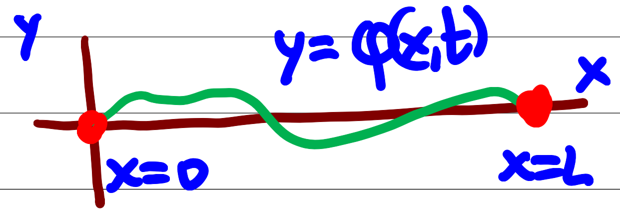
Minilecture 7D Further examples of separation of variables

Example



Find Separation of Variables (SoV) solutions for

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \frac{\partial^2 \varphi}{\partial x^2}$$

Let

$$\varphi(x,t) = X(x)\,T(t)$$

$$\Rightarrow X T'' = c^2 X'' T$$

$$\Rightarrow \underbrace{T''/T}_{\text{fn of } t} = \underbrace{c^2 X''/X}_{\text{fn of } x} = \alpha$$

$$\Rightarrow T''/T = \alpha \qquad c^2 X''/X = \alpha$$

$$\Rightarrow T'' - \alpha T = 0 \qquad X'' - \frac{\alpha}{c^2} X = 0$$

Alternative separation constant: $\lambda = -\alpha/c^2$

$$X'' + \lambda X = 0 \qquad T'' + c^2 \lambda T = 0$$

$\ulcorner$ auxiliary eqn $m^2 + \lambda = 0$ $\lrcorner$

Solution type depends on (sign of) λ:

(i) $\lambda > 0 \Rightarrow X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

$T = C\cos(c\sqrt{\lambda}t) + D\sin(c\sqrt{\lambda}t)$

sin missed in recording

(ii) $\lambda = 0 \Rightarrow X = A + Bx \quad T = C + Dt$

(iii) $\lambda < 0 \Rightarrow X, T =$ exponentials.

(will be eliminated by bc's)

Example Telegrapher's equation:

$$c^2 \frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial t^2} + a\frac{\partial \varphi}{\partial t} + b\varphi$$

a, b constants

Let

$$\varphi(x,t) = X(x)T(t)$$

$$\Rightarrow \quad c^2 X'' T = X T'' + a X T' + b X T$$

$$\Rightarrow \quad c^2 X''/X = \frac{T'' + aT' + b}{T} = -\lambda c^2$$

$$\Rightarrow \quad \underbrace{X'' + \lambda X = 0}_{\text{same as before}}, \quad \underbrace{T'' + aT' + (b + \lambda c^2)T = 0}_{\text{depends on } \lambda \text{ in a more complicated way}}.$$

Example The heat equation:

$$\partial\varphi/\partial t = D\,\partial^2\varphi/\partial x^2$$

Let

$$\varphi(x,t) = X(x)\,T(t)$$

$$\Rightarrow X T' = D X'' T$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{DT} = -\lambda$$

$$\Rightarrow X'' + \lambda X = 0 \qquad T' + D\lambda T = 0$$

(i) $\lambda > 0$ $\quad X = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t)$

$T = e^{-D\lambda t}$ ✓

(ii) $\lambda = 0$ $\quad X = A + Bx$ steady state ✓

$T = \text{const}$

(ii) $\lambda < 0$ $\quad X = \text{exponentials}$ ✗

$T = e^{-D\lambda t}$ is growing!

Example Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

Let

$$\varphi(x,y) = X(x) Y(y)$$

$$\Rightarrow \quad X'' Y + X Y'' = 0$$

$$\Rightarrow \quad X''/X = -Y''/Y = -\lambda$$

$$\Rightarrow \quad X'' + \lambda X = 0 \qquad Y'' - \lambda Y = 0$$

(i) $\lambda > 0$ $\quad X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

$\quad Y = C e^{-\sqrt{\lambda}y} + D e^{+\sqrt{\lambda}y}$ OK!

(ii) $\lambda = 0$ $\quad X = A + Bx$

$\quad Y = C + Dy$ OK!

(iii) $\lambda < 0$ same as (i) but x, y are exchanged

OK!

Example 2D wave equation

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right)$$

Let

$$\varphi(x,y,t) = \psi(x,y)\, T(t)$$

$$\Rightarrow \quad \psi T'' = c^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) T$$

$$\Rightarrow \quad \frac{T''}{T} = c^2 \, \frac{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}}{\psi} = -c^2 \lambda$$

$$\Rightarrow \quad T'' + c^2 \lambda T = 0 \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \lambda \psi = 0$$

(i) $\lambda > 0$ $\quad T = A\cos(c\sqrt{\lambda}\, t) + B\sin(c\sqrt{\lambda}\, t)$ ✓

(ii) $\lambda = 0$ $\quad T = A + Bt$ OK

(iii) $\lambda < 0$ $\quad T =$ exponentials ✗

(will be eliminated by bc's)

Example Helmholtz equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y} + k^2\psi = 0$$

Let

$$\psi(x,y) = X(x)\,Y(y)$$

$$\Rightarrow \quad X''Y + XY'' + k^2 XY = 0$$

$$\Rightarrow \quad X''/X + Y''/Y + k^2 = 0$$

$$\Rightarrow \quad X''/X = -k^2 - Y''/Y = -\lambda$$

$$\Rightarrow \quad X'' + \lambda X = 0 \qquad Y'' + \underbrace{(k^2 - \lambda)}_{\lambda' = k^2 - \lambda} Y = 0$$

(i) $\lambda > 0$ $\lambda' > 0$ $(0 < \lambda < k^2)$ — X, Y both sines cosines ✓

(ii) $\lambda < 0$, $\lambda' > 0$ — X exponentials, Y sines and cosines

(iii) $\lambda > 0$ $\lambda' < 0$ $(\lambda > k^2)$ — X sines cosines, Y exponentials

(iv) $\lambda = 0$ (v) $\lambda' = 0$ $(\lambda = k'')$ leave these out