DYNAMICS AND CONTROL

CONTROL SEMINAR 1

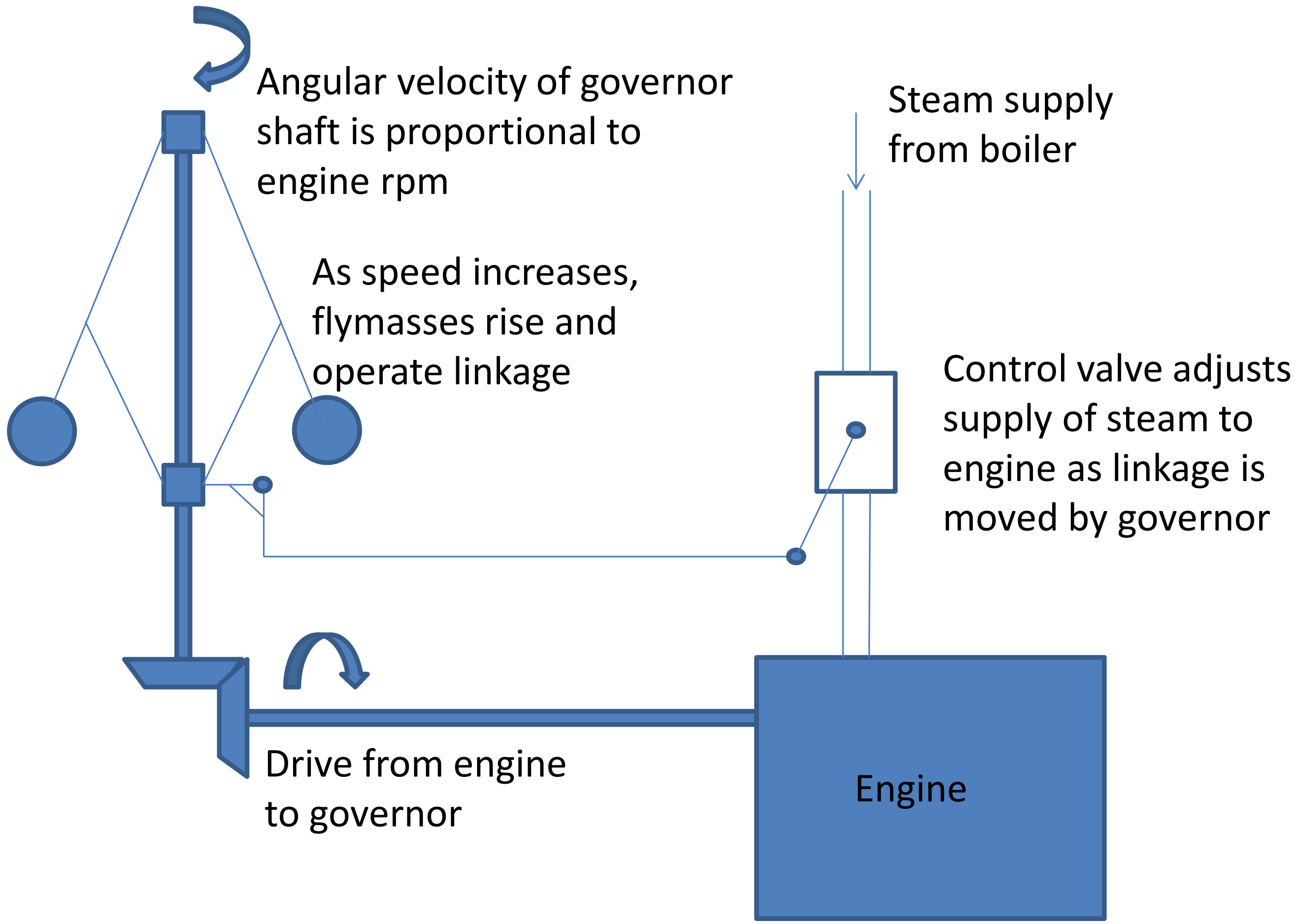
GENERAL RECAP – SESSION 1

- Why do we need control systems?
- Why is it important to be able to predict how a system behaves?

Lecture recap:

- Why do we model control?
  - Understanding behaviour of controlled systems
  - Tuning for optimum performance (speed, efficiency, rapid response)
  - Preventing out-of control behaviour (resonance, oscillation, run-away)

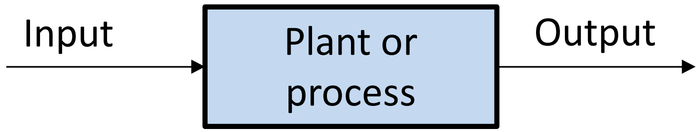
Method of operation for centrifugal governor



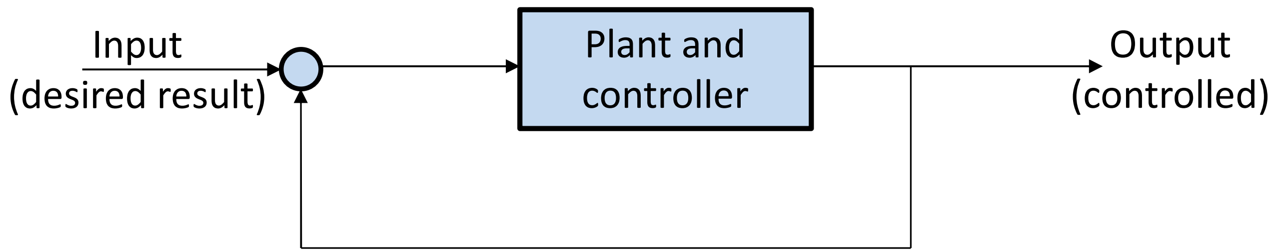
Video: https://www.youtube.com/watch?v=OG1AiaNTT6s

Systems and block diagrams

- Open-Loop system

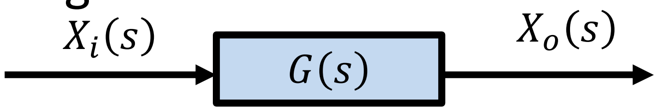


- Closed-Loop (feedback) system



Representation of control systems

- The block diagram for an element is drawn as follows:



$$X_o(s) = G(s)\, X_i(s)$$

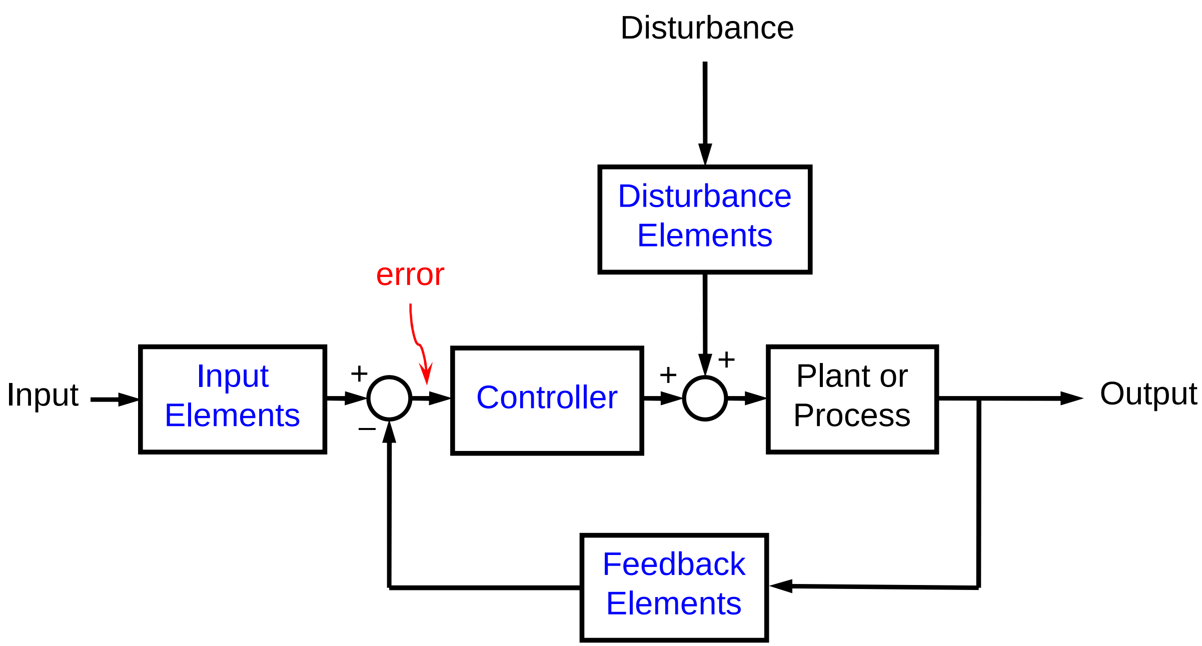
- The transfer function $G(s)$ is thus given by:

$$G(s) = \frac{X_o(s)}{X_i(s)} = \frac{P(s)}{Q(s)}$$

- Where $Q(s)$ is known as the *characteristic function*, and $Q(s) = 0$ is the *characteristic equation*.

Representation of Control Systems

A typical system has a block diagram of the following form

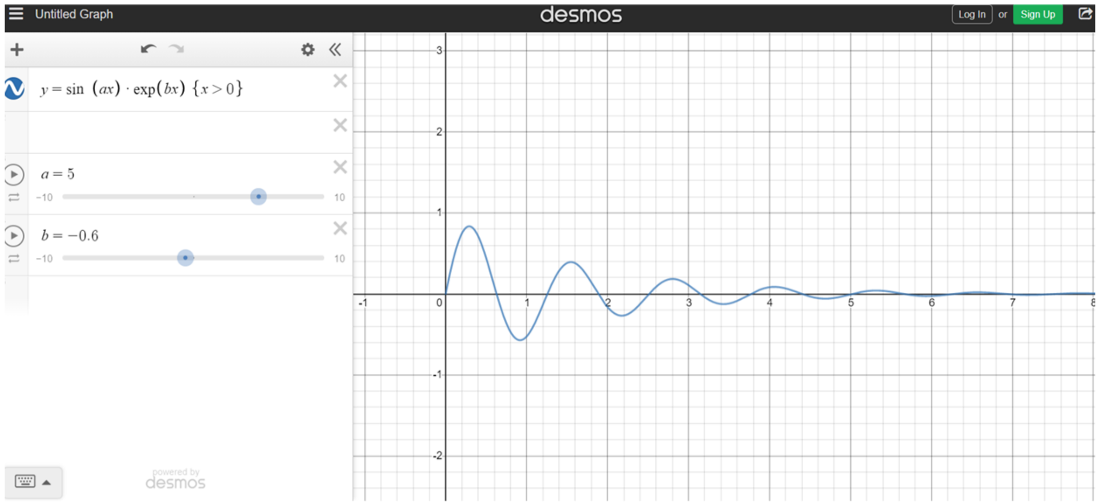


Each box will then contain the transfer function of the element contained in the box.

Laplace Transforms

- All based on the insight that any time dependent function can be written as

$$x(t) = Ae^{\alpha t} \times e^{j\omega t} = Ae^{(\alpha + j\omega)t} = Ae^{st}$$



https://www.youtube.com/watch?v=n2y7n6jw5d0

https://www.youtube.com/watch?v=3gjJDuCAEQQ

Laplace Transforms

- Differentiate: multiply by s:

$$\mathcal{L}\left[\frac{dx}{dt}\right] = sX(s)$$

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] = s^2X(s)$$

- Integrate: divide by s:

$$\mathcal{L}\left[\int x(t)dt\right] = \frac{1}{s}X(s)$$

https://www.youtube.com/watch?v=n2y7n6jw5d0

https://www.youtube.com/watch?v=3gjJDuCAEQQ

Laplace Transforms: Linear functions

i) Addition and Subtraction (superposition applies)

$$l[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

ii) Multiplication by a constant

$$l[Kf(t)] = KF(s)$$

iii) Final Value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

This theorem is only valid if the final value is finite and constant.

A table of Laplace transform pairs will be provided (also at the exam).

Table of Laplace Transforms

	$f(t)$	$F(S)$
1	$\frac{df(t)}{dt}$	$sF(s) - f(0)$
2	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - \ldots. - f^{n-1}(0)$
3	$\int f(t)dt$	$\frac{1}{s}F(s)$
4	Unit impulse $\delta(t)$ at t=0	1
5	Unit step at t=0	$\frac{1}{s}$
6	Unit ramp $f(t) = t$	$\frac{1}{s^2}$
7	e^{-at}	$\frac{1}{s+a}$
8	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
9	$t - \frac{1}{a}(1 - e^{-at})$	$\frac{a}{s^2(s+a)}$

Table of Laplace Transforms (continued)

	$f(t)$	$F(S)$
10	$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$
11	$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
12	$\dfrac{1}{(\omega^2-p^2)}\left[\sin(pt)-\dfrac{p}{\omega}\sin(\omega t)\right]$	$\dfrac{p}{(s^2+p^2)(s^2+\omega^2)}$
13	$\dfrac{1}{(\omega^2-p^2)}[\cos(pt)-\cos(\omega t)]$	$\dfrac{s}{(s^2+p^2)(s^2+\omega^2)}$
14	$\dfrac{\omega}{\sqrt{1-\gamma^2}}e^{-\gamma\omega t}\sin\left(\omega t\sqrt{1-\gamma^2}\right)$	$\dfrac{\omega^2}{s^2+2\omega\gamma s+\omega^2}$
15	$1-\dfrac{e^{-\gamma\omega t}}{\sqrt{1-\gamma^2}}\sin\left(\omega t\sqrt{1-\gamma^2}+\phi\right)$	$\dfrac{\omega^2}{s(s^2+2\omega\gamma s+\omega^2)}$
16	$t-\dfrac{2\gamma}{\omega}-\dfrac{e^{-\gamma\omega t}}{\omega\sqrt{1-\gamma^2}}\sin\left(\omega t\sqrt{1-\gamma^2}+\phi\right)$ where $\cos\phi=\gamma$	$\dfrac{\omega^2}{s^2(s^2+2\omega\gamma s+\omega^2)}$

This Table will be provided in the exam handout.

LAPLACE TRANSFORMS REVISION

- Example 2 from Example sheet 0

a) Use Laplace transforms to determine the solution to the following differential equation in the time domain (i.e. $x(t)$)

$$\frac{d^2x}{dt^2} + 0.1\frac{dx}{dt} + x = f(t)$$

Where $f(t)$ is a unit step and the initial conditions are taken to be zero

b) Determine the transfer function $G(s)$ of the system analysed in (a) taking $f(t)$ to be the input and $x(t)$ to be the output of the system.

PART (A)

Step 1: transform the differential equation from the time domain to the s domain:

$$\frac{d^2x}{dt^2} + 0.1\frac{dx}{dt} + x = f(t)$$

Becomes:

$$(s^2 + 0.1s + 1)X(s) = F(s)$$

- *F(s)* is $\frac{1}{s}$ (Unit step from the table of Laplace transforms)

So: $$(s^2 + 0.1s + 1)X(s) = \frac{1}{s}$$

PART (A)

$$(s^2 + 0.1s + 1)X(s) = \frac{1}{s}$$

$$X(s) = \frac{1}{s(s^2 + 0.1s + 1)}$$

Back to table of Laplace transforms (entry 17 in the table, page 11)

$$1 - \frac{e^{-\gamma\omega t}}{\sqrt{1-\gamma^2}} \sin\left(\omega t\sqrt{1-\gamma^2} + \varphi\right) \qquad \frac{\omega^2}{s(s^2 + 2\gamma\omega s + \omega^2)}$$

$$x(t) = 1 - \frac{e^{-0.05t}}{\sqrt{1-(0.05)^2}} \sin\left(t\sqrt{1-(0.05)^2} + \varphi\right) \qquad \varphi = \cos^{-1}\gamma$$

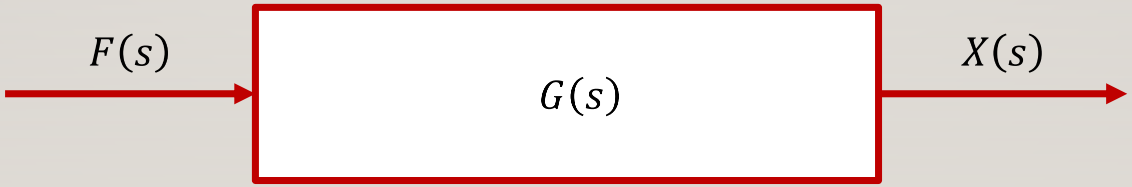
PART (B)

To get the transfer function:

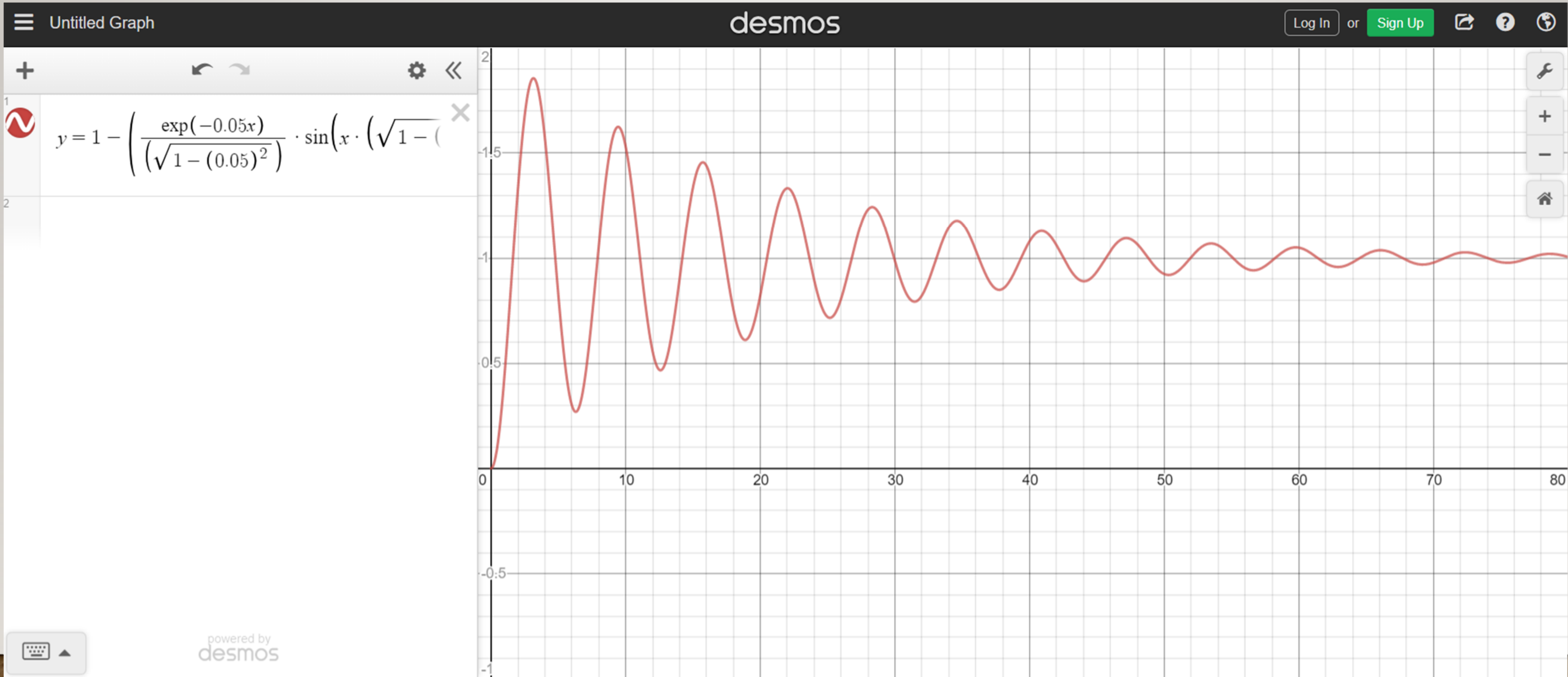
$$(s^2 + 0.1s + 1)X(s) = F(s)$$

Rearrange so that:

$$\frac{X(s)}{F(s)} = G(s) = \frac{1}{(s^2 + 0.1s + 1)}$$



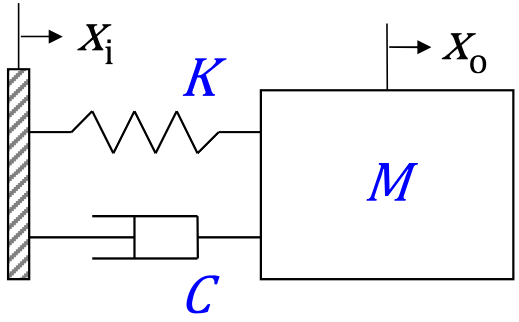
And this is the system response:



Modelling of Simple Components

- Lever Systems
- Rotor with Viscous Drag
- Mass-Spring-Damper System (Exercise)
- Hydraulic Ram

d) Spring-Mass-Damper System



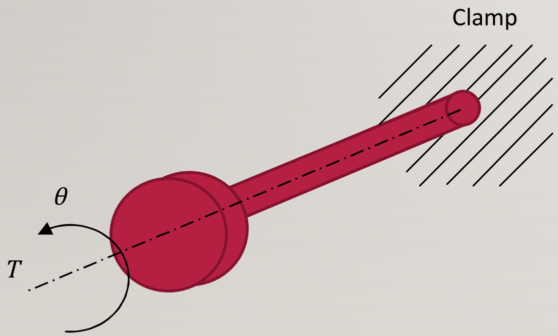
Exercise: Noting that the input to the above system is a displacement, show that the **transfer function** for the system is given by

$$G(s) = \frac{X_o(s)}{X_i(s)} = \frac{Cs + K}{Ms^2 + Cs + K} = \frac{2\gamma\omega_n s + \omega_n^2}{s^2 + 2\gamma\omega_n s + \omega_n^2}$$

where

$$\omega_n^2 = \frac{K}{M} \quad \text{and} \quad \gamma = \frac{C}{2\sqrt{KM}}$$

HOW TO – EXAMPLE SHEET 1, NO. 2

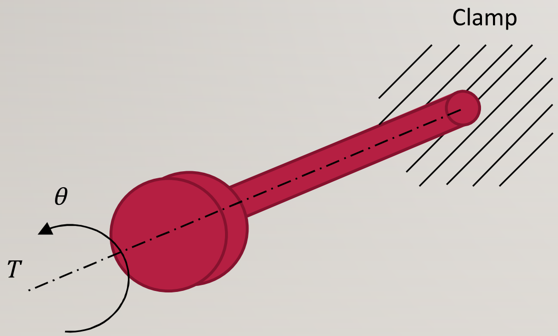


- Derive expressions for the transfer functions that relate input torque $T(t)$ and output angular displacement $\theta(t)$ of the torsional system shown for the following cases:

1. The block has negligible mass
2. The block has a moment of inertia I

Note that the torsional stiffness of the mass-less bar is k and the directions of T and θ are similar, as shown in the figure.

Always think about the physics!



- Case 1: The block has negligible mass
- Time domain: Torque T causes bar to twist through an angle θ.

$$T = k\theta$$

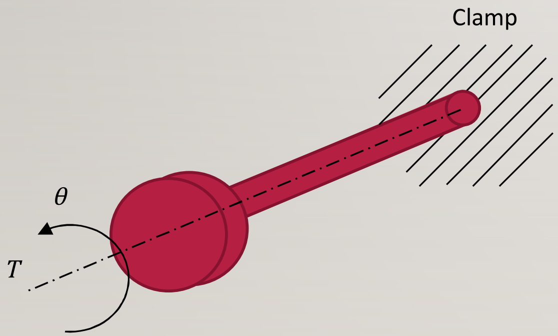
- Laplace domain – remember that k is a scalar constant so

$$T(s) = k\Theta(s)$$

- Transfer function:

$$\frac{\Theta(s)}{T(s)} = \frac{1}{k}$$

Case 2: Block has moment of inertia I



- Equation of motion: input torque causes block to move

$$T = I\ddot{\theta} + k\theta$$

- Laplace domain
  - $T(t) \xrightarrow{\mathcal{L}} T(s)$
  - $I\ddot{\theta}(t) \xrightarrow{\mathcal{L}} s^2 I\Theta(s)$
  - $k\theta(t) \xrightarrow{\mathcal{L}} k\Theta(s)$

$$T(s) = (Is^2 + k)\Theta(s)$$

Bonus: What does this mean?

$$T(s) = (Is^2 + k)\Theta(s)$$

Impulse response (wind up the clock and start the pendulum):

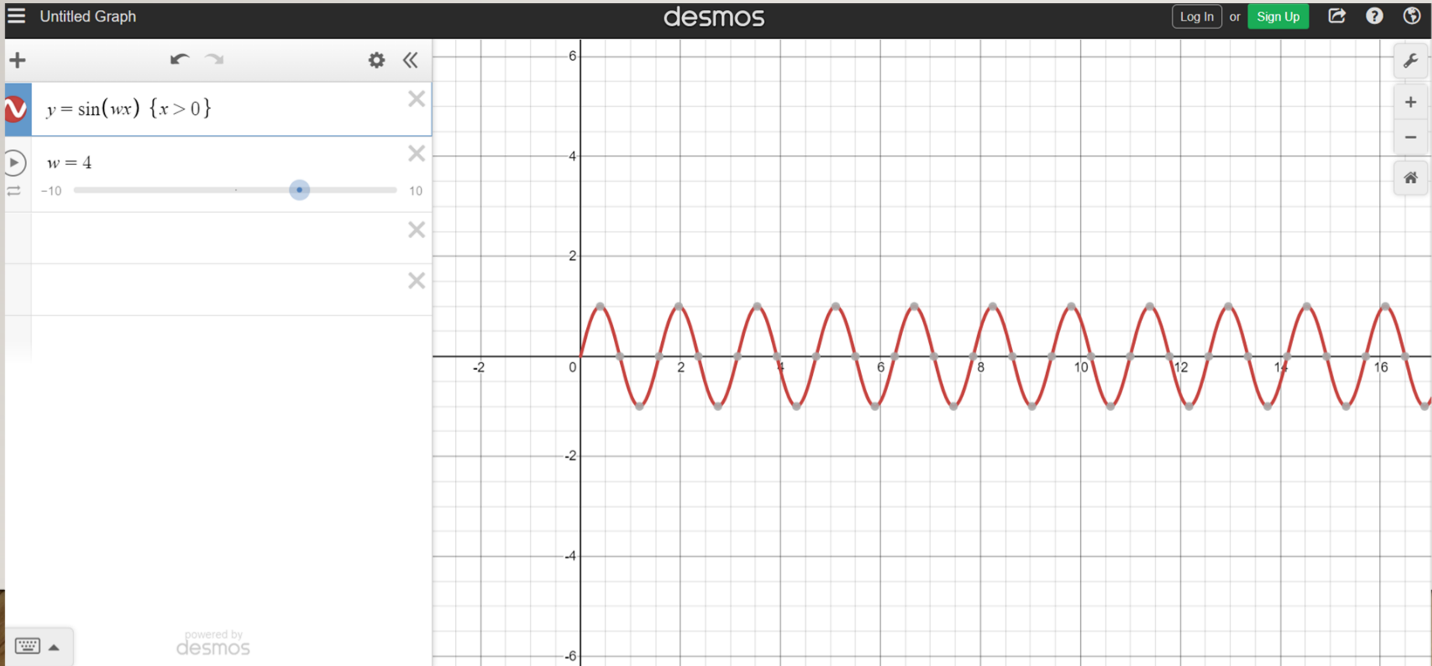
$$\Theta(s) = \frac{1}{Is^2 + k}$$

Inverse Laplace transforms give:

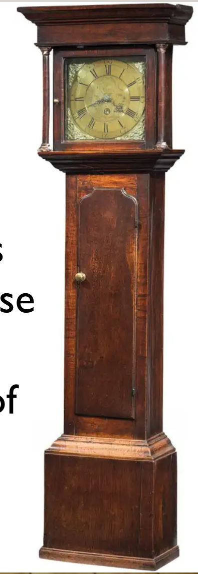
$$\theta(t) = \frac{1}{\sqrt{kI}} \sin \omega t \quad \text{where} \quad \omega = \sqrt{\frac{k}{I}}$$

Bonus: What does this mean?

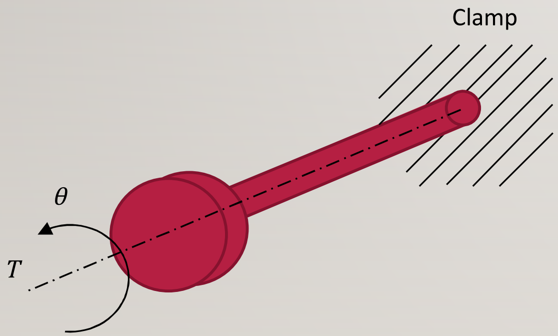
Harrison's chronometer – rotary pendulum not affected by the motion of the ship – solved the longitude problem.



My ancestors made these things (Ritchie of Methven)



HOW TO – EXAMPLE SHEET 1, NO. 2



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Note that the torsional stiffness of the mass-less bar is k and the directions of T and θ are similar, as shown in the figure.

IMPORTANT TAKE-AWAYS

- Laplace transforms make solving differential equations much more straightforward.
- Control modelling is about stability: the denominator of the transfer function tells you a lot about the system behaviour (more later)