DYNAMICS AND CONTROL

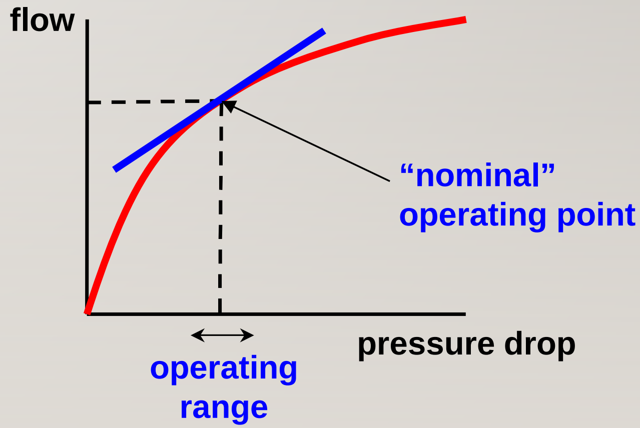
CONTROL SEMINAR 2

GENERAL INTRODUCTION – SESSION 2

- Non-linearity and the operating point
- 1st order systems
- Characterising the system response

NON-LINEARITY AND THE OPERATING POINT

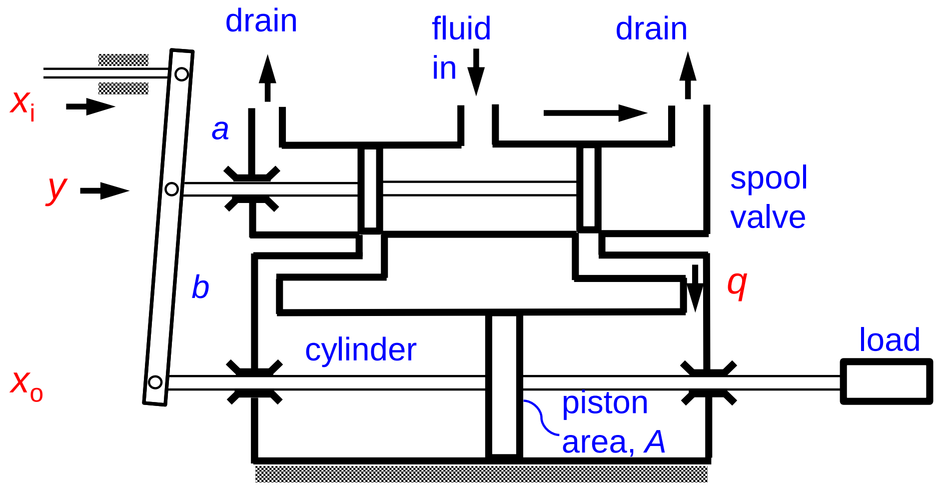
- Remember that the control system is maintaining the operating condition close to an optimum efficiency – only small variations from this will occur in practice.
  - Speed up, slow down, no reversal of direction.
- Non-linearities such as: backlash, coulomb friction, clearance, and saturation should not come into play around this point.



Brilliant idea no. 2

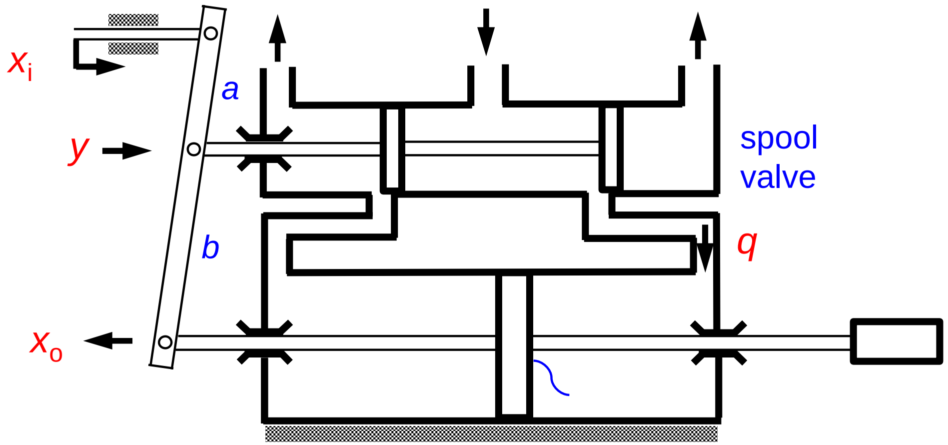
- Hydraulic Position Control
  - Also known as servo-assistance
  - Aeroplane flaps
  - Car brakes
  - Power steering (some cars)
  - Tractors and JCBs!

Hydraulic Position Control System



- How it works
  - Operator changes setting (x_i)
  - Piston is fulcrum – spool valve (y) translates
  - Spool valve admits fluid into cylinder

Case Study: Hydraulic Position Control System

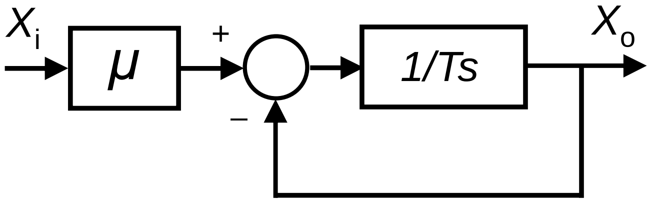


We showed that the **transfer function is:**

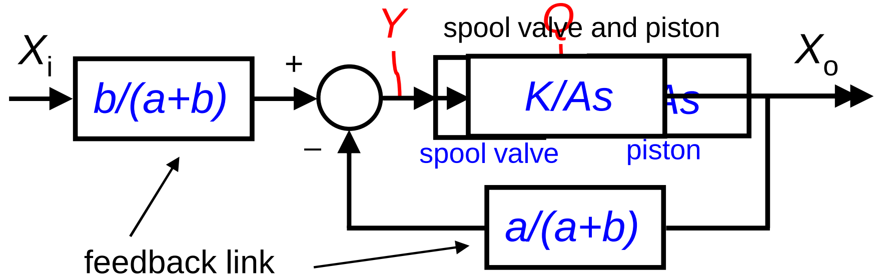
$$\frac{X_0(s)}{X_i(s)} = \frac{\frac{b}{a}}{1+\left(\frac{A(a+b)}{Ka}\right)s} \quad \text{OR} \quad G(s) = \frac{X_o(s)}{X_i(s)} = \frac{\mu}{1+Ts}$$

1st order system

with the **block diagram**



Hydraulic Position Control System: Overall Transfer Function



From the block diagram

$$X_o(s) = \left[X_i(s)\frac{b}{a+b} - X_o(s)\frac{a}{a+b}\right]\frac{K}{As}$$

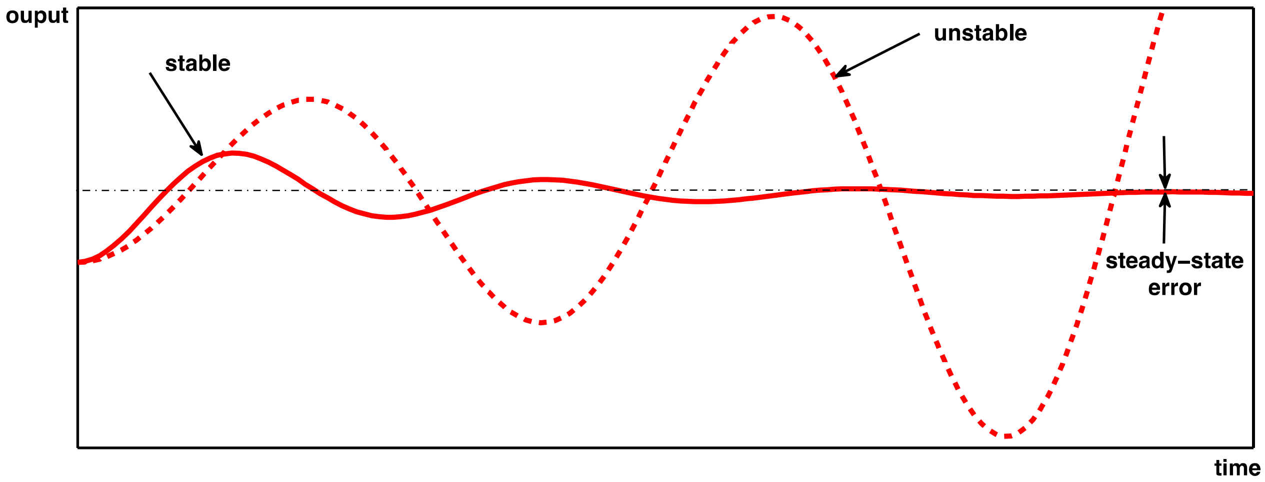
rearranging

$$\left[1 + \frac{A(a+b)s}{Ka}\right]X_o(s) = \frac{b}{a}X_i(s)$$

$$\frac{X_0(s)}{X_i(s)} = \frac{\frac{b}{a}}{1 + \left(\frac{A(a+b)}{Ka}\right)s}$$

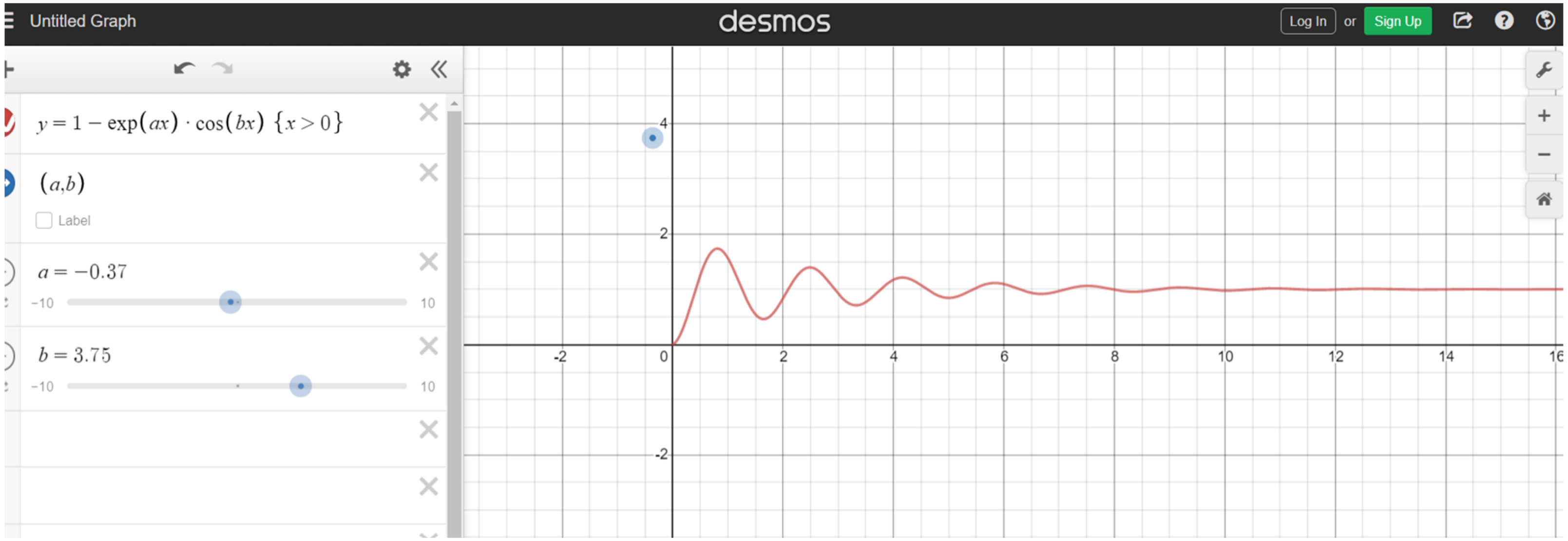
Stability

i) **Is the System Stable?**

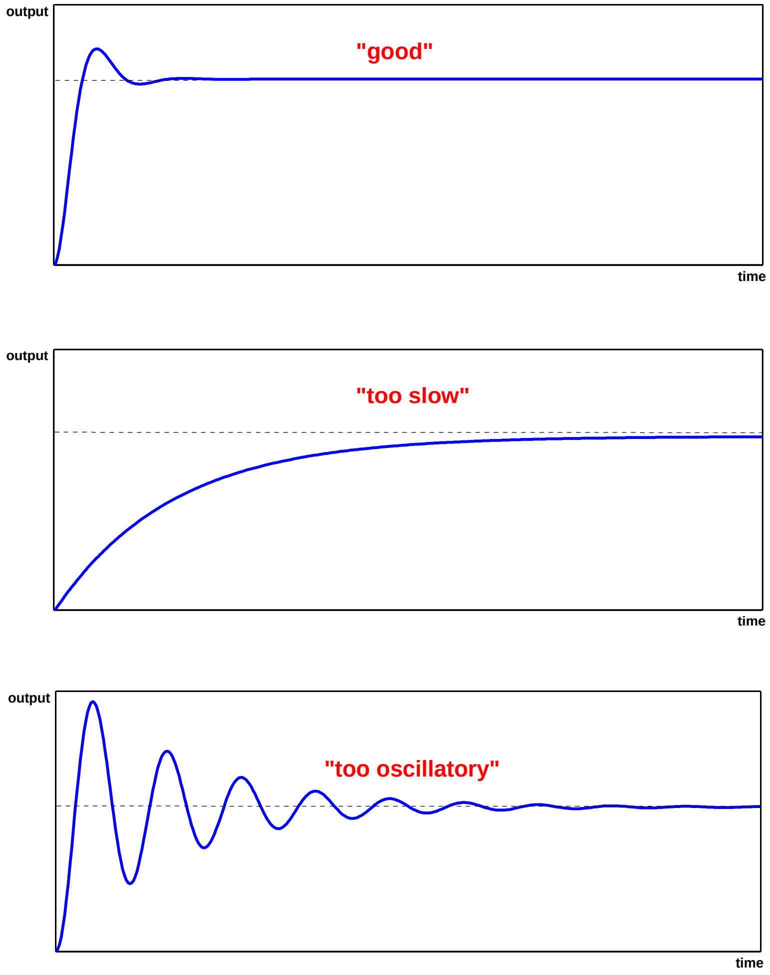


ii) **How Accurate is the System in Steady State?**

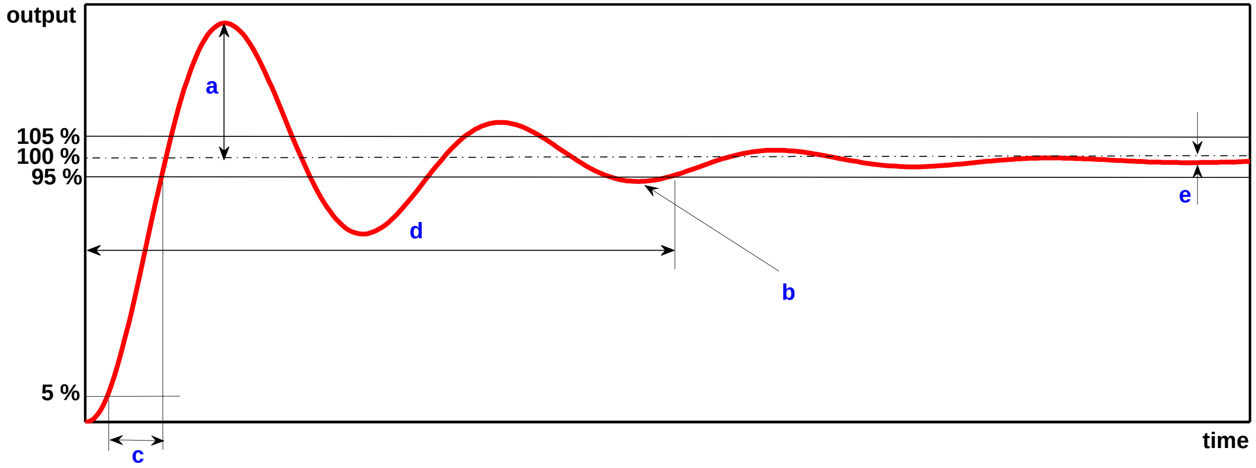
Oscillatory step response



iii) **How Quickly Does the System Reach a Steady State?**



Practical Measures of Transient Response



a) Maximum Overshoot as a percentage of step size.

b) Number of Oscillations before system settles to within a fixed percentage (5% say) of its steady state value.

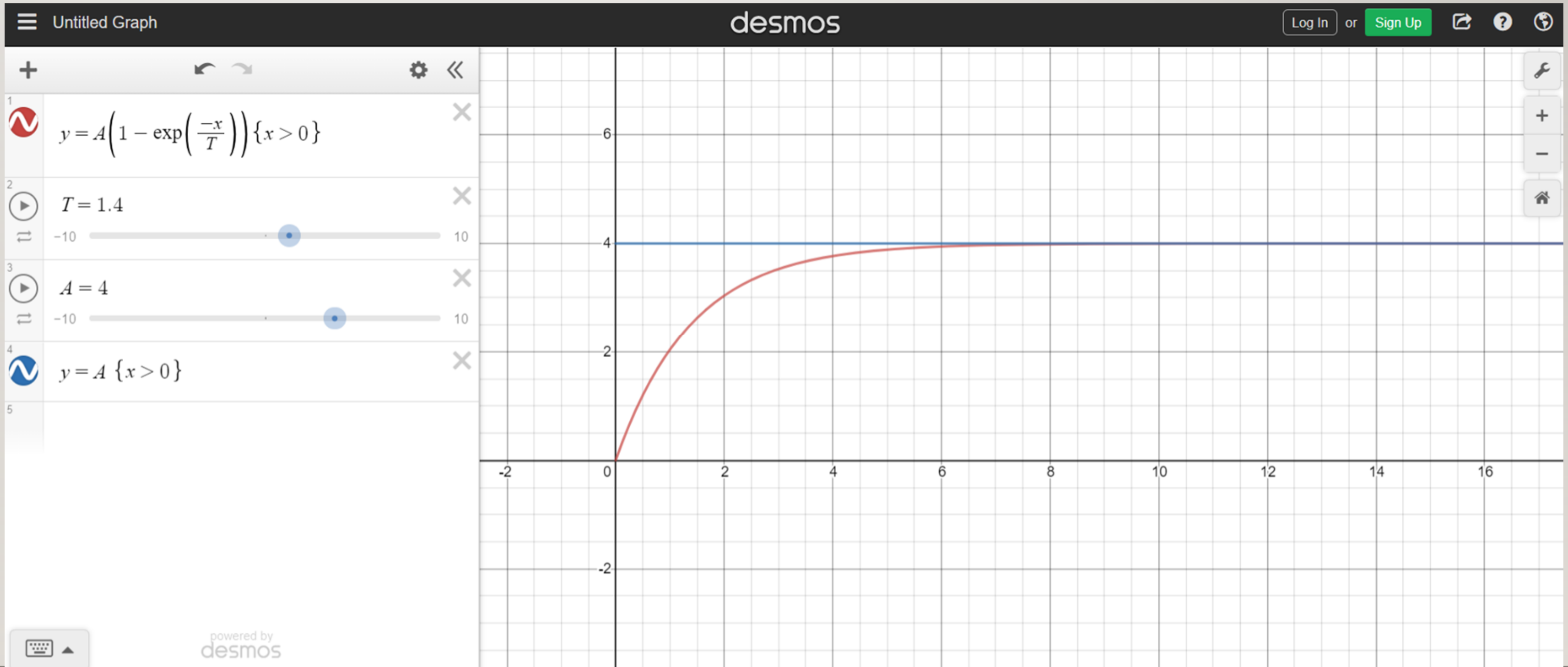
c) Rise Time: The time taken for output to rise from 5% to 95% of step size.

d) Settling Time: The time taken for output to reach and remain within ±5% of steady state value.

e) Steady State Error

1ST ORDER SYSTEMS

- Characteristic transfer function:
- $G(s) = \frac{A}{MS+C} \equiv \frac{\mu}{1+ST}$
- Step response:
  - $y = \mu(1 - e^{-t/T})$
  - (stable system!)

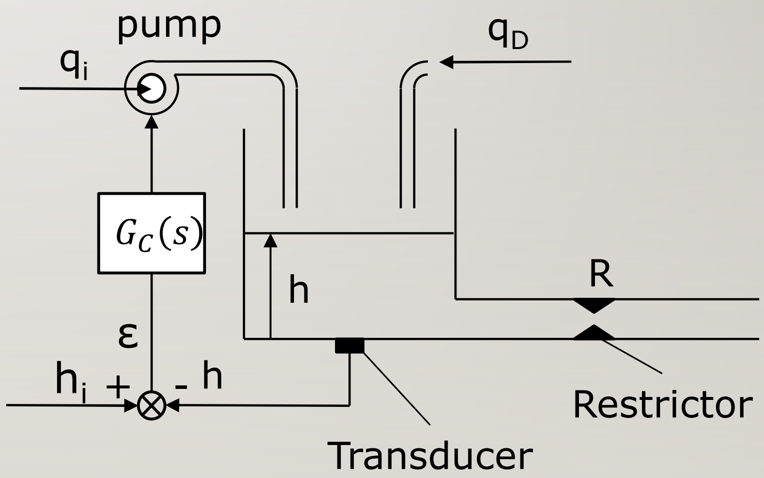


EXAMPLE SHEET 3 QUESTION 2

Figure Q2 illustrates a simple system for controlling the level of liquid in a tank with uniform cross-sectional area A. The error signal ε is derived by comparing the actual height h with the desired level h_i, and is fed to a controller which drives a variable speed pump such that the controlled volumetric inflow rate q_i to the tank is given by:

$$Q_i(s) = G_C(s)\varepsilon(s)$$

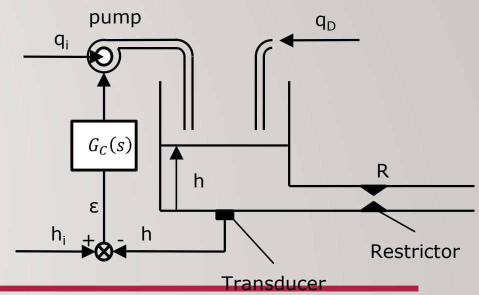
where $G_C(s)$ is the transfer function of the controller. In addition, there is an uncontrolled disturbance inflow to the tank given by $Q_D(s)$. The tank outflow passes through a restriction with linearised flow resistance R.



For the case when the controller is a proportional controller with gain K, such that $G_C(s) = K$

a) Derive the overall transfer function relating h to h_i and Q_D and show that the system is first order;

STAGE I

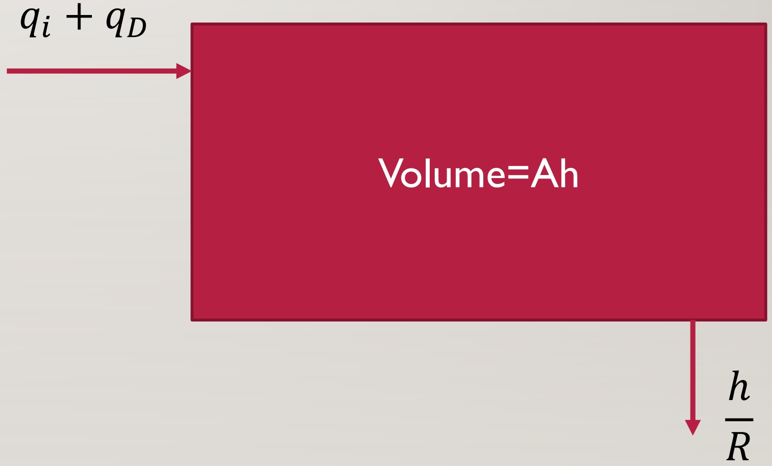


- Dynamics for the tank – we need an expression for h:
- In the time domain:
- Volume of tank = Ah(t)

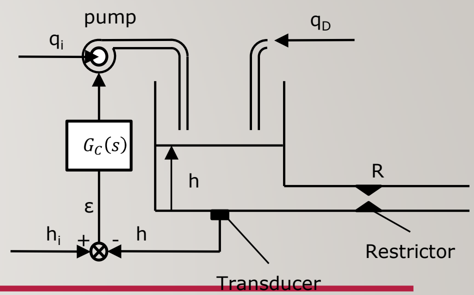
$$\frac{d}{dt}\big(Ah(t)\big) = q_i + q_D - \frac{h}{R}$$

In the Laplace domain,

$$sAH(s) = Q_i(s) + Q_D(s) - \frac{H(s)}{R}$$

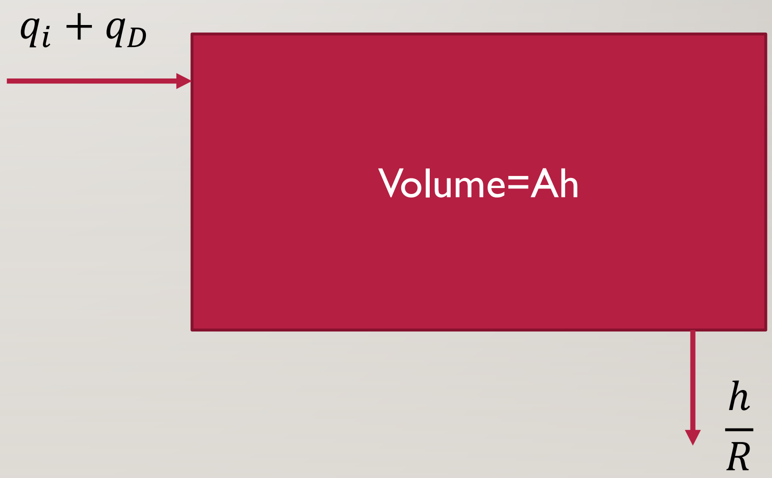


STAGE I



$$sAH(s) = Q_i(s) + Q_D(s) - \frac{H(s)}{R}$$

$$\frac{sARH(s) + H(s)}{R} = Q_i(s) + Q_D(s)$$

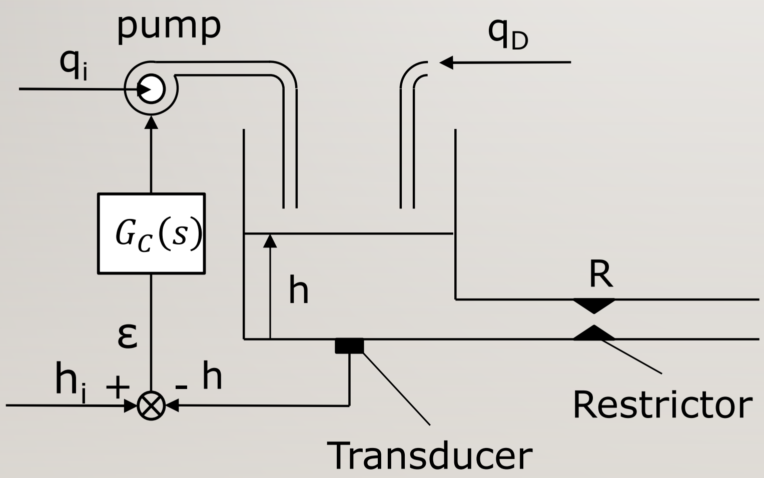


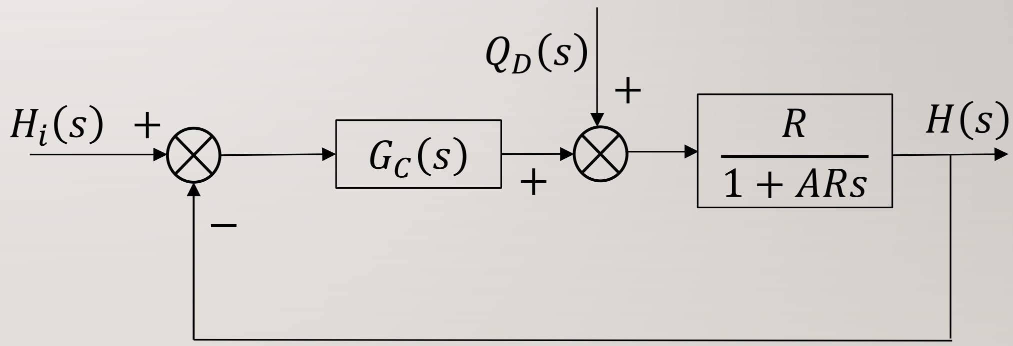
Transfer functions:

$$\frac{H(S)}{Q_i(s)} = \frac{R}{1 + ARs}$$

$$\frac{H(S)}{Q_D(s)} = \frac{R}{1 + ARs}$$

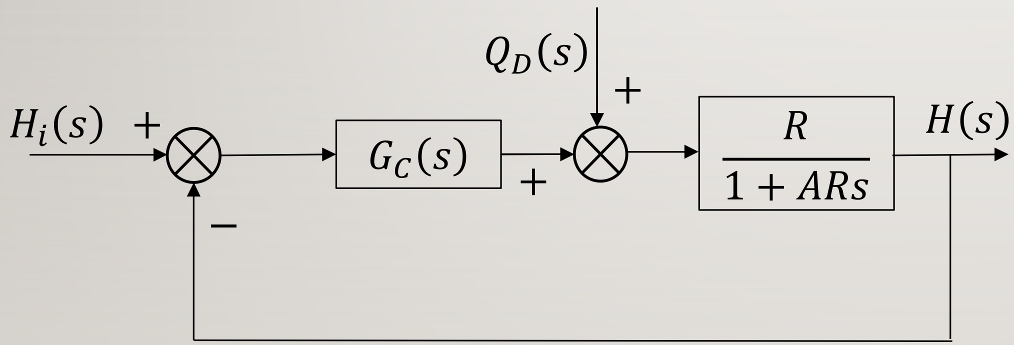
STAGE 2: MAKE THE BLOCK DIAGRAM





Note: Transfer function derived previously describes relationship between Q and H – schematic shows that H is fed back to the summing junction

STAGE 3: OVERALL TRANSFER FUNCTIONS



$$H(s) = \left(H_i(s) - H(s)\right)\left(\frac{G_C(s)R}{1+ARs}\right) \qquad (1)$$

$$H(s) = \left(Q_D(s) - G_C(s)H(s)\right)\left(\frac{R}{1+ARs}\right) \qquad (2)$$

Rearrange to give Transfer functions:

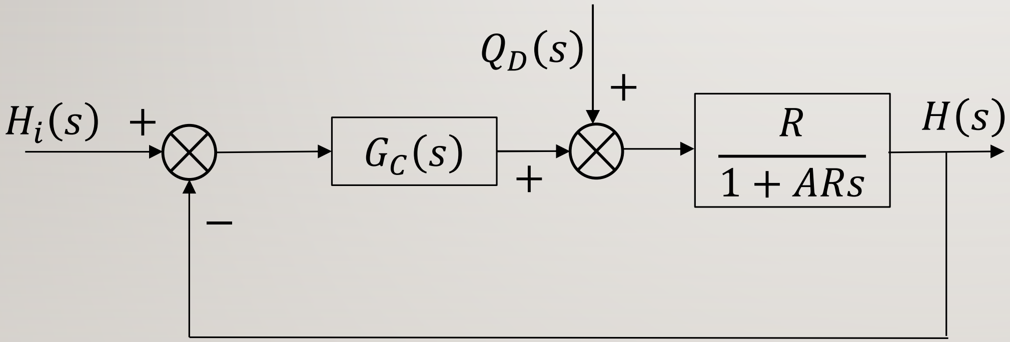
From (1):

$$H(s)\left(1+\frac{G_C(s)R}{1+ARs}\right) = \frac{H_i(s)G_C(s)R}{1+ARs}$$

$$H(s)(1+ARs+G_C(s)R) = H_i(s)G_C(s)R$$

$$\frac{H(s)}{H_i(s)} = \frac{G_C(s)R}{(1+ARs+G_C(s)R)}$$

STAGE 3: OVERALL TRANSFER FUNCTIONS



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$$H(s) = \left(Q_D(s) - G_C(s)H(s)\right)\left(\frac{R}{1+ARs}\right) \qquad (2)$$

Rearrange to give Transfer functions:

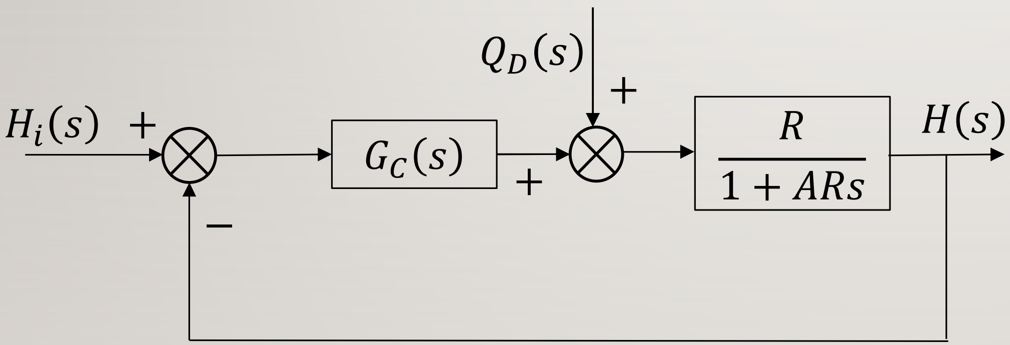
From (2):

$$H(s)\left(1 + \frac{G_C(s)R}{1+ARs}\right) = \frac{Q_D(s)R}{1+ARs}$$

$$H(s)(1 + ARs + G_C(s)R) = Q_D(s)R$$

$$\frac{H(s)}{Q_D(s)} = \frac{R}{(1 + ARs + G_C(s)R)}$$

STAGE 3: OVERALL TRANSFER FUNCTIONS



$$\frac{H(s)}{H_i(s)} = \frac{G_C(s)R}{(1 + ARs + G_C(s)R)}$$

$$\frac{H(s)}{Q_D(s)} = \frac{R}{(1 + ARs + G_C(s)R)}$$

For combined input and disturbance: $G_C(s) = K$

$$H(s) = \frac{KRH_i(s) + Q_D(s)R}{(1 + ARs + KR)}$$

Have we shown that the system is first order?

EXAMPLE SHEET 3 QUESTION 2

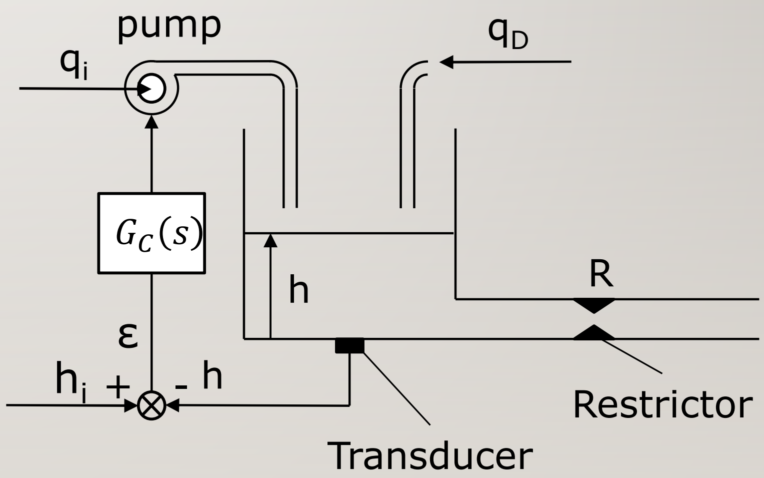
Figure Q2 illustrates a simple system for controlling the level of liquid in a tank with uniform cross-sectional area A. The error signal ε is derived by comparing the actual height h with the desired level h_i, and is fed to a controller which drives a variable speed pump such that the controlled volumetric inflow rate q_i to the tank is given by:

$$Q_i(s) = G_C(s)\varepsilon(s)$$

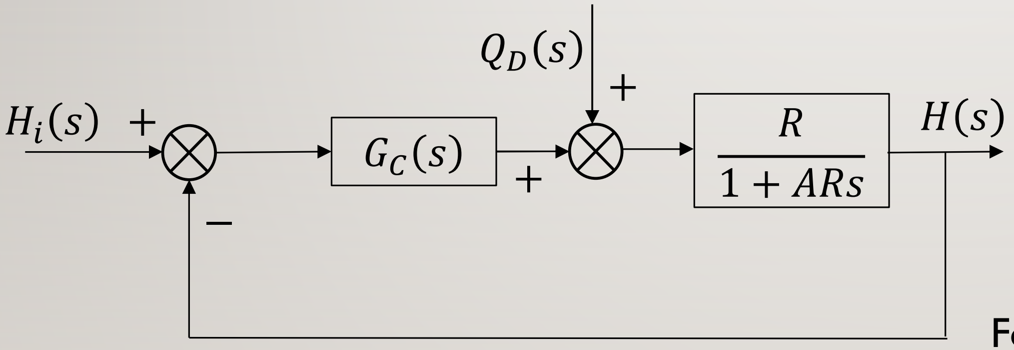
where $G_C(s)$ is the transfer function of the controller. In addition, there is an uncontrolled disturbance inflow to the tank given by $Q_D(s)$. The tank outflow passes through a restriction with linearised flow resistance R.

For the case when the controller is a proportional controller with gain K, such that $G_C(s) = K$

b) If the tank area $A = 2$ and the flow resistance $R = 10$ in consistent units, find the required value of the controller gain K to give a system time constant of 5 seconds.



STAGE 3: OVERALL TRANSFER FUNCTIONS



$$H(s) = \frac{KRH_i(s)}{(1 + ARs + KR)}$$

$A = 2 \quad R = 10$

$$\frac{H(s)}{H_i(s)} = \frac{10K}{(1 + 20s + 10K)}$$

For time constant, T, we must rearrange to give the form:

$$\frac{H(s)}{H_i(s)} = \frac{\mu}{(1 + Ts)}$$

So $\mu = \frac{10K}{10K+1}$ and $T = \frac{20}{1+10K}$

For $T = \frac{20}{1+10K} = 5s, K = 0.3$

EXAM 2019 QUESTION 4

- Compulsory part
- Written in a different style – it was set by colleagues at the Ningbo campus
- Mathematically similar …

4. Figure Q4 shows a block diagram for a simple control system comprising a proportional controller with $G_C(s) = K$. The input and output signals are $x_i(t)$ and $x_o(t)$, respectively, and their corresponding Laplace Transforms are $X_i(s)$ and $X_o(s)$, respectively,

   i. Determine the overall transfer function of the closed loop system relating $X_o(s)$ to $X_i(s)$. [3]

   ii. For a constant step input signal $X_i(s)$, find the expression for the steady-state output $x_o(t)$ in terms of K using the Final Value Theorem. [2]

   iii. Find the conditions for K for which the steady-state error $|x_o(t) - x_i(t)|$ is within 10% of a step input. [3]

   iv. Determine expressions for the natural frequency and damping ratio of the closed loop system. Calculate the value of K at which the system has a damping ratio of 0.5. Obtain the corresponding value of the natural frequency. [4]

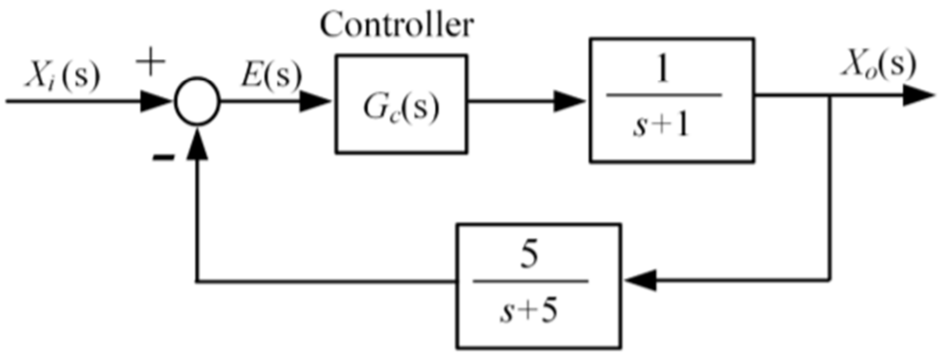


FIGURE Q4

PART I

i. Determine the overall transfer function of the closed loop system relating $X_o(s)$ to $X_i(s)$.

[3]

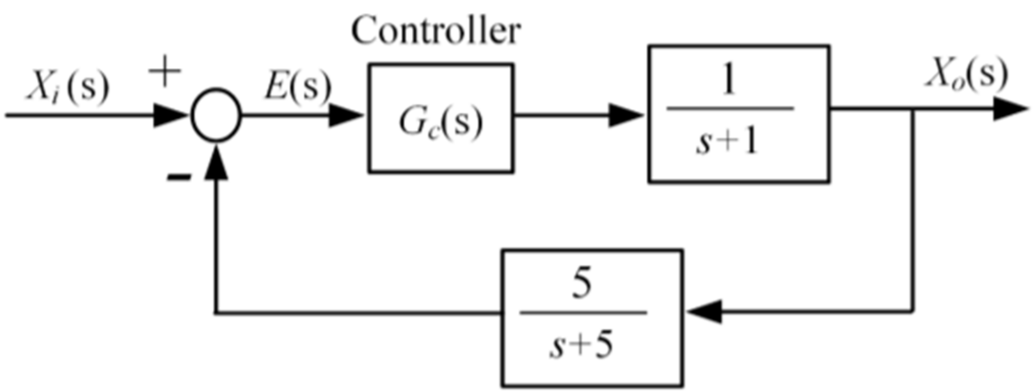
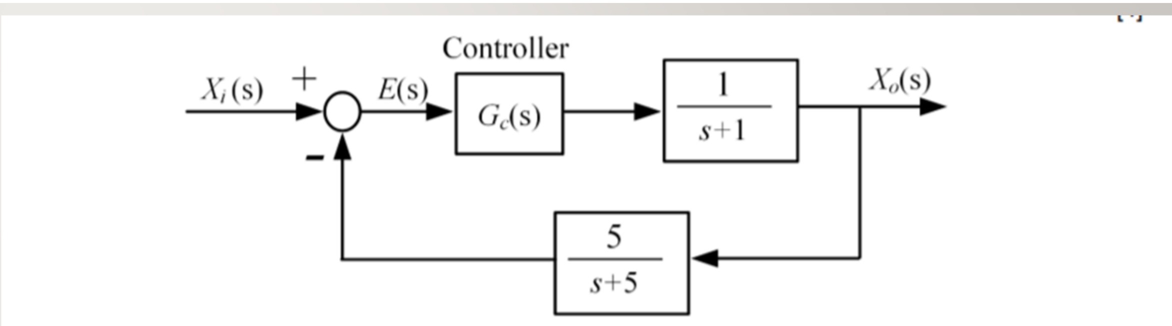


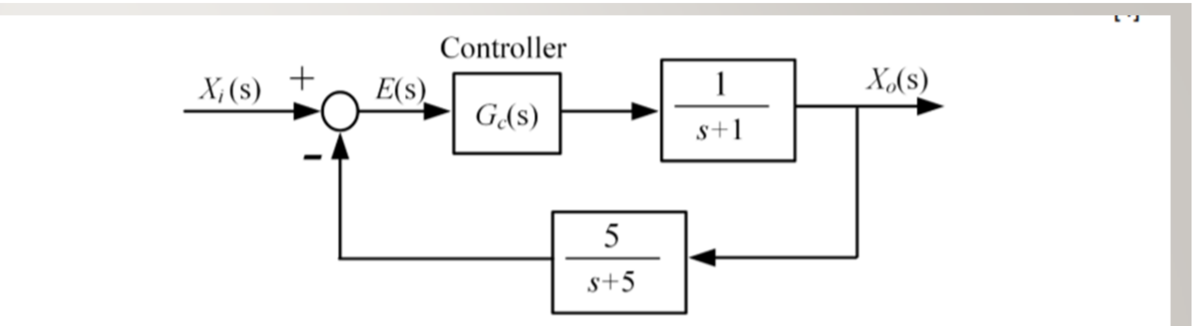
FIGURE Q4

PART I



- The question tells us that $G_c(s) = K$, so we can write the forward transfer function as $\frac{K}{s+1}$
- Error, E(s) is given by: $E(s) = X_i(s) - \left(\frac{5}{s+5}\right) X_o(s)$
- Using the blocks in the upper part of the diagram,
- $X_o(s) = E(s)\left(\frac{K}{s+1}\right) = \left(X_i(s) - \left(\frac{5}{s+5}\right) X_o(s)\right)\left(\frac{K}{s+1}\right)$

PART I

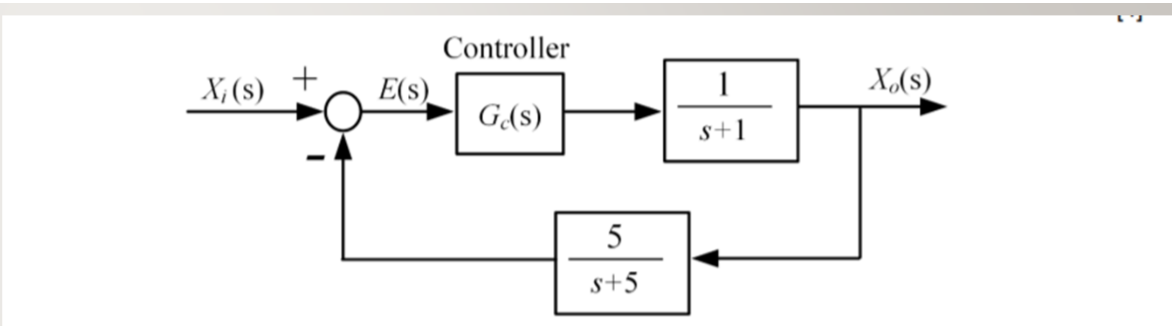


$$X_o(s) = E(s)\left(\frac{K}{s+1}\right) = \left(X_i(s) - \left(\frac{5}{s+5}\right)X_o(s)\right)\left(\frac{K}{s+1}\right)$$

- Multiply left and right by s+1: $(s+1)X_o(s) = KE(s) = K\left(X_i(s) - \left(\frac{5}{s+5}\right)X_o(s)\right)$
- Multiply both sides of the equation by s+5:
- $(s+1)(s+5)X_o(s) = K(s+5)X_i(s) - 5KX_o(s)$
- Rearrange to put terms in X_o on the left and X_i on the right:

$$(s+1)(s+5)X_o(s) + 5KX_o(s) = K(s+5)X_i(s)$$

PART I

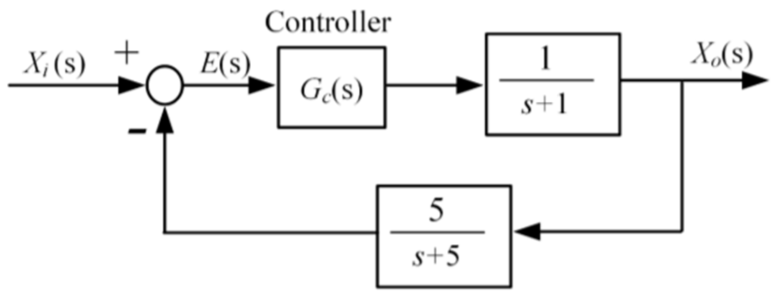


$$(s+1)(s+5)X_o(s) + 5KX_o(s) = K(s+5)X_i(s)$$

- The overall transfer function is therefore:

$$G(s) = \frac{X_o(s)}{X_i(s)} = \frac{K(s+5)}{(s+1)(s+5)+5K} = \frac{K(s+5)}{(s^2+6s+5+5K)}$$

PART II



ii. For a constant step input signal $X_i(s)$, find the expression for the steady-state output $x_o(t)$ in terms of K using the Final Value Theorem.

[2]

- Part ii): A unit step input is given by: $\frac{1}{s}$ and using the transfer function from (i) the output is:

$$X_o(s) = X_i(s)G(s) = \frac{K(s+5)}{s(s^2+6s+5+5K)}$$

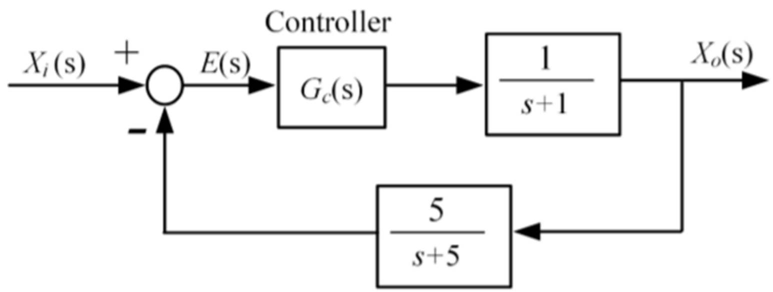
- Using the final value theorem **(remember to multiply by s!):**

$$\lim_{t\to\infty} x_o(t) = \lim_{s\to 0} sX_o(s) = \frac{sK(s+5)}{s(s^2+6s+5+5K)}$$

- In the limit, s tends to zero so this becomes:

$$\lim_{t\to\infty} x_o(t) = \lim_{s\to 0} sX_o(s) = \frac{sK(s+5)}{s(s^2+6s+5+5K)} = \frac{5K}{5+5K} = \frac{K}{1+K}$$

PART III



iii. Find the conditions for K for which the steady-state error $|x_o(t) - x_i(t)|$ is within 10% of a step input.

[3]

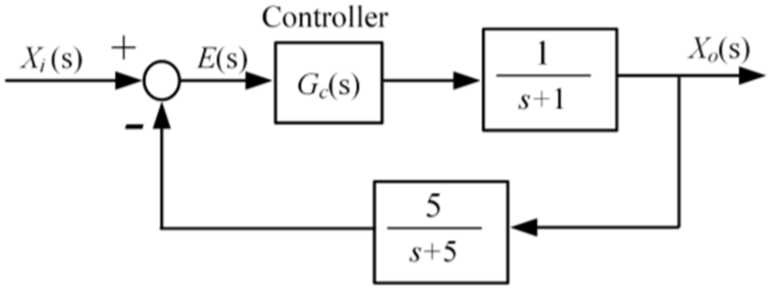
There are two ways to do this:

- The first is by following this reasoning: If the steady state error is within ten percent, then $0.9 \leq \lim_{t\to\infty} x_o(t) \leq 1.1$. So using the result from part (ii):

$$0.9 \leq \frac{K}{1+K} \leq 1.1$$

- This holds true for K>9 and K<-11. We would normally only consider positive values for K so K>9 is an acceptable answer.

PART III



iii. Find the conditions for K for which the steady-state error $|x_o(t) - x_i(t)|$ is within 10% of a step input.

[3]

- The second method is more formal: Begin by defining the error as a function of s:

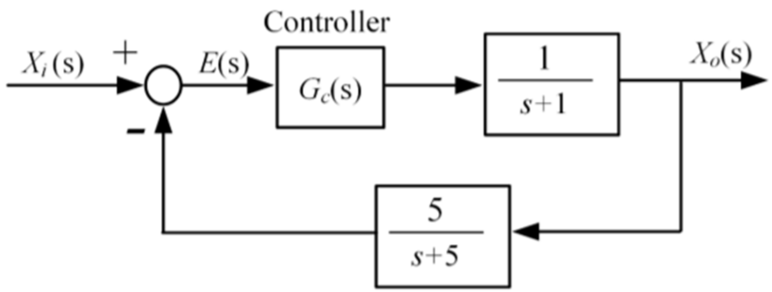
$$E(s) = X_o(s) - X_i(s) = \frac{K(s+5)}{s(s^2+6s+5+5K)} - \frac{1}{s}$$

$$E(s) = \frac{K(s+5) - (s^2+6s+5+5K)}{s(s^2+6s+5+5K)}$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{-s(s^2+(6-K)s+5)}{s(s^2+6s+5+5K)} = \frac{5}{5+5K} = \frac{1}{1+K}$$

- The error at steady state must be less than 10%, giving K>9

PART IV



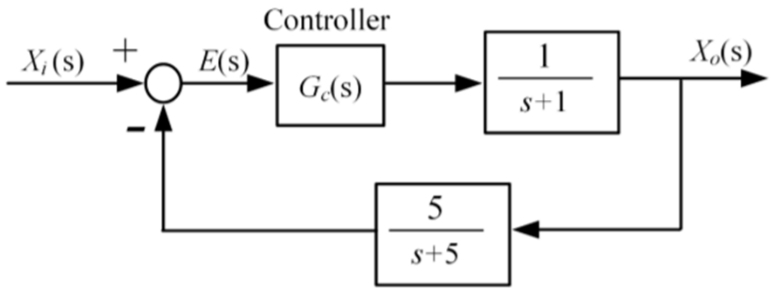
iv. Determine expressions for the natural frequency and damping ratio of the closed loop system. Calculate the value of K at which the system has a damping ratio of 0.5. Obtain the corresponding value of the natural frequency.

[4]

- My first tip is not to get involved in the numerator of the transfer function. All the information you need is in the denominator – as a reminder, here are the relevant transforms from the table.

16	$\frac{\omega}{\sqrt{1-\gamma^2}} e^{-\gamma\omega t} \sin\left(\omega t\sqrt{1-\gamma^2}\right)$	$\frac{\omega^2}{s^2+2\gamma\omega s+\omega^2}$
17	$1-\frac{e^{-\gamma\omega t}}{\sqrt{1-\gamma^2}} \sin\left(\omega t\sqrt{1-\gamma^2}+\varphi\right)$	$\frac{\omega^2}{s(s^2+2\gamma\omega s+\omega^2)}$
18	$t-\frac{2\gamma}{\omega}-\frac{e^{-\gamma\omega t}}{\omega\sqrt{1-\gamma^2}} \sin\left(\omega t\sqrt{1-\gamma^2}+\varphi\right)$	$\frac{\omega^2}{s^2(s^2+2\gamma\omega s+\omega^2)}$

PART IV



iv. Determine expressions for the natural frequency and damping ratio of the closed loop system. Calculate the value of K at which the system has a damping ratio of 0.5. Obtain the corresponding value of the natural frequency.

[4]

- The denominator (remember its other name, the characteristic equation) of a closed loop second order transfer function has the form:

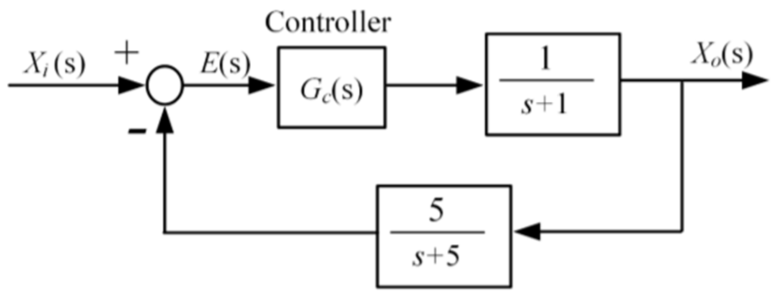
$$(s^2 + 2\zeta\omega_n s + \omega_n^2)$$

- Where ω_n is the natural frequency and ζ is the damping ratio.

- From the answer to part (i):

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 6s + 5 + 5K$$

PART IV



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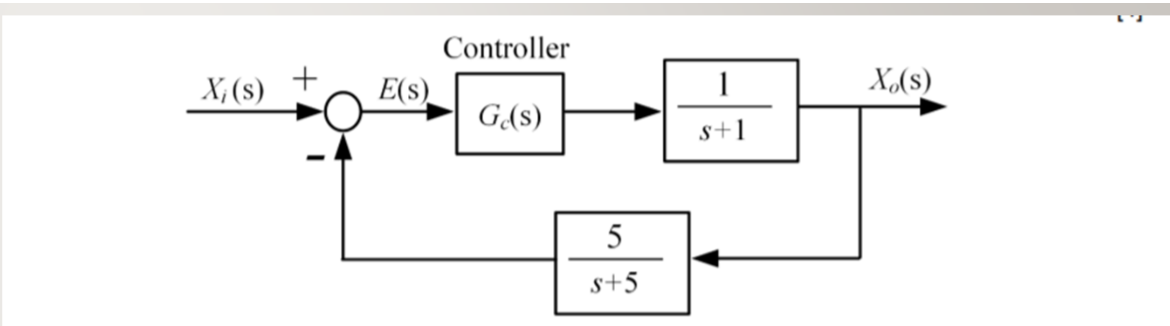
[4]

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 6s + 5 + 5K$$

- So the unit (s^0) terms give the natural frequency: $\omega_n^2 = 5 + 5K$
- Terms in s give the damping ratio: $2\zeta\omega_n = 6$
- $\zeta = \frac{3}{\omega_n} = \frac{3}{\sqrt{5+5K}} = 0.5$ (from the question). $\frac{3}{\sqrt{5+5K}} = 0.5$ and $\sqrt{5+5K} = 6$.
- It follows that $5 + 5K = 36$ and K=6.2

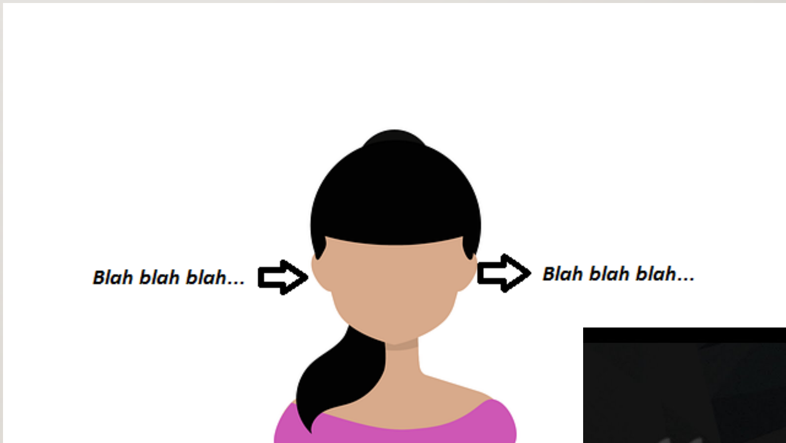
PART IV

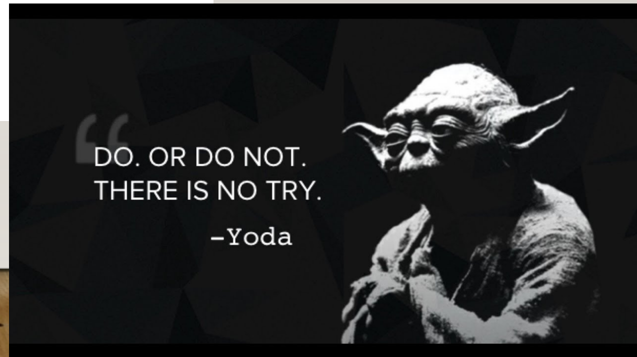


iv. Determine expressions for the natural frequency and damping ratio of the closed loop system. Calculate the value of K at which the system has a damping ratio of 0.5. Obtain the corresponding value of the natural frequency.

[4]

- K=6.2. To find ω_n:
- Either:
- $\omega_n^2 = 5 + 5K = 5 + 31 = 36$ and $\omega_n = 6$
- Or $2\zeta\omega_n = 6$, $\zeta = 0.5$ so $\omega_n = 6$.





THE END?

Any questions?