MM2DYN Dynamics

Control Revision

Dr Alastair Campbell Ritchie

MM2DYN CONTROL TOPICS

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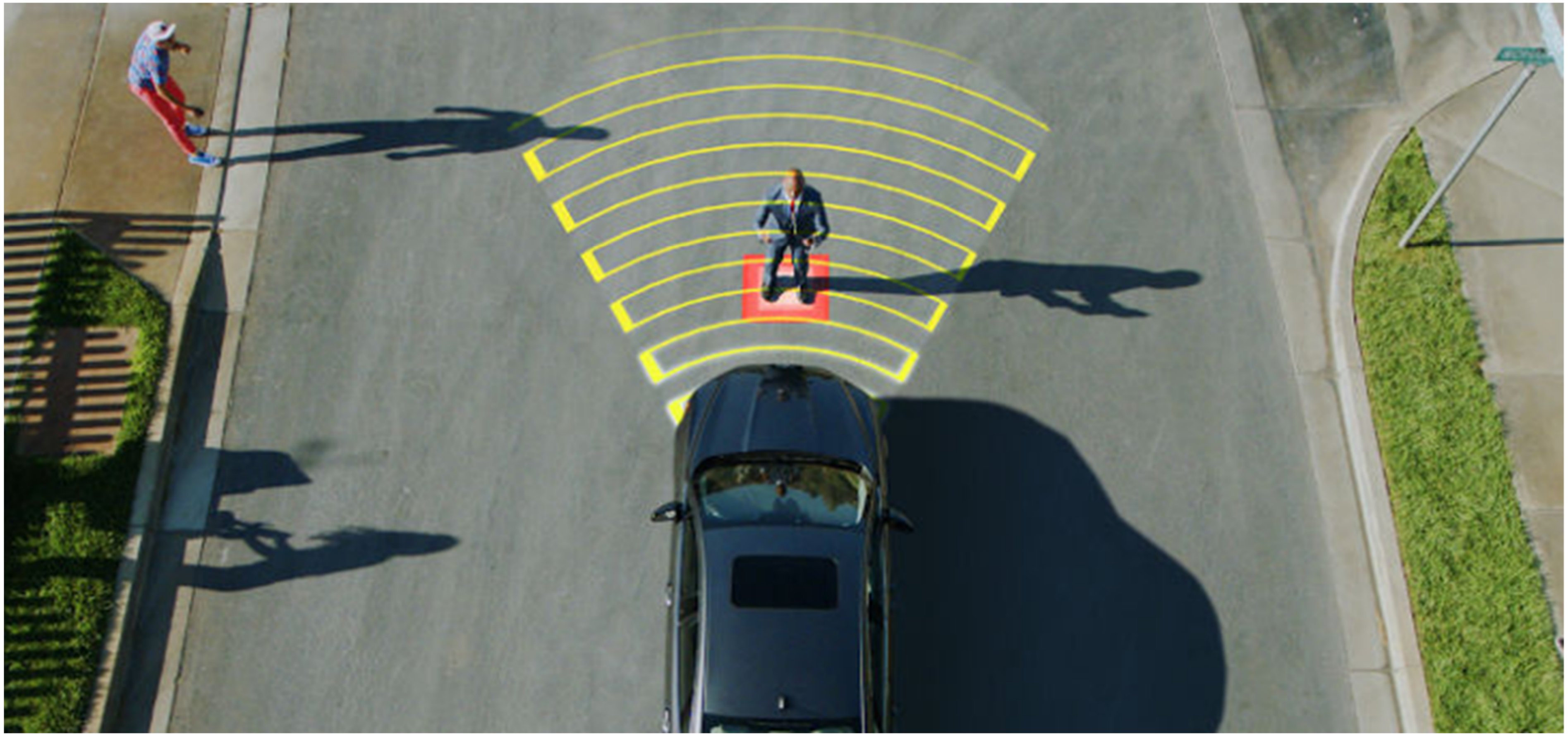
Learning Outcomes

- At the end of the lecture, you should:
  - Understand system modelling using Laplace Transforms
  - Know the difference between Open Loop and Closed Loop Feedback Control
  - Understand how to derive transfer functions using Block Diagram Manipulation or Algebraic Methods
  - Understand the concept of Root Locus and Stability
  - Be able to apply the Routh-Hurwitz Stability Criteria to determine if a system will be stable

Open Loop Control

https://www.youtube.com/watch?v=TOM-IGaI-7k

Closed Loop Control

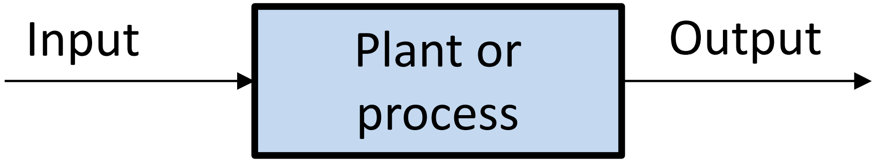


The key is feedback!

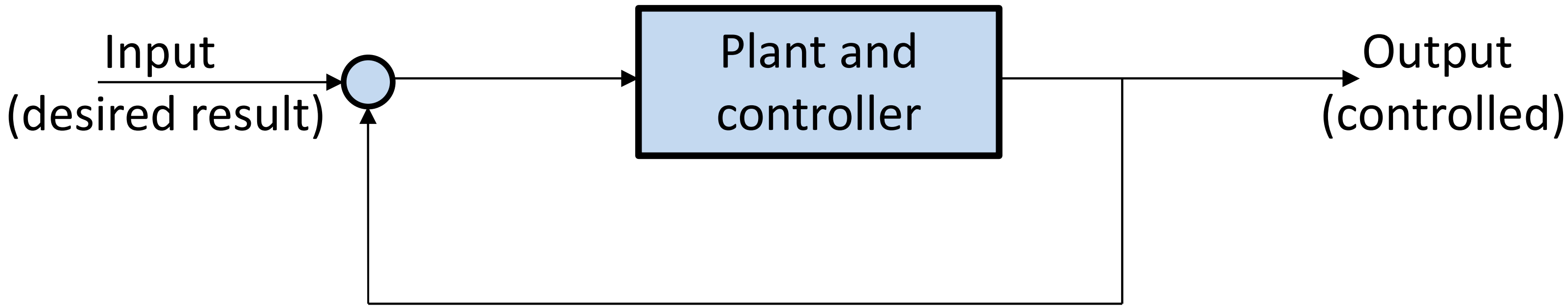
https://www.youtube.com/watch?v=TJgUiZgX5rE

Systems and block diagrams

- Open-Loop system

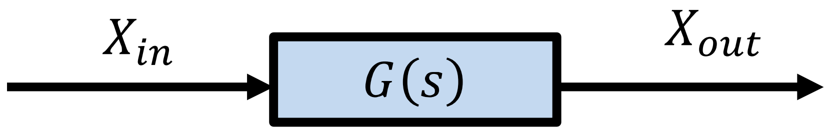


- Closed-Loop (feedback) system



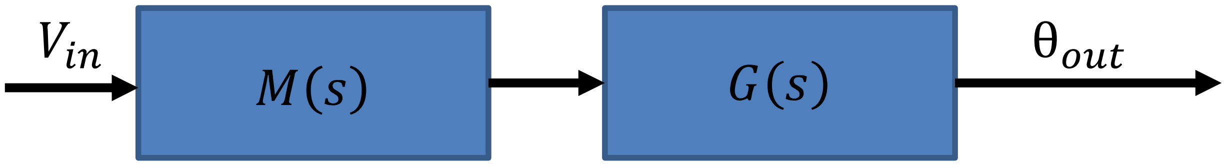
Representation of control systems

- What comes out = What goes in × transfer function.
- The block diagram for an element is drawn as follows:



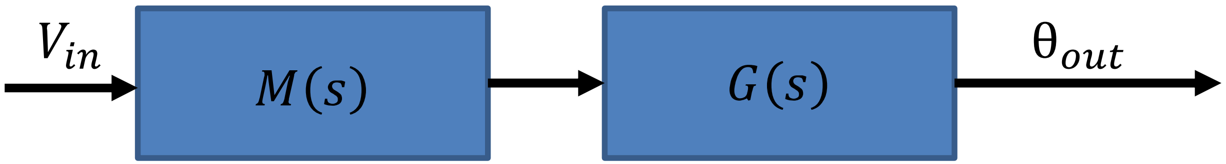
$$X_{out}(s) = G(s) \times X_{in}(s)$$

- Multiple elements: Geared Motor



Representation of control systems

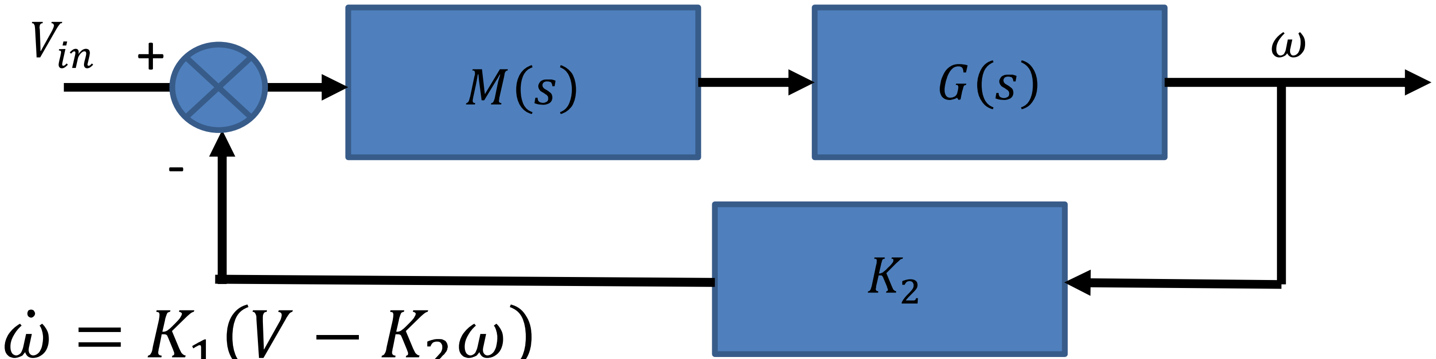
- Multiple elements: Geared Motor



- Motor – armature resistance, efficiency, inertia
- Gearbox – Gear Ratio, efficiency, inertia, viscous drag

Representation of control systems

- Multiple elements: Geared Motor



- $J_e\dot{\omega} = K_1(V - K_2\omega)$
  - V is the input voltage
  - J_e is the effective inertia of the system
  - K_1 is the combined gear ratio and armature characteristics, relating input voltage to acceleration
  - K_2 is the combined back EMF of the motor and viscous drag of the gearbox and motor
- Laplace Transform: $J_e s\Omega(s) = K_1\big(V(s) - K_2\Omega(s)\big)$

System Modelling

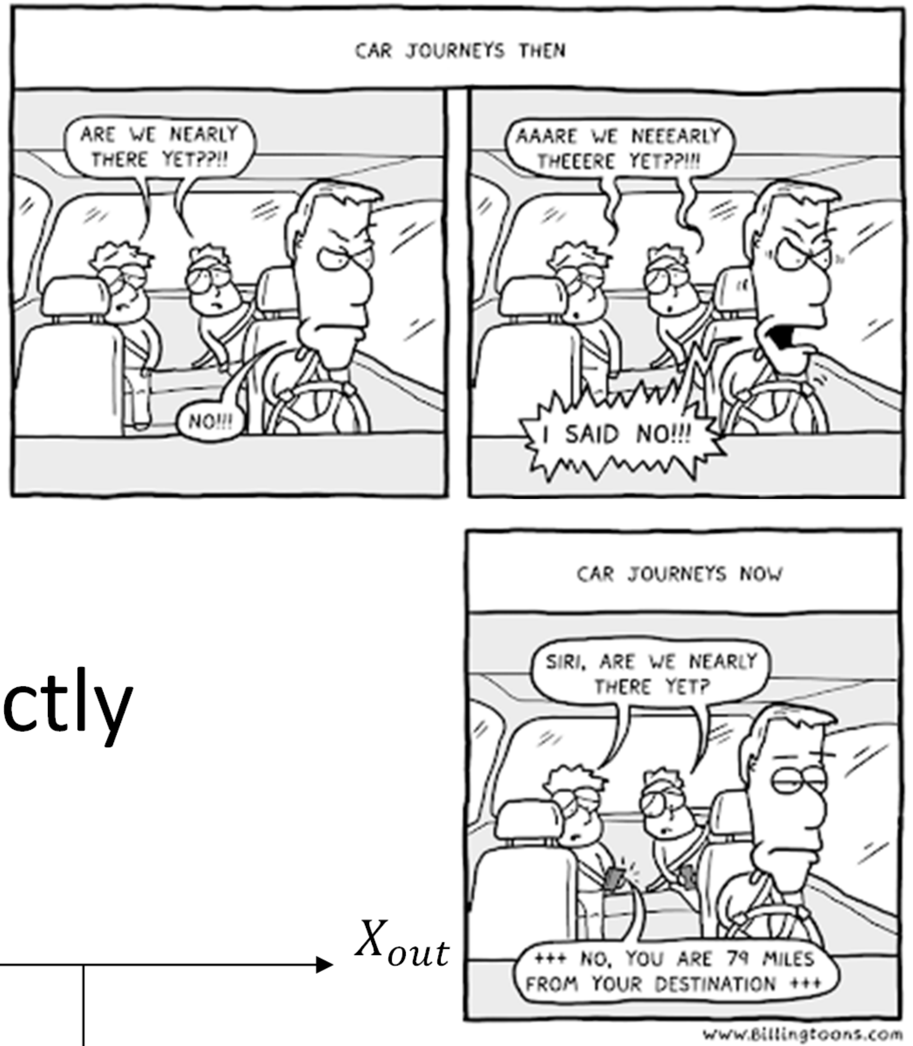
- Transfer function:

$$J_e s\Omega(s) = K_1\big(V(s) - K_2\Omega(s)\big)$$
$$J_e s\Omega(s) + K_1K_2\Omega(s) = K_1V(s)$$
$$\frac{\Omega(s)}{V(s)} = H(s) = \frac{K_1}{J_e s + K_1K_2}$$

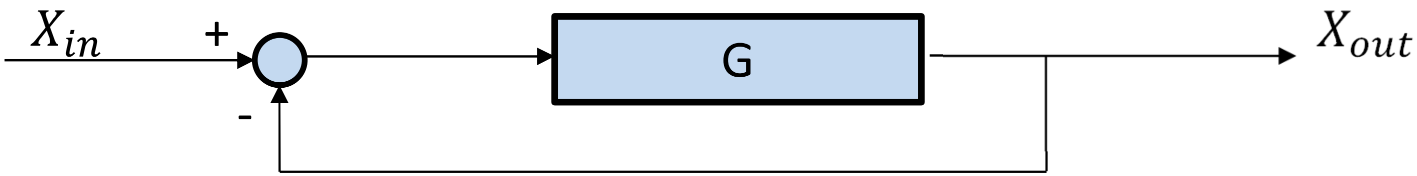
Note: Angular velocity is related to input voltage – output torque would need a different TF.

Load on output shaft – need to add to transfer function (separate input)

Feedback Control



- How the system knows:
  - Where you currently are
  - Where you need to go
- When output can be directly compared to input:



- More commonly:

http://www.billingtoons.com/2016/01/are-we-nearly-there-yet.html

Response to common inputs

- Switching the system on:
  - Unit step: $X(s) = \frac{1}{s}$
  - Ramp function: $x(t) = at \quad X(s) = \frac{a}{s^2}$
- For our geared motor: response to step voltage input:

$$\Omega(s) = H(s)V(s) = \frac{K_1}{J_e s + K_1 K_2} \times \frac{1}{s} = \frac{K_1}{s(J_e s + K_1 K_2)}$$

Response to common inputs

- Response to step voltage input:

$$\Omega(s) = \frac{K_1}{s(J_e s + K_1 K_2)}$$

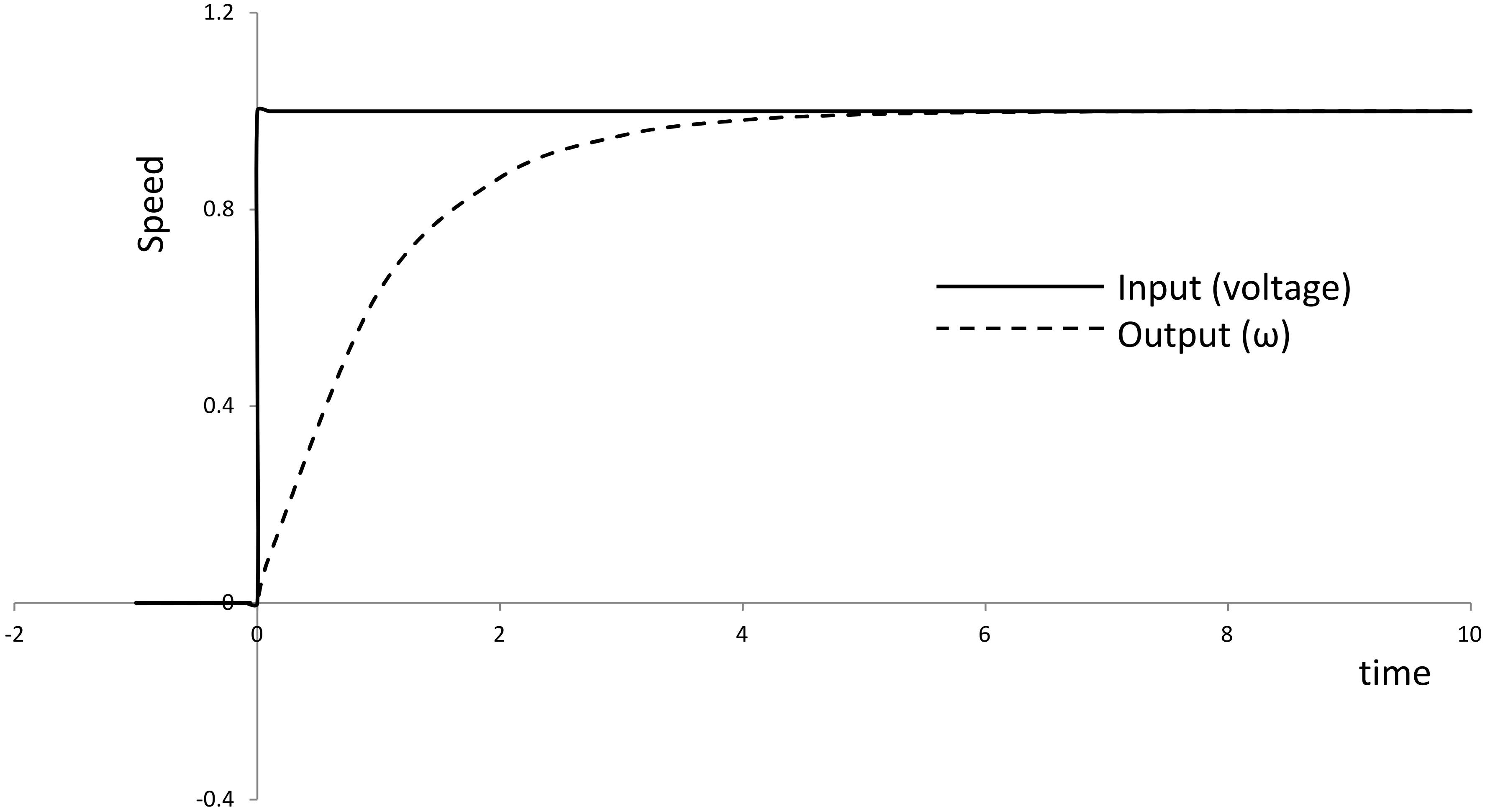
From Table of reverse Laplace transforms:

$$\frac{1}{a}(1 - e^{-at}) \rightarrow \frac{1}{s(s+a)}$$

$$\Omega(s) = \frac{K_1}{J_e}\frac{1}{s(s+b)} \ldots b = \frac{K_1 K_2}{J_e}$$

$$\omega(t) = \frac{1}{K_2}\left(1 - e^{\frac{-K_1 K_2 t}{J_e}}\right)$$

Response of geared motor to unit step input



Steady State Error

- Difference between input and output at

  $t = \infty$

  – Corresponds to $s = 0$

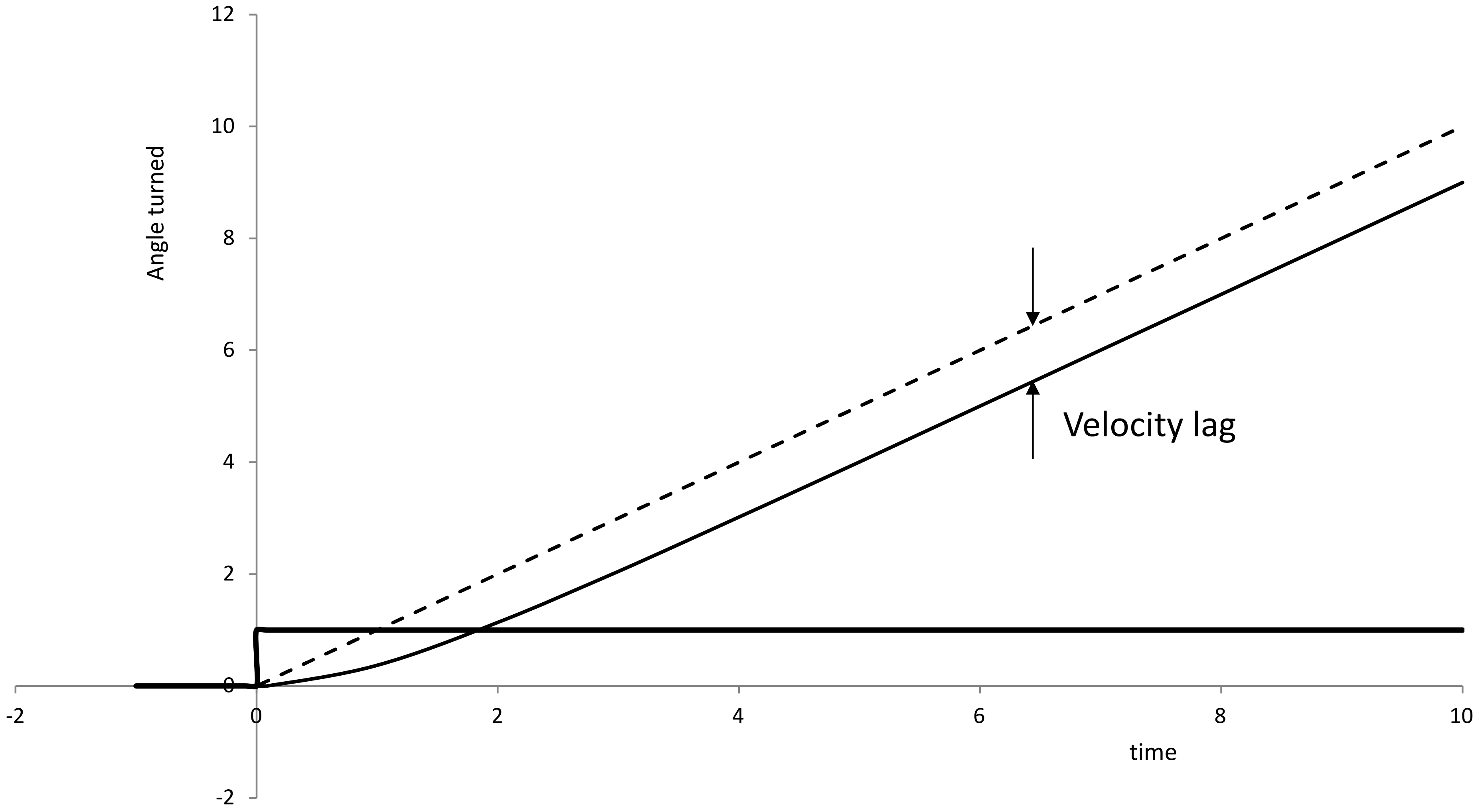
  $$E(s) = X(s) - Y(s) = \big(1 - G(s)\big)X(s)$$

- Final Value Theorem:

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s\big(1 - G(s)\big)X(s)$$

For Rotary systems (continuously moving) – velocity lag is equivalent to steady state error

Response of geared motor to unit step input

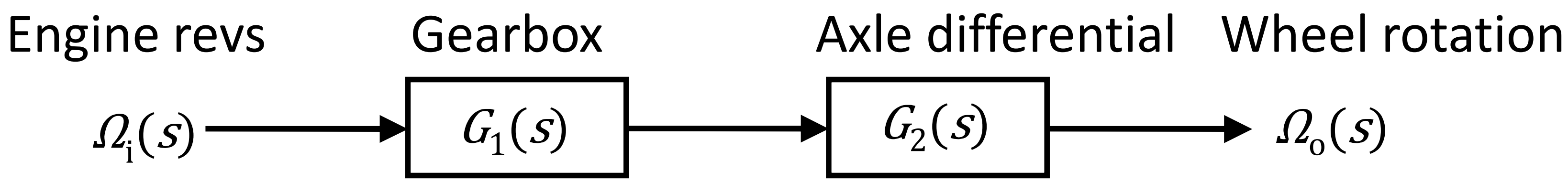


Improving System response

- PID Controller
  - Adds Proportional, Integrator and Derivative terms
  - Reduces steady state error and response lag
  - May cause Oscillation

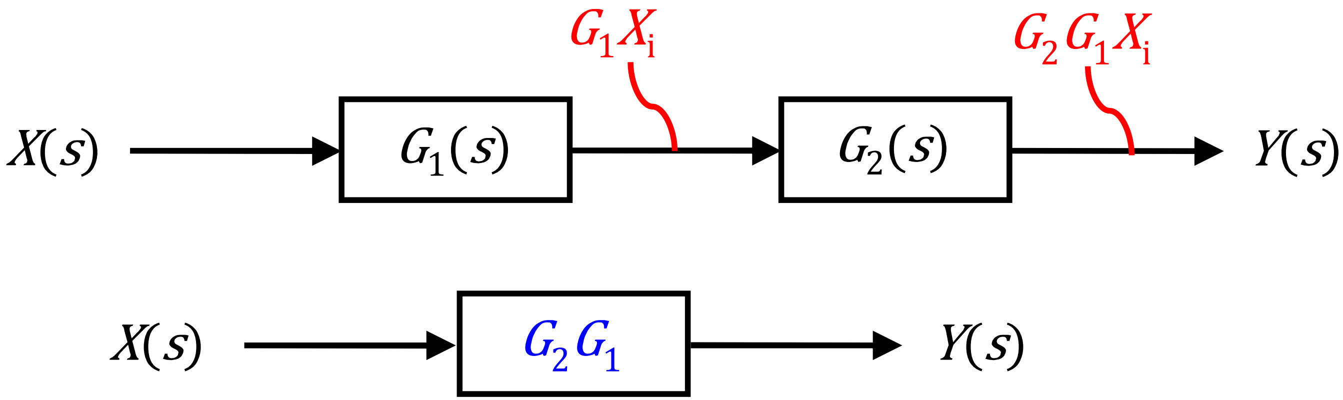
Part 2 – Block Diagram Manipulation

- As an example, think about a car transmission:



Block Diagram Manipulation: Basic Rules

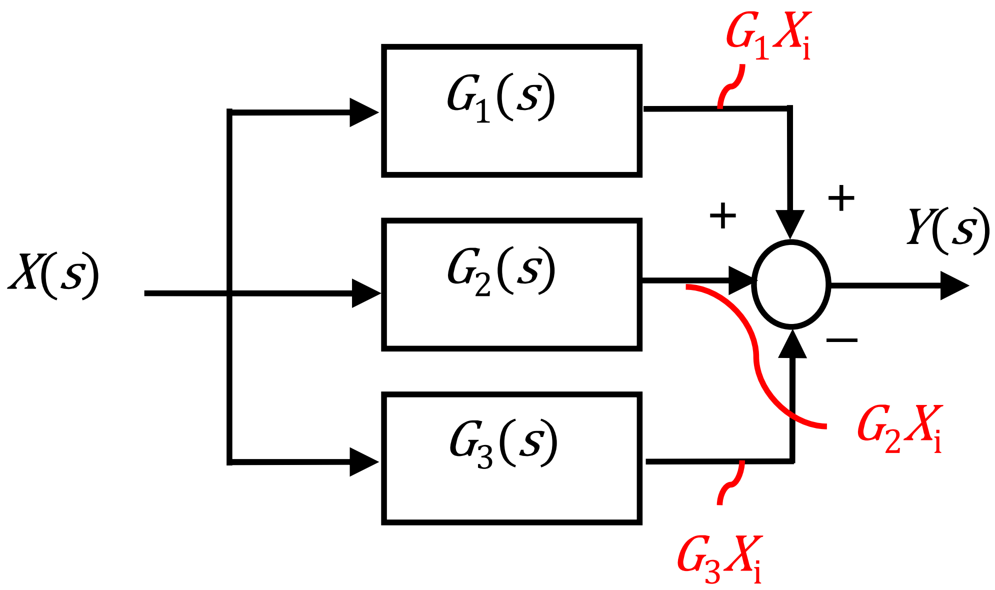
a) Elements in Series: Multiplication



$$Y(s) = G_1(s)G_2(S)X(s)$$

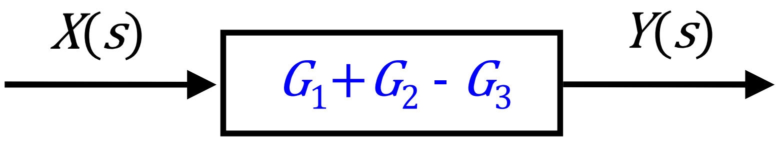
Block Diagram Manipulation: Basic Rules

b) Elements in Parallel



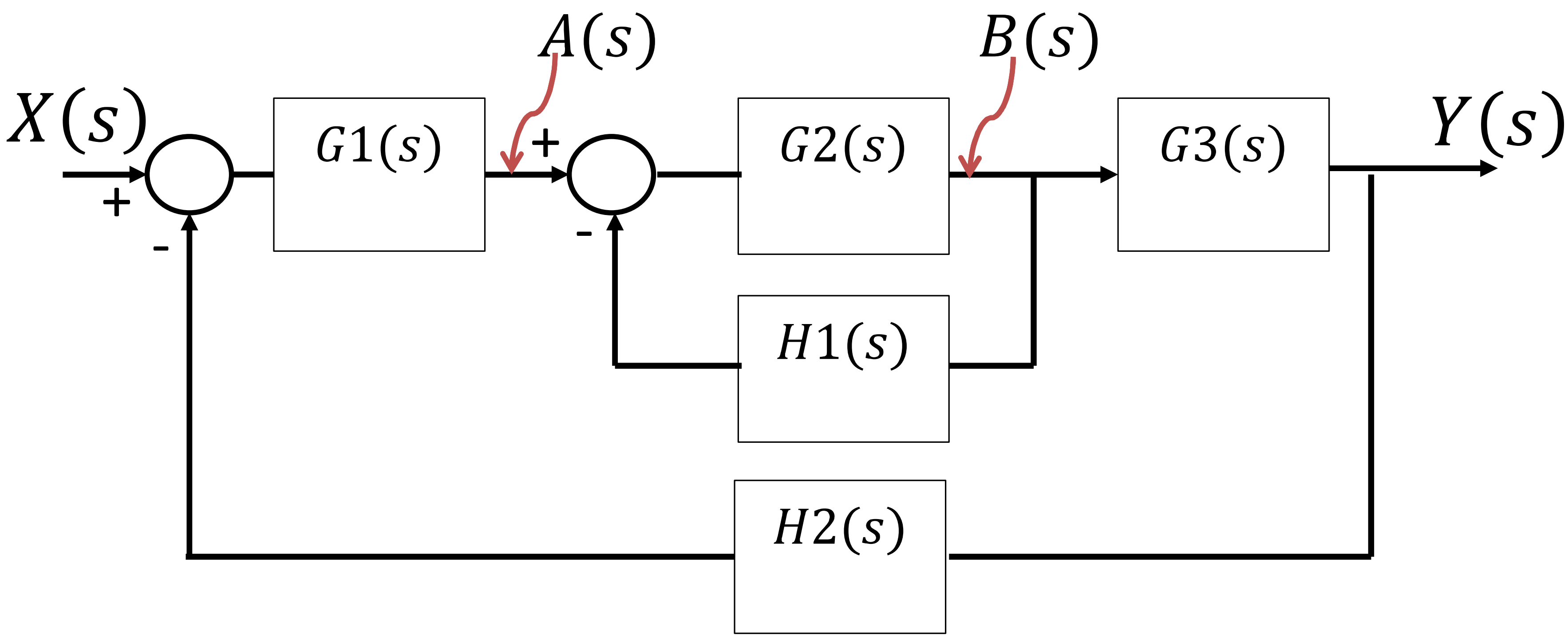
Split: X is unaffected. Summing junction follows signs given.
After summing junction:

$$Y(s) = \left(G_1(s) + G_2(s) - G_3(s)\right) \times X(s)$$



Intermediate Signals

- In a complex block diagram, it can help to calculate the value of an intermediate signal as you work your way through the system.



Intermediate Signals

$$A(s) = (X(s) - H2(s)Y(s)) \times G1(s)$$
$$B(s) = \left(A(s) - H1(s)B(s)\right) \times G2(s)$$
$$Y(s) = \left(B(s)\right) \times G3(s)$$

- Methodical substitution:

$$\frac{Y(s)}{G3} = \left(A(s) - H1(s)\frac{Y(s)}{G3}\right) \times G2(s)$$
$$\frac{Y(s)\left(1 + H1(s)G2(s)\right)}{G3} = A(s)G2(s)$$
$$\frac{Y(s)\left(1 + H1(s)G2(s)\right)}{G3} = (X(s) - H2(s)Y(s))G1(s)G2(s)$$

Intermediate Signals

$$Y(s)\big(1 + H1(s)G2(s)$$
$$+ H2(s)G1(s)G2(s)G3(s)\big)$$
$$= \big(X(s)\big)G1(s)G2(s)G3(s)$$

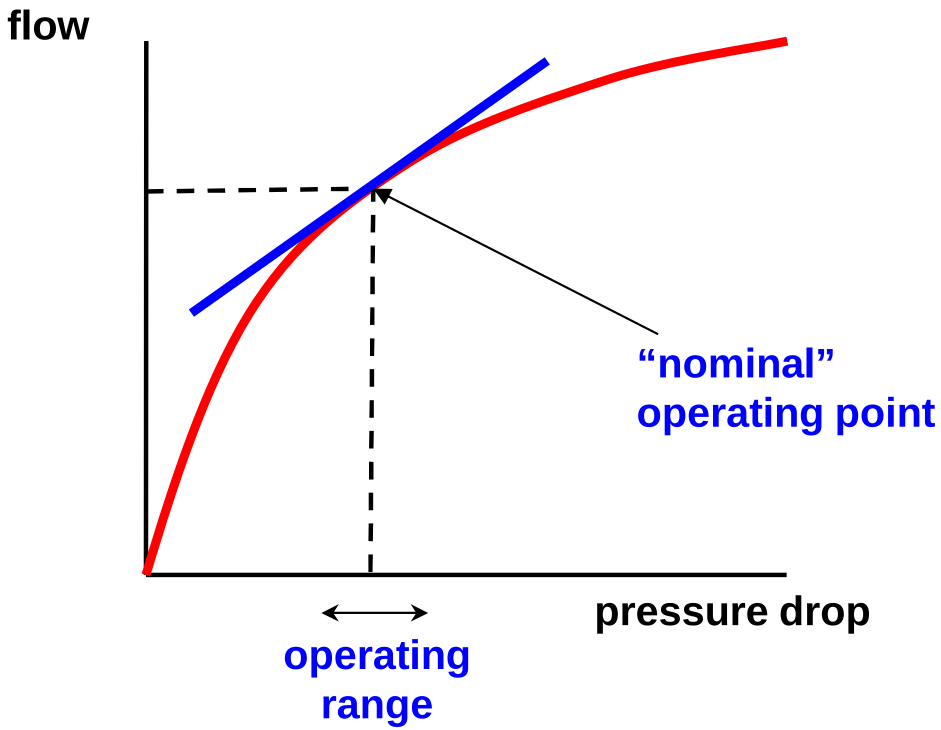
Intermediate Signals

$$Y(s)\big(1 + H1(s)G2(s) + H2(s)G1(s)G2(s)G3(s)\big) = \big(X(s)\big)G1(s)G2(s)G3(s)$$

$$\frac{Y(s)}{X(s)} = \frac{G1G2G3}{1 + H1G2 + G1G2G3H2}$$

Non-linearisation

- Most of the time, we are modelling responses around an operating point



Blue line shows operating range

Normally composed of a tangent equal to that found at the operating point:
find $y = mx + b$
for the blue line

Transient response – Third and higher order systems

- Generalised transfer function for the system:

$$G(s) = \frac{Q(s)}{P(s)}$$

$$\mathrm{G}(s) = \frac{Q(s)}{(s-p_1)(s-p_S)\ldots(s-p_N)}$$

Transient Response – Higher order systems

- Values for which $Q(s)$ is zero are zeros of the transfer function
- Values for which $P(s)$is zero (i.e. $G(s)$becomes infinite) are the poles:
  - $p_1, p_2, \ldots, p_N$ for an Nth order system
  - These poles are either real (singular) or complex (pairs)

$$s = \sigma_r \text{ or } s = \sigma_c \pm \omega_c$$

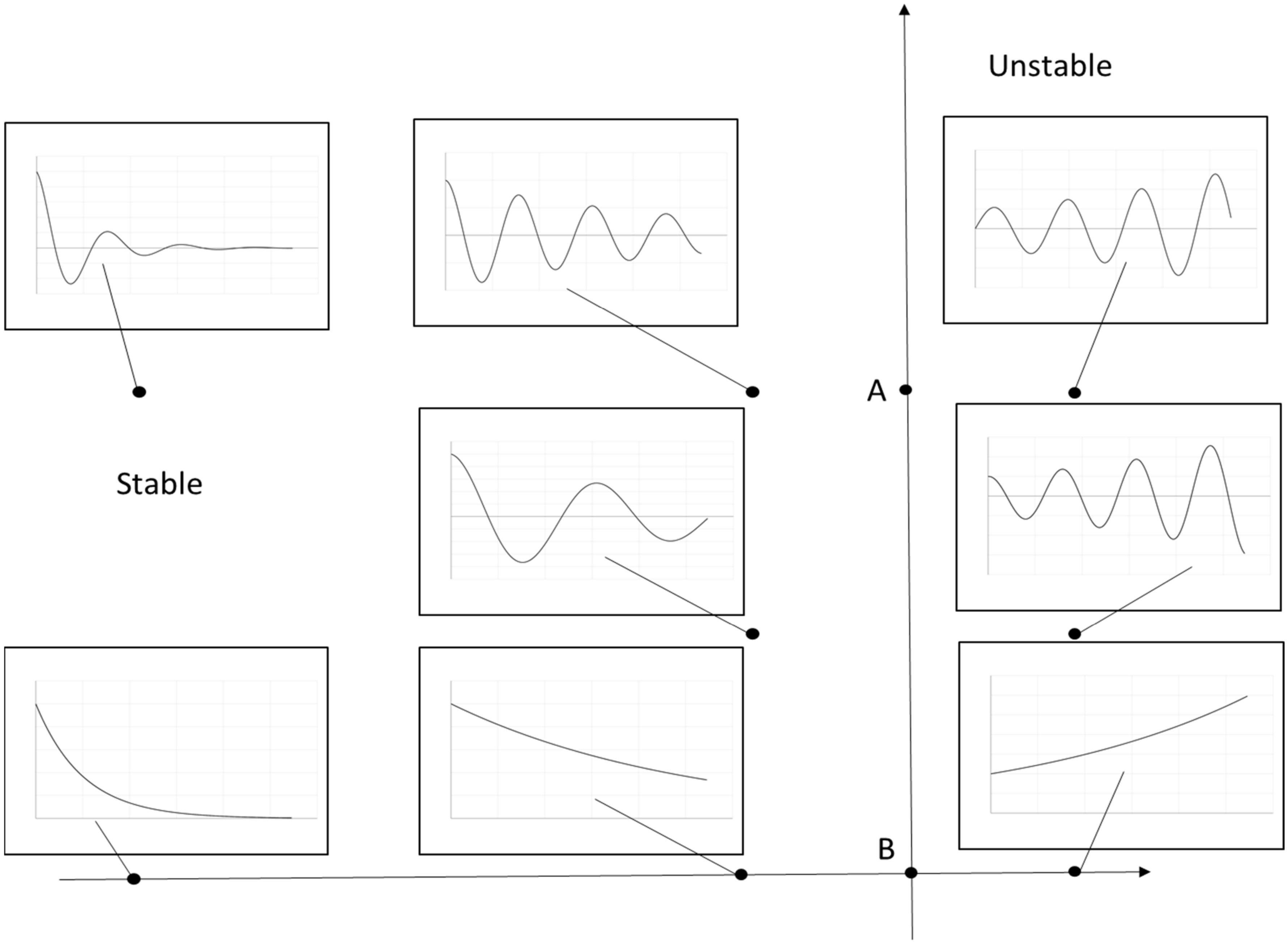
Transient Response – Higher order systems

If the input is a unit step: $X_i(s) = \frac{1}{s}$

Then:

$$X_0(s) = \frac{1}{s} + \sum_{r=1}^{N_R} \frac{A_r}{s - \sigma_r} + \sum_{c=1}^{N_c} \frac{A_c}{(s - \sigma_c)^2 + \omega_c^2}$$

$$x_0(t) = 1 + \sum_{r=1}^{N_R} B_r e^{\sigma_r t} + \sum_{c=1}^{N_c} B_c e^{\sigma_c t} sin(\omega_c t)$$


Unstable
A
Stable
B

Routh-Hurwitz Stability Criteria

$$P(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = 0$$

Routh Hurwitz criteria for stability:

i) Necessary: All coefficients $a_0, a_1, a_2, \ldots, a_n$ are non-zero and have the same sign.
   - i.e. if there is a change of sign in the denominator, the system will be unstable. No need to proceed to condition ii).
   - However, it is possible for the system to be unstable without a change of sign …

Routh-Hurwitz Stability Criteria

$$P(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = 0$$

Routh Hurwitz criteria for stability:

i) Necessary: All coefficients $a_0, a_1, a_2, \ldots, a_n$ are non-zero and have the same sign.

ii) Necessary and sufficient: if i) is satisfied, then the Hurwitz determinants $D_1, D_2, \ldots, D_n$ must be positive.

- This very quickly becomes laborious …
- Better to use a Routh Array

Routh-Hurwitz Stability Criteria (Routh Array)

s^n	a_0	a_2	a_4	a_6	…
s^{n-1}	a_1	a_3	a_5	a_7	…
s^{n-2}	b_1	b_2	b_3	…	…
s^{n-3}	c_1	c_2	c_3	…	…
…	…	…	…	…	…
s^0	…	…	…	…	…

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \qquad b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{\mathrm{b}_1}$$

Routh-Hurwitz Stability Criteria

Using the Routh Array:

- If there is a change of sign in the ***first*** column, there is a root on the real, positive side of the s-plane. For every change of sign, there is another positive root.
- Thus, for the system to be stable, all values in the first column must be positive.
  – There is an issue if there is a zero in the first column, or there is a complete row of zeros so that the array cannot be completed.
  – Beyond the scope of MM2DYN!

Example 1

- The characteristic equation of a system is:

$$2s^3 + 4s^2 + 4s + 12 = 0$$

- Is the system stable or unstable? If it is unstable, how many roots lie in the right half of the s-plane?
- Given that the coefficients of the characteristic equation are non-zero and have the same sign, the stability of the system must be investigated using criterion (2):
- Provided that condition (1) is satisfied, then the *necessary* and *sufficient* condition that no root of equation (1) lies on the right hand side of the s-plane is that the Hurwitz determinants of the polynomial must be positive.