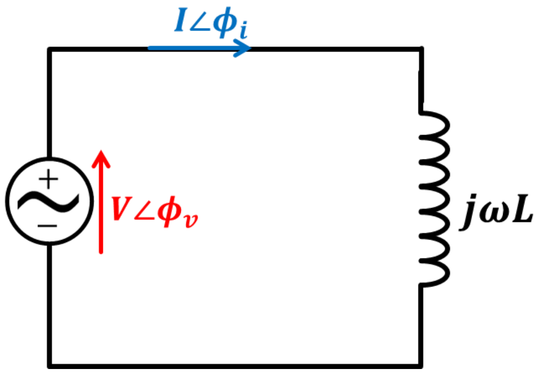
$$Z_R = R$$

$$\frac{V}{R}\angle\phi_v = I\angle\phi_i$$



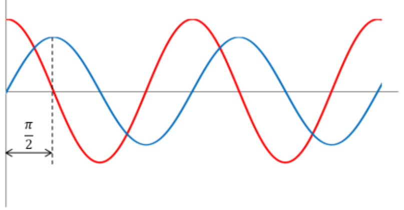
current in phase with **voltage**



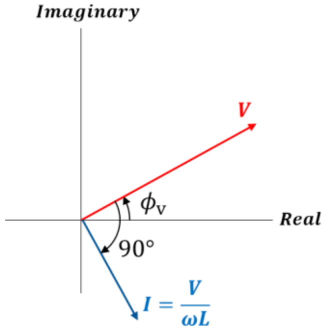


$$Z_L = j\omega L$$

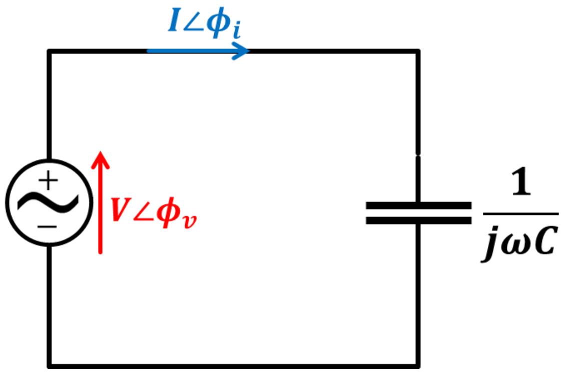
$$\frac{V}{\omega L}\angle(\phi_v - 90°) = I\angle\phi_i$$



current LAGS **voltage** by 90° or $\frac{\pi}{2} rad$

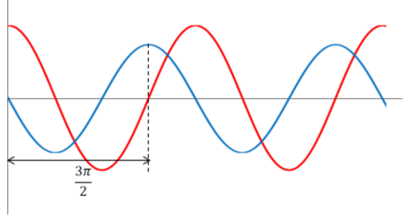


CIVIL

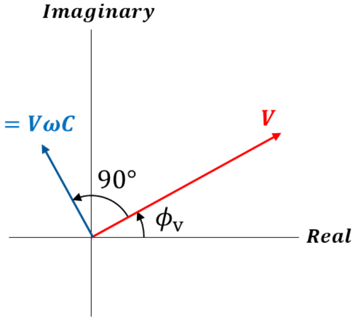


$$Z_C = \frac{1}{j\omega C}$$

$$V\omega C\angle(\phi_v + 90°) = I\angle\phi_i$$



current LEADS **voltage** by 90° or $\frac{\pi}{2} rad$



General Rule to Solve Circuits

Step 1 – Convert voltage and current to phasor form

Step 2 – Convert R, L, C to impedance (Z_R, Z_L, Z_C)

Step 3 – Apply circuit simplification (series/parallel) depending on what is required to be solved

Step 3A – You may need to use the "loop current" & "branch current" method depending on the problem – however, most likely you can do without if you smartly apply the series/parallel rules of circuit simplification

Step 4 – Apply KVL to the required loop(s)

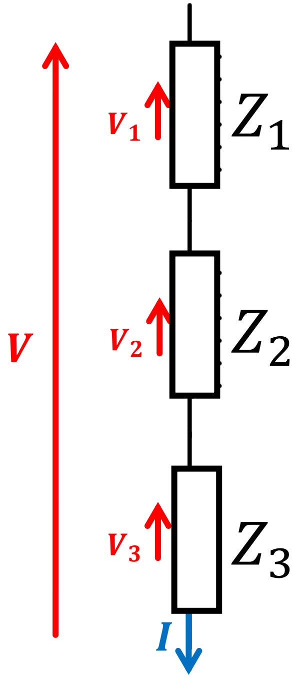
Step 5 – Apply Ohm's Law

Step 6 – Solve the linear system of equations – you can solve for n unknowns with n equations

The Real Circuit (Resistive + Reactive)

Series

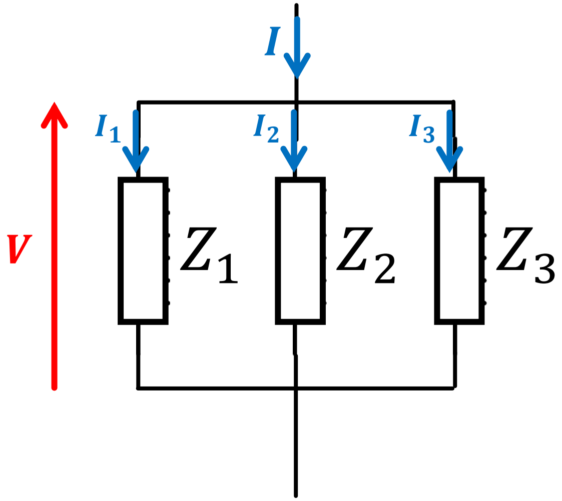
When two (or more) elements are connected together head-to-toe



$$Z = \sum Z_i$$

Parallel

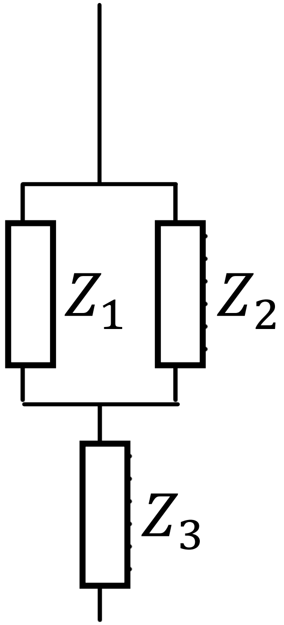
When two (or more) elements are connected head-to-head and toe-to-toe



$$\frac{1}{Z} = \sum \frac{1}{Z_i}$$

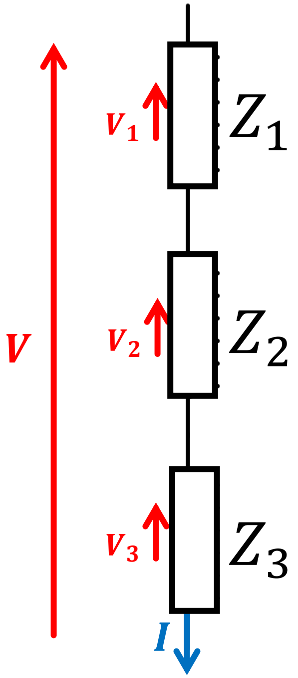
Series-Parallel

Combination of the both



Break the circuit up into series and parallel and solve individually

Potential (& Current) Divider Rule



$$\boldsymbol{V_1 = \frac{Z_1}{Z_1 + Z_2 + Z_3} V}$$

$$V = V_1 + V_2 + V_3$$

$$V = IZ_1 + IZ_2 + IZ_3$$

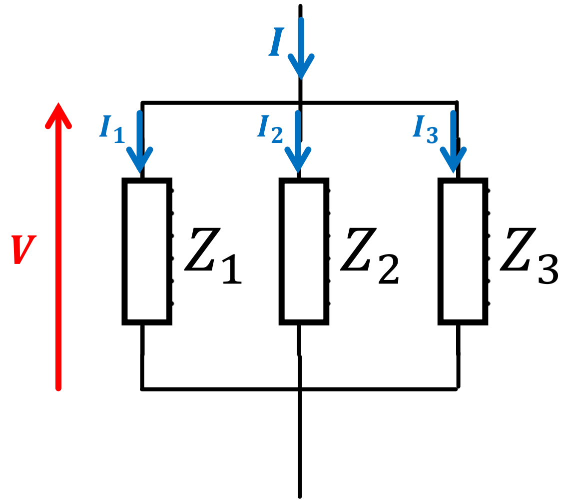
$$\boldsymbol{V = I(Z_1 + Z_2 + Z_3)}$$

$$V_1 = IZ_1$$

$$\boldsymbol{I = \frac{V_1}{Z_1}}$$

$$V = \frac{V_1}{Z_1}(Z_1 + Z_2 + Z_3)$$

$$\boldsymbol{V_1 = \frac{Z_1}{Z_1 + Z_2 + Z_3} V}$$

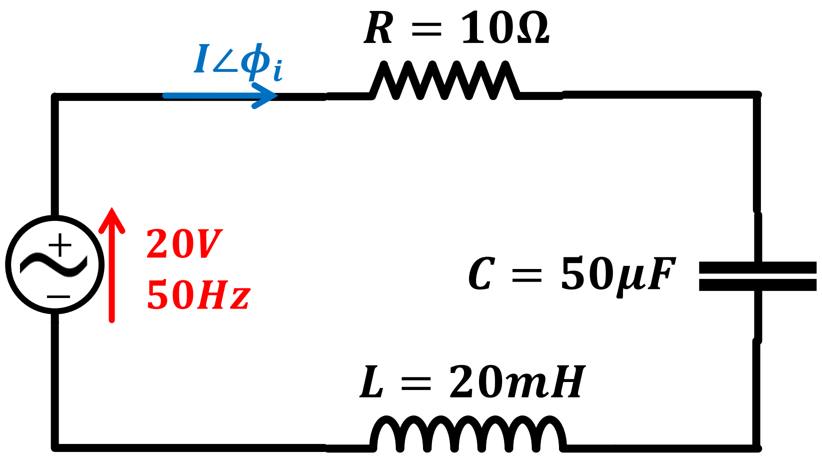


$$\boldsymbol{I_1 = \frac{\frac{1}{Z_1}}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}} I}$$

$$\boldsymbol{I_1 = \frac{Y_1}{Y_1 + Y_2 + Y_3} I}$$

Admittance (symbol Y unit mho or ℧) is the reciprocal of Impedance

Example of Real Circuit



$$\boldsymbol{Z_R = 10\Omega}$$

$$\boldsymbol{Z_L} = j\omega L = j2\pi f L = j2 \times 3.14 \times 50 \times 20 \times 10^{-3} = \boldsymbol{j6.28\Omega}$$

$$\boldsymbol{Z_C} = \frac{1}{j\omega C} = \frac{1}{j2\pi f C} = \frac{-j}{2 \times 3.14 \times 50 \times 50 \times 10^{-6}} = \boldsymbol{-j63.66\Omega}$$

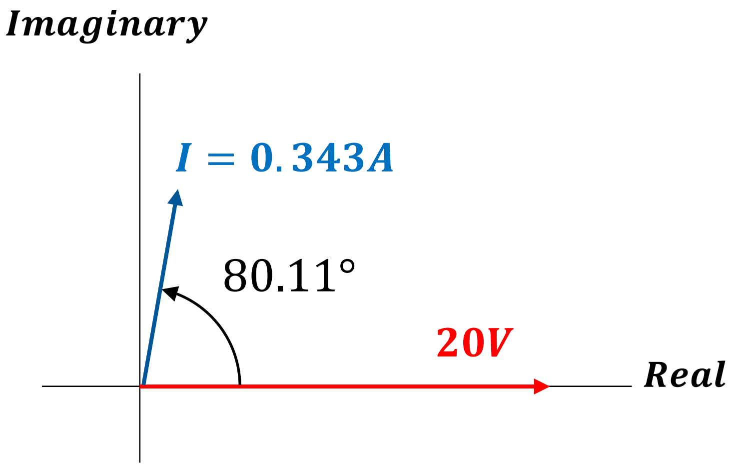
The **three elements** are clearly in **series**

$$\boldsymbol{Z} = Z_R + Z_L + Z_C = 10 + j(6.28 - 63.69) = \boldsymbol{10 - j57.38}$$

Applying **Ohm's Law**, we need to divide V by Z, remember, for division, we need complex numbers in **polar form**

$$\boldsymbol{|Z| = \sqrt{10^2 + 57.41^2} = \sqrt{3395.91} = 58.24}$$

$$\boldsymbol{\angle Z = \tan^{-1}\frac{-57.41}{10} = -80.11^\circ}$$



Applying **Ohm's Law**

$$\boldsymbol{I = \frac{V}{Z} = \frac{20\angle 0^\circ}{58.24\angle - 80.11^\circ} = 0.343\angle 80.11^\circ}$$

When no info on phase offset for voltage provided, no harm in setting it to 0°, makes calculations easier!

Learning Outcomes

- Revision of circuits
  - **Sinusoidal** waveform – phase **angle** v **time**?
  - **Phasors**
  - Series/Parallel **clarification**
  - **Impedance**
  - **General Rule to solve circuits**
  - **Potential Divider Rule**
- Power in AC circuits
  - **Active** v **Reactive** v **Apparent** Power
  - **Power Factor**
  - **Resonance**

University of Nottingham
UK | CHINA | MALAYSIA

Root Mean Square (RMS)

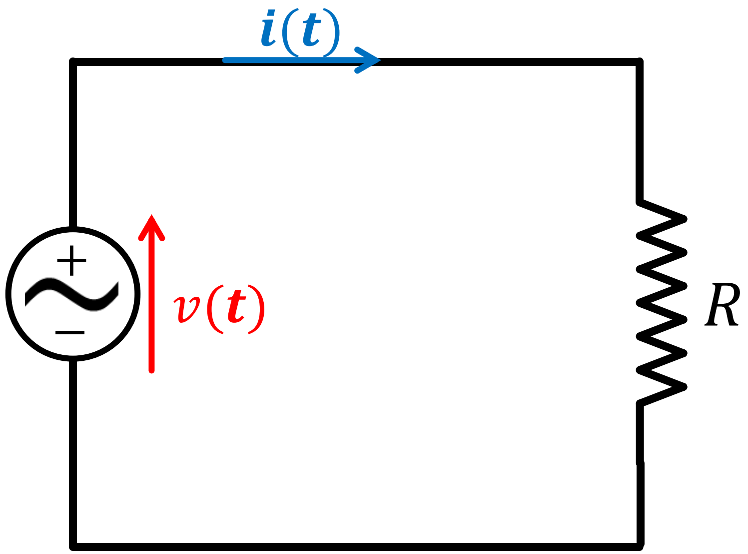
- In mathematics, the **root-mean-square** (or **RMS**) of a set of numbers x_i is defined as the square root of the arithmetic mean of the squares of the set

$$\underline{x} = \sqrt{\frac{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}{n}} = \sqrt{\frac{\sum x_i^2}{n}}$$

- When dealing with AC applications, the amplitude of voltage or current is seldom used (we will see shortly why – power)
- Hence, AC ammeters/voltmeters are invariably calibrated for RMS value – not peak/amplitude
- For all **sinusoidal waves**, the RMS value is $\frac{1}{\sqrt{2}} = 0.707$ times the amplitude
- It is much more convenient to make the **length of phasors** represent **RMS** instead of amplitude
- Going forward, we will deal with **only RMS values** when studying AC

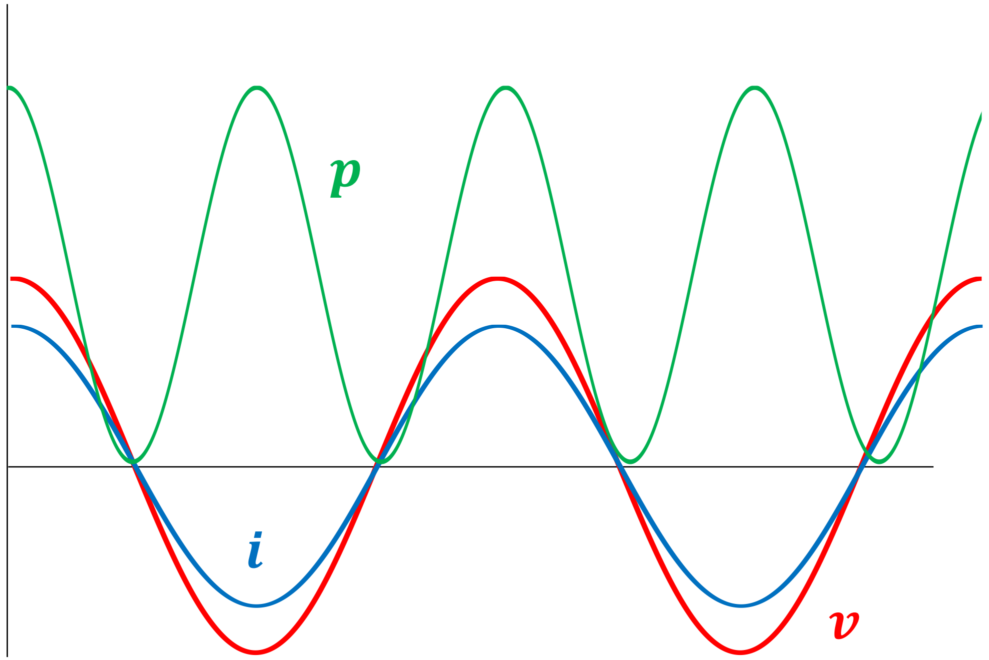
$$\boldsymbol{RMS\ value\ of\ V = V_{rms} = \frac{|V|}{\sqrt{2}} = 0.707V}$$

Power in Resistive Circuit



$$v(t) = V\cos\omega t$$

$$i(t) = I\cos\omega t$$



Instantaneous power

$$p(t) = v(t) \times i(t)$$

$$p(t) = V\cos\omega t \times I\cos\omega t$$

$$p(t) = \frac{VI}{2}(1+\cos 2\omega t) = \frac{I^2R}{2}(1+\cos 2\omega t) = \frac{V^2}{2R}(1+\cos 2\omega t)$$

Average power – integrate over full cycle

$$P_{avg} = \int \frac{VI}{2}(1+\cos 2\omega t)$$

$$P_{avg} = \frac{VI}{2} + 0$$

$$P_{avg} = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}} = V_{rms}I_{rms}$$

Power in Resistive Circuit

Proof (don't learn)

$$\begin{aligned}
P_{avg} &= \frac{1}{T}\int_0^T p(t)\,dt \\
&= \frac{1}{T}\int_0^T v(t)i(t)\,dt \\
&= \frac{1}{T}\int_0^T V\cos(\omega t) I\cos(\omega t)\,dt \\
&= \frac{1}{T}\int_0^T \frac{VI}{2}\{1+\cos(2\omega t)\}\,dt \\
&= \frac{1}{T}\int_0^T \frac{VI}{2}\,dt + \frac{1}{T}\int_0^T \frac{V_m I_m}{2}\{\cos(2\omega t)\}\,dt \\
&= \frac{VI}{2} - \frac{1}{\omega T}\int_0^{2\pi} \frac{V_m I_m}{2}\{\cos(2\omega t)\}\,d\omega t \\
&= \frac{VI}{2} = V_{rms} I_{rms}
\end{aligned}$$

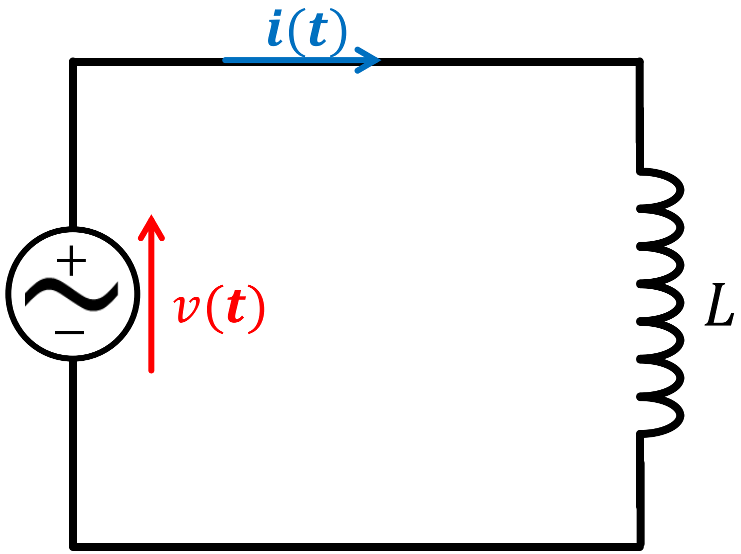
$$\boldsymbol{P_{avg} = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}} = V_{rms} I_{rms}}$$

Remember that power in DC circuits
$\boldsymbol{P = V_{dc} \times I_{dc}}$

Equivalently, the AC counterparts for
$\boldsymbol{V_{dc}}$ **is** $\boldsymbol{V_{rms}}$ **and** $\boldsymbol{I_{dc}}$ **is** $\boldsymbol{I_{rms}}$

That is why we always use the RMS value of voltage and current

Power in Inductive Circuit



$$v(t) = V\cos\omega t$$

$$i(t) = I\sin\omega t$$

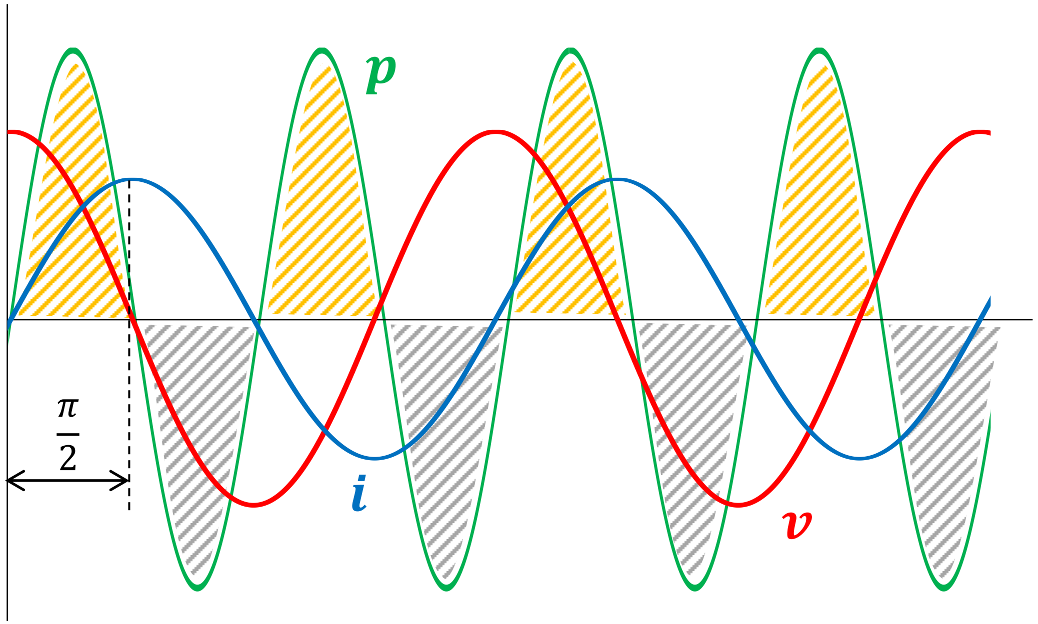
Do you know why?

Instantaneous power

$$p(t) = v(t) \times i(t)$$

$$p(t) = V\cos\omega t \times I\sin\omega t$$

$$p(t) = \frac{VI}{2}\sin 2\omega t = \frac{\omega L I^2}{2}\sin 2\omega t = \frac{V^2}{2\omega L}\sin 2\omega t$$

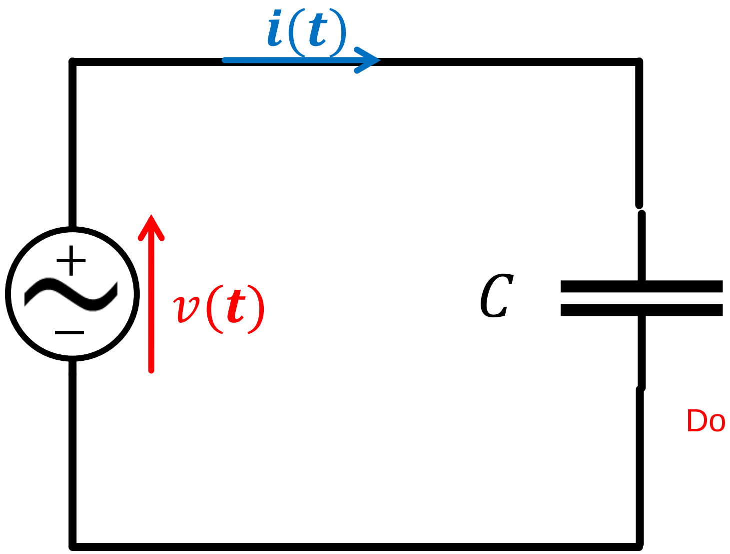


▨ Energy absorbed from the source

▨ Energy released to the source

Average power is ZERO!

Power in Capacitive Circuit



$$v(t) = V\cos\omega t$$

$$i(t) = -I\sin\omega t$$

Do you know why?

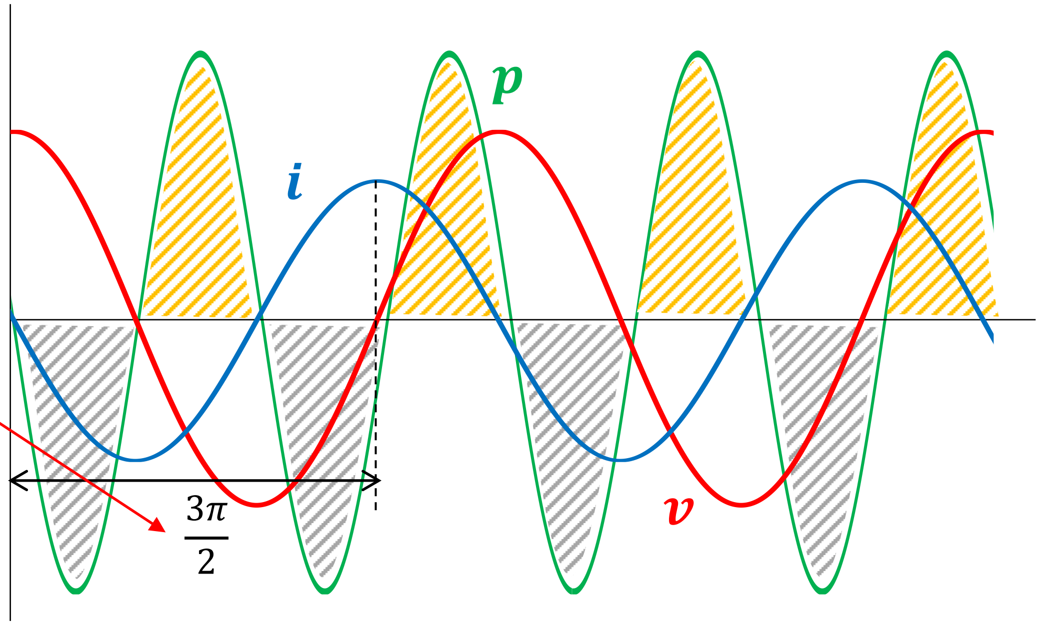
Instantaneous power

$$p(t) = v(t) \times i(t)$$

$$p(t) = -V\cos\omega t \times I\sin\omega t$$

$$p(t) = \frac{-VI}{2}\sin 2\omega t = \frac{I^2}{2\omega C}\sin 2\omega t = \frac{\omega C V^2}{2}\sin 2\omega t$$

Do you know why?

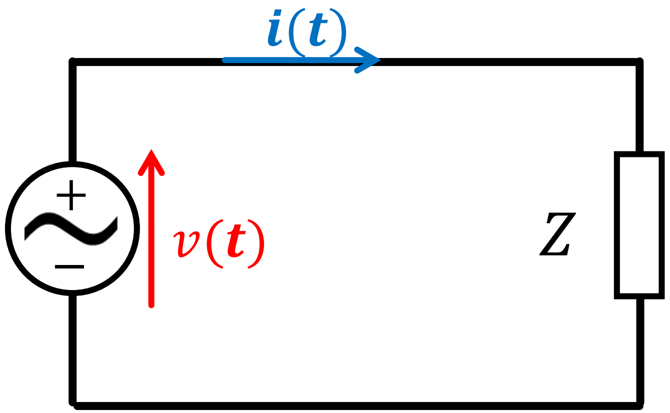


▨ Energy absorbed from the source

▨ Energy released to the source

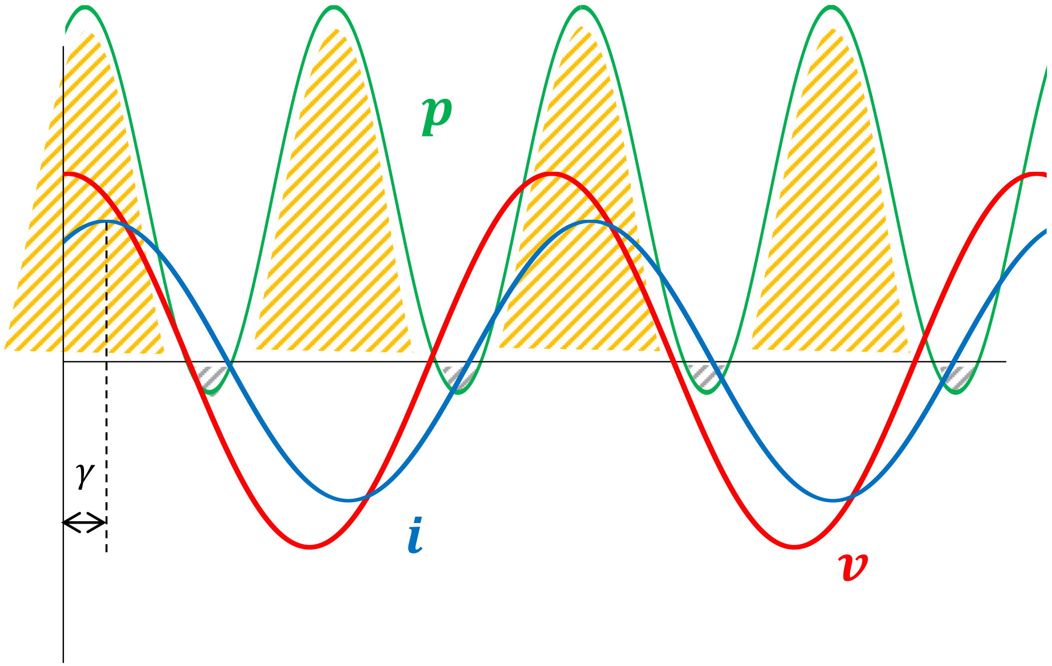
Average power is ZERO!

Power in Real Circuit (Resistive + Reactive)



$$v(t) = V\cos\omega t$$

$$i(t) = I\cos(\omega t + \gamma)$$



Instantaneous power

$$p(t) = v(t) \times i(t)$$

$$p(t) = V\cos\omega t \times I\cos(\omega t + \gamma)$$

$$p(t) = \frac{VI}{2}\{\cos(\omega t - \omega t - \gamma) + \cos(\omega t + \omega t + \gamma)\}$$

$$p(t) = \boxed{V_{rms}I_{rms}\cos\gamma} + \boxed{V_{rms}I_{rms}\cos(2\omega t + \gamma)}$$

Average Power

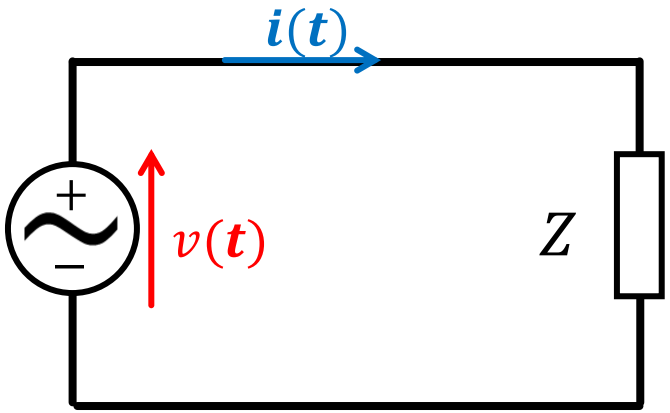
This term averages to zero over a cycle

▨ Energy absorbed from the source

▨ Energy released to the source

Average power is $V_{rms}I_{rms}\boxed{\cos\gamma}$ **Power Factor**

Power Factor



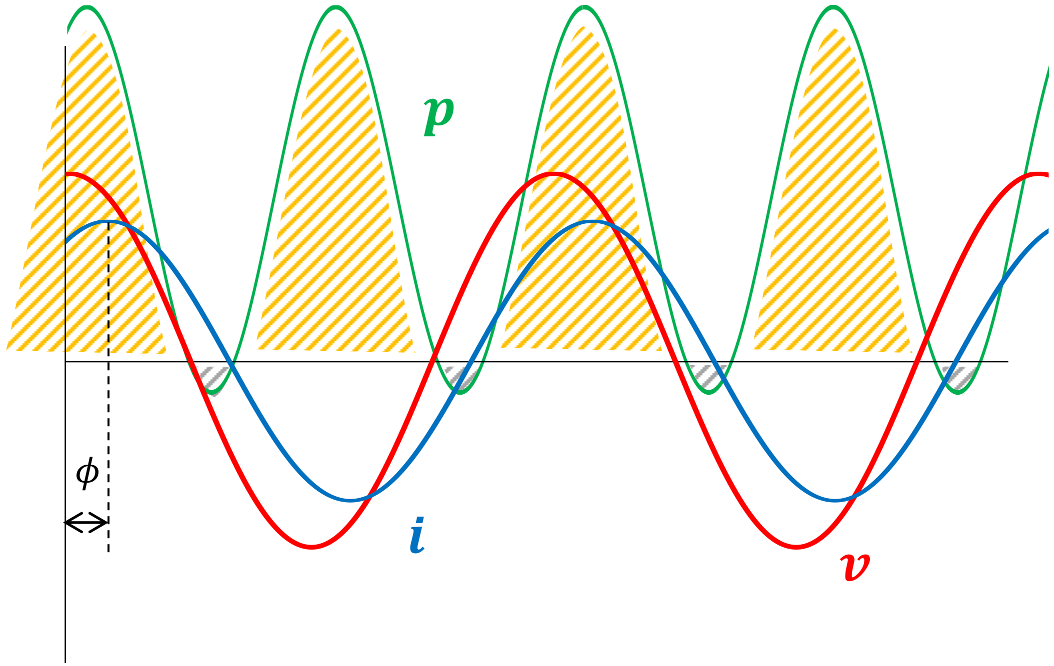
$$P_{avg} = V_{rms} I_{rms} \cos\gamma$$

$$\cos\gamma = Power\ Factor = PF$$

γ is the **phase deviation** between voltage & current

PF tells us **what fraction of the current does useful work**

Is it phase **advance/delay**? ***Does it matter?***



▨ Energy absorbed from the source

▨ Energy released to the source

Power Factor

Purely Resistive Load R	$\gamma = 0°$ $\cos\gamma = 1$	All power consumed
Purely Reactive Load L or C	$\gamma = \pm 90°$ $\cos\gamma = 0$	No **real power** consumed
Real Inductive Load $RL \text{ or } RLC$	$-90° < \gamma < 0°$ $0 < \cos\gamma < 1$	Part of **apparent power** consumed
Real Capacitive Load $RC \text{ or } RLC$	$0° < \gamma < 90°$ $0 < \cos\gamma < 1$	

Active v Reactive v Apparent Power

Apparent Power (symbol **S** unit **VA)**

$$S = V_{rms}I_{rms}$$

- As the name suggests, this is the amount of power that appears to be flowing from source to load
- This is not the case as over a cycle, some (or all) of this power gets returned back to source
- As the power still flows (even if it is simply thrown back-forth between source and load), losses still occur
- A good circuit should have PF very close to unity
- However, AC equipment are rated for Apparent Power as it handles both used and unused power

Active Power (symbol **P** unit **W)**

$$P = V_{rms}I_{rms}\cos\gamma = V_{rms}I_{rms}PF = S \times PF$$

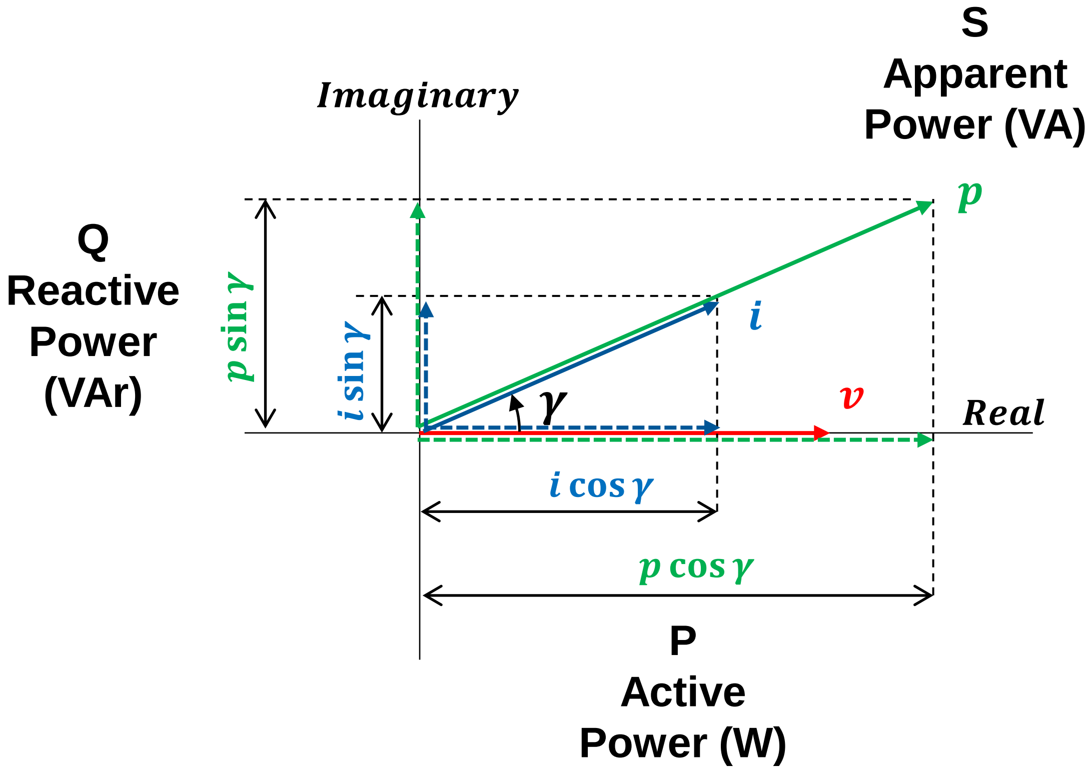
- This is the real power transferred to the load

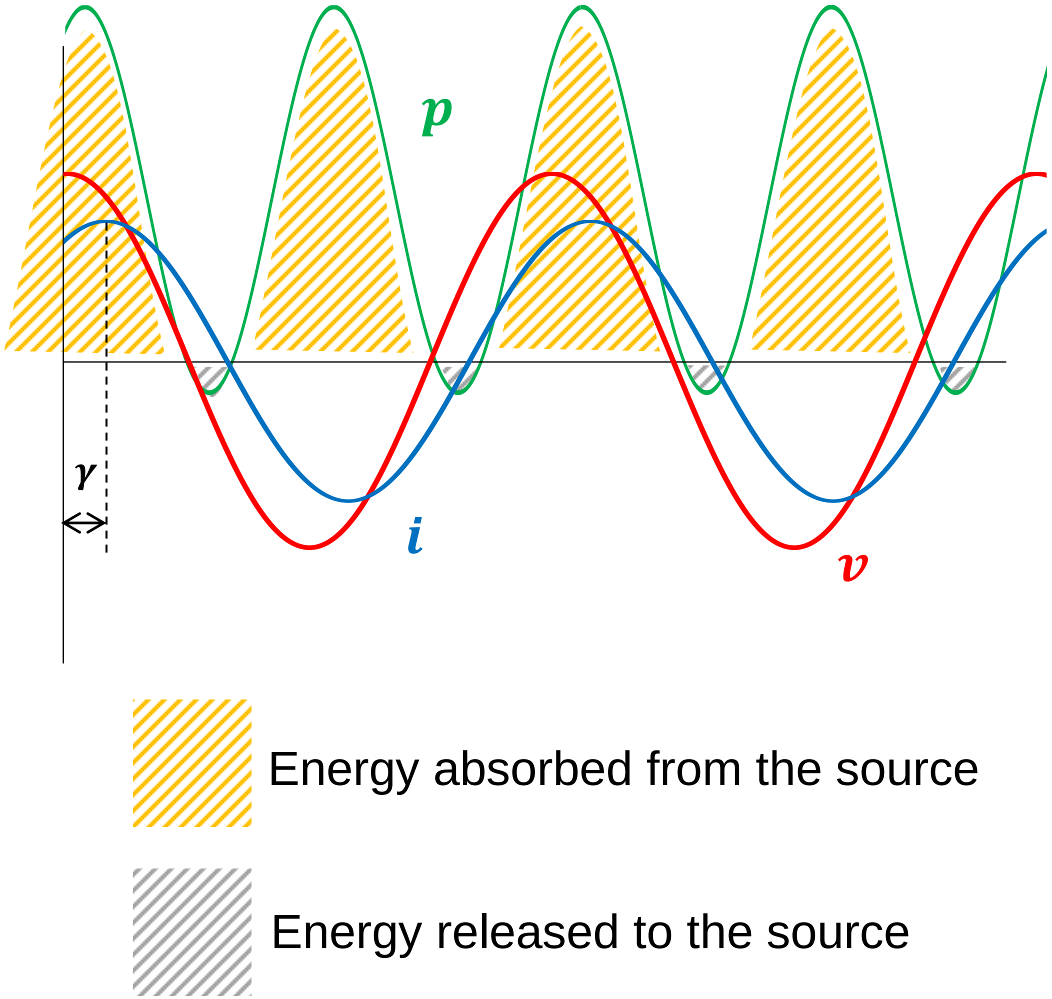
Reactive Power (symbol **Q** unit **VAr)**

$$P = V_{rms}I_{rms}\sin\gamma = V_{rms}I_{rms}\,sin\,\gamma = S\,sin\,\gamma$$

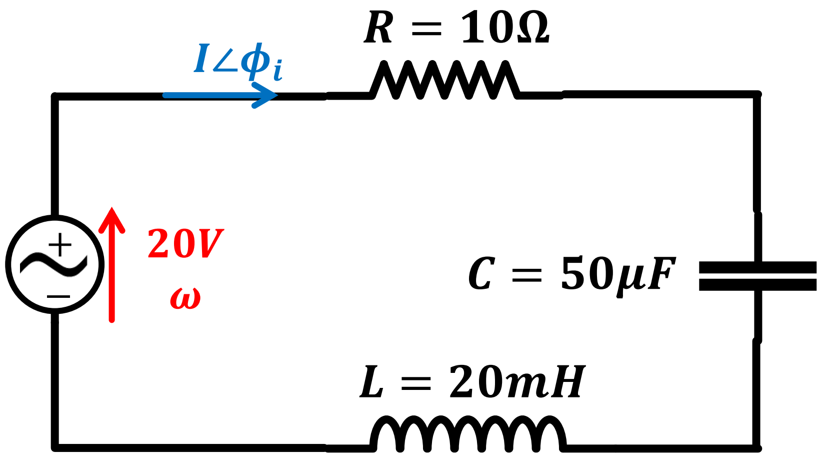
- This is the purely unused power exchanged between the source and load

Active v Reactive v Apparent Power





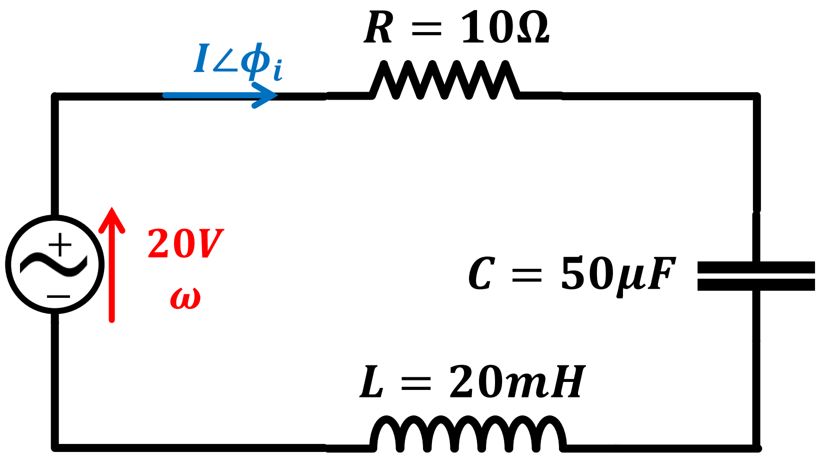
Resonance



- We have seen that inductor and capacitor **individually contribute** to delaying and advancing (respectively) the current waveform w/r/t the voltage
- When the inductance and capacitance value are equal (and opposite, inherently) they **nullify** each other – **Resonance**
- $Z_L = j\omega L$ **increases** with **increasing frequency**
- $Z_C = \frac{1}{j\omega C}$ **decreases** with **increasing frequency**

- We did this example earlier with frequency ($50\ Hz$), we saw that the **overall circuit was capacitive** (i.e., capacitance was overpowering inductance and resultant current was $80°$ leading)
- What happens if we **increase the frequency**?
- There will come a frequency when **inductance just matches capacitance** – this is **resonance**
- When this happens, you will be left with a **purely resistive circuit**, i.e., **overall impedance drops**!
- As you increase the frequency (from $50\ Hz$), you would see current rising gradually, then sharply at resonance, then again start falling

Resonance



$$Z_L = Z_C$$

$$j\omega L = \frac{1}{j\omega C}$$

$$\omega = \frac{1}{\sqrt{LC}}$$

$$\omega = \frac{1}{\sqrt{20 \times 10^{-3} \times 50 \times 10^{-6}}} = \frac{1}{\sqrt{10^{-6}}}$$

$$\omega_{res} = 1000\,\frac{rad}{s}$$

Lets find out the current at resonant frequency and plot the phasor diagram

$$Z_L = j\omega_{res}L = j \times 1000 \times 20 \times 10^{-3} = j20\Omega$$

$$Z_C = \frac{1}{j\omega_{res}C} = \frac{-j}{1000 \times 50 \times 10^{-6}} = -j20\Omega$$

The **three elements** are clearly in **series**

$$Z = Z_R + Z_L + Z_C = 10 + j(20 - 20) = 10\Omega$$

Applying **Ohm's Law**

$$I = \frac{V}{Z} = \frac{20\angle 0°}{10\angle 0°} = 2\angle 0° A$$

$2\ A$ is significantly higher than $343\ mA$ that we calculated at $50\ Hz$ frequency

This is because the at resonance, inductive and capacitive impedances nullify each other

Example 1

A coil is connected to a 50 V AC supply at 400 Hz. If the current supplied to the coil is 200 mA and the coil has a resistance of 60 Ω , determine the value of inductance.

Like most practical forms of inductor, the coil in this example has both resistance *and* reactance. We can find the impedance of the coil from:

$$|Z| = \frac{V}{I} = \frac{50}{0.2} = 250\Omega$$

Since

$$|Z| = \sqrt{R^2 + X^2}$$

$$X = \sqrt{|Z|^2 - R^2}$$

$$X = \sqrt{250^2 - 60^2} = 243\Omega$$

Now since $XL = 2\pi f L$,

$$L = \frac{X}{2\pi f} = \frac{243}{100\pi} = 0.097H$$

Example 2

An AC load has a power factor of 0.8. Determine the active power dissipated in the load if it consumes a current of 2 A at 110 V.

Since active power

$$P = PF \times V_{rms} \times I_{rms}$$

$$P = 0.8 \times 110 \times 2$$

$$P = 176\ W$$

Example 3

A coil having an inductance of $150\ mH$ and resistance of $250\ \Omega$ is connected to a $115\ V\ 400\ Hz$ AC supply. Determine:

(a) the power factor of the coil
(b) the current I_{rms} taken from the supply
(c) the power dissipated as heat in the coil.

(a) First we must find the reactance of the inductor, X_L, and the impedance, Z, of the coil at 400 Hz.

$$X_L = 2\pi \times 400 \times 0.015 = 376\ \Omega$$

Thus

$$Z = R + jX_L = 250 + j376\ \Omega$$

The power factor is

$$\cos\gamma = \frac{R}{|Z|}$$

Since

$$|Z| = \sqrt{R^2 + X_L^2} = \sqrt{250^2 + 376^2} = 452\ \Omega$$

Thus $$\boldsymbol{\cos\gamma} = \frac{R}{|Z|} = \frac{250}{452} = \mathbf{0.553}$$

(b)

$$\boldsymbol{I_{rms}} = \frac{V_{rms}}{|Z|} = \frac{115}{452} = \mathbf{0.254\ A}$$

(c) The power dissipated as heat is the active power

$$P = V_{rms} I_{rms} \cos\gamma = 0.254 \times 115 \times 0.553$$

$$\boldsymbol{P = 16.15\ W}$$

Summary

- Revision of circuits
  - **Sinusoidal** waveform – phase **angle** v **time**?
  - **Phasors**
  - Series/Parallel **clarification**
  - **Impedance**
  - **General Rule to solve circuits**
  - **Potential Divider Rule**
- Power in AC circuits
  - **Active** v **Reactive** v **Apparent** Power
  - **Power Factor**
  - **Resonance**