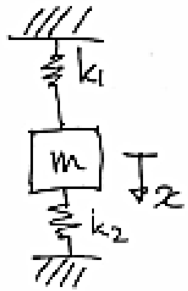
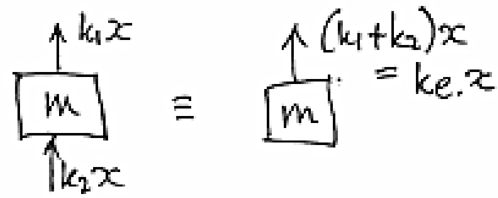
Solutions for Exercise Sheet 1 : VIBRATION
Single-Degree-of-Freedom Systems.

①



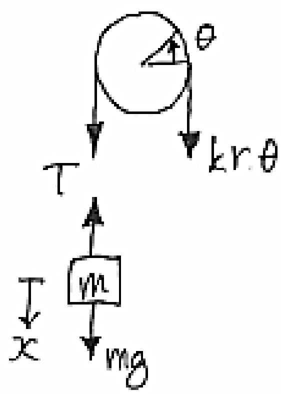
F.B.D.



The equivalent spring stiffness, $k_e = k_1 + k_2$

②

F.B.D.



T, tension in the string.

Using the Newton's second law:

- for the mass, m :

$$mg - T = m\ddot{x} \quad \rightarrow T = mg - m\ddot{x} \quad \text{—— (1)}$$

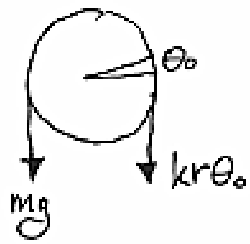
- for the pulley, the moment of inertia $J = \frac{1}{2}Mr^2$

$$J\ddot{\theta} = T.r - kr\theta.r \quad \text{—— (2)}$$

Substituting (1) into (2):

$$(J + mr^2)\ddot{\theta} + kr^2\theta = mgr \quad \text{—— (3)}$$

If $\theta = \theta_0 + \theta_1$, where θ_0 is the static angular displacement caused by the weight, mg :



$$mg = kr\theta_0$$
$$\Leftrightarrow mgr = kr^2\theta_0$$

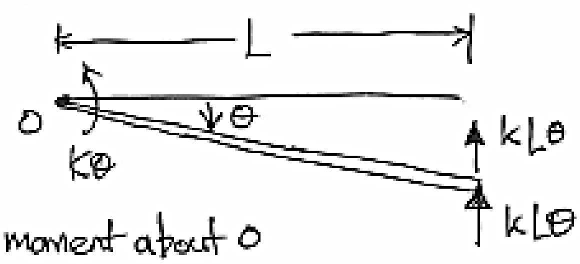
Equation (3) becomes:

$$(J+mr^2)\ddot{\theta}_1 + kr^2(\theta_0+\theta_1) = mgr$$

$$\Leftrightarrow (J+mr^2)\ddot{\theta}_1 + kr^2\theta_1 = 0 \quad \left(\text{Note that } \ddot{\theta} = \ddot{\theta}_0 + \ddot{\theta}_1 \right.$$
$$\left. \Leftrightarrow \ddot{\theta} = \ddot{\theta}_1 \right)$$

$$\omega_n = \sqrt{\frac{k}{M/2+m}} \qquad \text{since } J = \frac{1}{2}Mr^2$$

(3) F.B.D.



Taking the moment about O

$$\sum \overset{\curvearrowright}{M_0} = J\ddot{\theta} = -K\theta - 2kL\theta \times L \quad \left(\text{Small } \theta \text{ is assumed}, \sin\theta \simeq \theta \right)$$

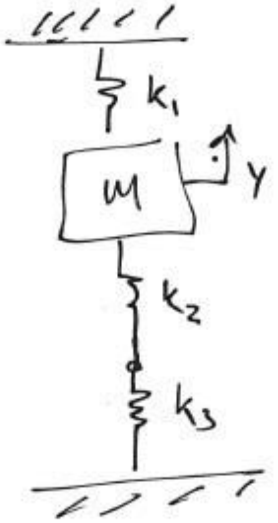
$$J = \frac{1}{3}mL^2$$

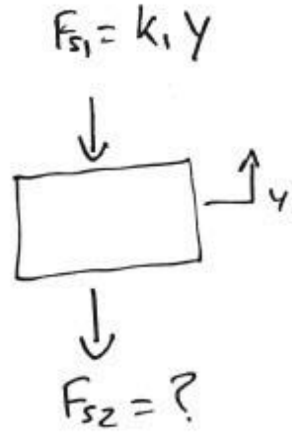
$$\Leftrightarrow \frac{mL^2}{3}\ddot{\theta} = -(K + 2kL^2)\theta$$

$$\Leftrightarrow \frac{mL^2}{3}\ddot{\theta} + (K+2kL^2)\theta = 0$$

$$\omega_n = \sqrt{\frac{3K+6kL^2}{mL^2}}$$

Q4





For F_{s2} find equivalent spring constant $F_{s2} = k_{eq} Y$

springs are in series

$$\Delta_{tot} = \Delta_2 + \Delta_3$$

$$\Delta_2 = \frac{F}{k_2} \qquad \Delta_3 = \frac{F}{k_3}$$

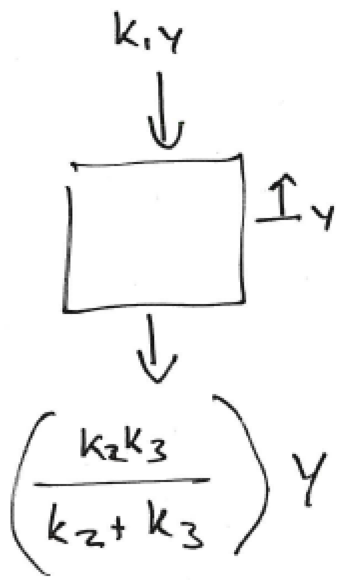
$$\Delta_{tot} = \frac{F}{k_2} + \frac{F}{k_3}$$

$$k_{eq} = \frac{F}{\Delta_{tot}} = \frac{F}{\frac{F}{k_2} + \frac{F}{k_3}}$$

$$\therefore k_{eq} = \left(\frac{1}{k_2} + \frac{1}{k_3}\right)^{-1} = \frac{k_2 k_3}{k_2 + k_3}$$ (springs in series)

In General $\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \dots + \frac{1}{k_n}$

FBD



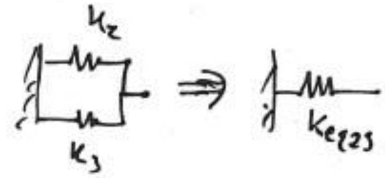
EOM

$$m\ddot{y} + ky = 0$$

$$\boxed{m\ddot{y} + K_1+\left(\frac{k2\,k3}{k2+k3}\right) y = 0}$$

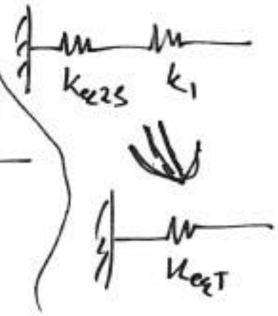
$$\boxed{\omega_n = \sqrt{\frac{k1 + \left(\frac{k2\,k3}{k2+k3}\right)}{m}}}$$

4b) k_2 and k_3 are in parallel



$$k_{eq23} = k_2 + k_3$$

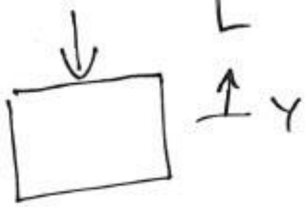
k_{eq23} and k_1 are in series



$$k_{eqT} = \frac{k_1 \, k_{eq23}}{k_1 + k_{eq23}} = \frac{k_1 (k_2 + k_3)}{k_1 + k_2 + k_3}$$

FBD

$$k_{eqT}\, y = \left[\frac{k_1 (k_2 + k_3)}{k_1 + k_2 + k_3}\right] y$$

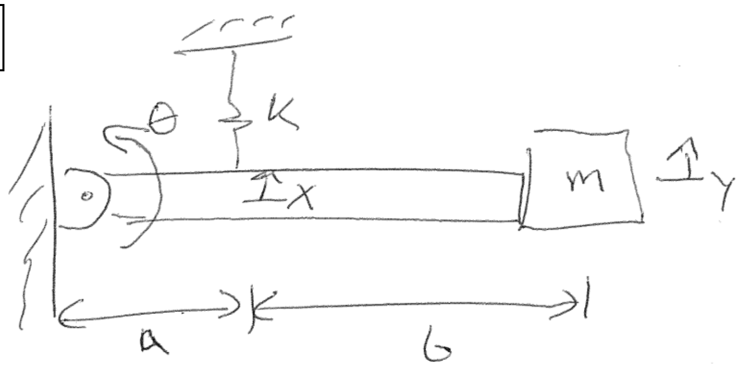


EOM

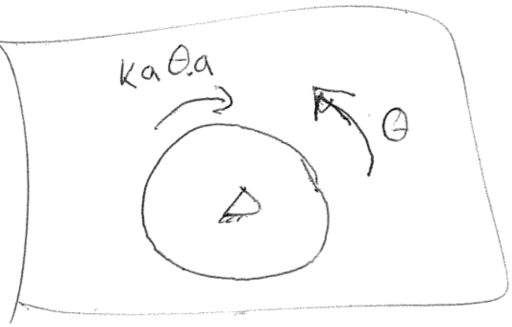
$$\boxed{m\ddot{y} + \left[\frac{k_1 (k_2 + k_3)}{k_1 + k_2 + k_3}\right] y = 0}$$

$$\boxed{\omega_n = \sqrt{\frac{k_1 (k_2 + k_3)}{(k_1 + k_2 + k_3) m}}}$$

4C



FBD



$$x = a\theta$$

EOM

$$-ka^2\theta = I\ddot{\theta}$$

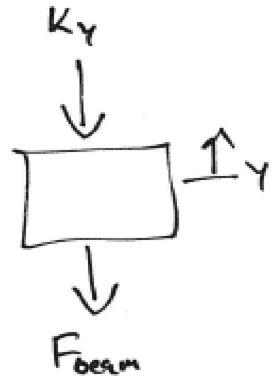
$$I\ddot{\theta} + ka^2\theta = 0$$

$$I = \cancel{I_o}^{0} + m(a+b)^2$$

$$m(a+b)^2\ddot{\theta} + ka^2\theta = 0$$

$$\omega_n = \sqrt{\frac{k}{m}} = \boxed{\sqrt{\frac{ka^2}{m(a+b)^2}}}$$

Q5a) FBD



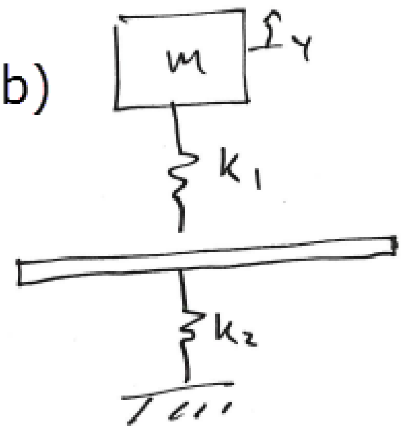
$$F_{beam} = \frac{3EI}{\ell^3} y$$

EOM

$$\boxed{m\ddot{y} + \left[\frac{3EI}{\ell^3} + k\right] y = 0}$$

$$\boxed{\omega_n = \sqrt{\frac{\frac{3EI}{\ell^3} + k}{m}}}$$

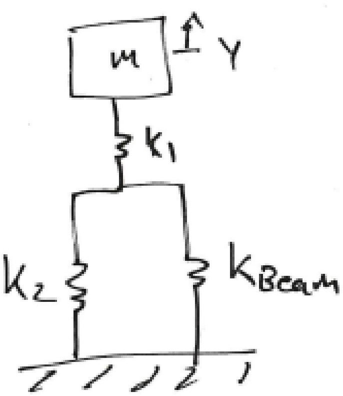
5b)



Beam is in parallel with k_2 because they see the same deflection

$$k_{beam} = \frac{48EI}{\ell^3}$$

$\therefore$ equivalent to



this is the same as 1b then

$$M\ddot{y} + \left[\frac{k_1(k_2 + k_{Beam})}{k_1 + k_2 + k_{Beam}}\right] Y = 0 \qquad \omega_n = \sqrt{\frac{k_1(k_2 + k_b)}{(k_1 + k_2 + k_b)m}}$$

Q6

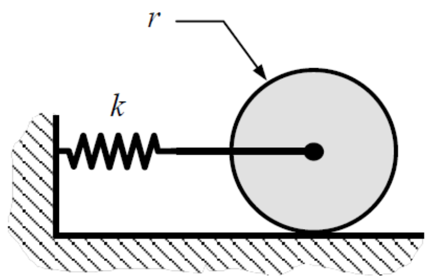
Steps:

1- Assign a coordinate system (x if the system is translational or θ if the system is rotational)to the problem
2- Draw the free body diagram of the system
3- Apply Newton second law of motion $\sum Forces = m\ddot{x}$ if the system is translational or $\sum Torques = I\ddot{\theta}$ if the system is rotational. In general the equation of motion should be in the form: $M\ddot{Z} + kZ = 0$, $M = mass, Z = x$ if the system is translational and $M = I, Z = \theta$ if the system is rotational, **make sure that all the terms in the equation of motion are positive**.
4- Obtain the natural frequency of the system by dividing the coefficient of the displacement by the coefficient of the acceleration in the equation of motion and taking the square root of the division.

Note: The solution given here very detailed, we detail the solution because we are at the beginning of the course. In future it will be much easier for you to solve the previous simple problem since they will be considered very simple in future.

Q7

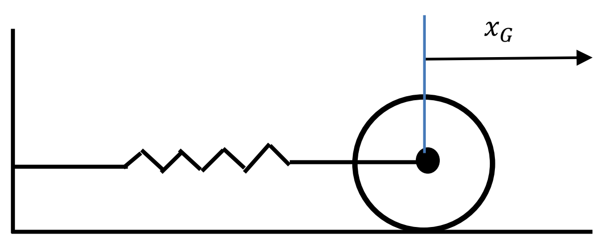
A wheel (radius r, mass m, moment of inertia about its centre I) can roll without slipping on a horizontal plane. It is restrained by a horizontal spring (stiffness k) attached at one end to the centre of the wheel and at the other end to a rigid vertical wall, as in Figure Q2. Derive the equation of motion and hence find the natural frequency for the system. What would the natural frequency be if there was no friction between the wheel and the plane?



Solution

Steps1 (Coordinate system)

If we study the translational motion of the centre of the disk we will use a coordinate x_G,



Step 2 (Free body diagram)

What is our interest here? It is the motion of the wheel centre, let's isolate the rotational inertia element (wheel) and draw all forces acting on it:

We have the sprnig force: kx_G and the friction force that cause rolling F

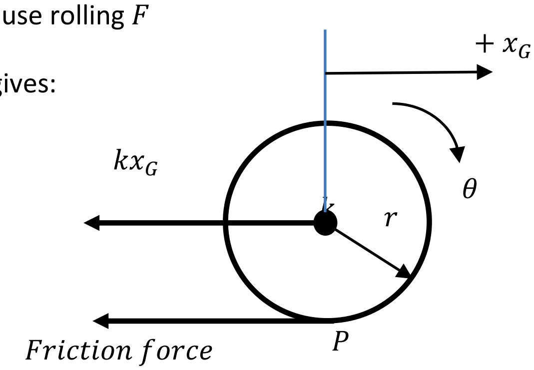
Applying Newton second law of motion in the $x_G$direction gives:

$\sum Forces = m\ddot{x} \quad \rightarrow \qquad -kx_G - F = m\ddot{x}_G$

$Fr = I\ddot{\theta} \longrightarrow F = I\ddot{\theta}/r$

$x_G = r\theta \longrightarrow \ddot{x}_G = r\ddot{\theta}$

$F = I\ddot{x}_G/r^2$



Finally the equation of motion will be:

$$\left(m+\frac{I}{r^2}\right)\ddot{x}_G + kx_G = 0$$

The natural frequency of the system is obtained by dividing the coefficient of the displacement by the coefficient of the acceleration in the equation of motion and taking the square root of the division.

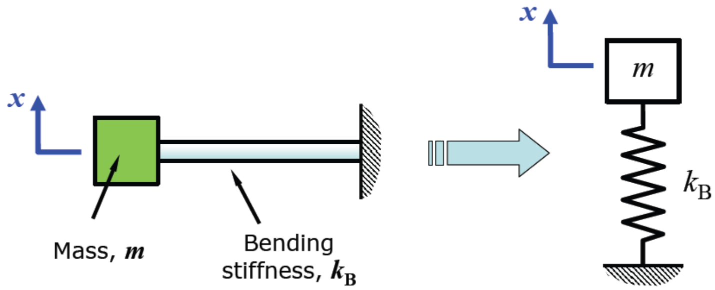
$$\omega_n = \sqrt{\frac{k}{m+\frac{I}{r^2}}}$$

If there is no friction the force $F = 0$ $\longrightarrow$

$$\omega_n = \sqrt{\frac{k}{m}}$$

Q8

Beam in bending



The best practice is to obtain the natural frequency from equation of motion

Equation of motion:

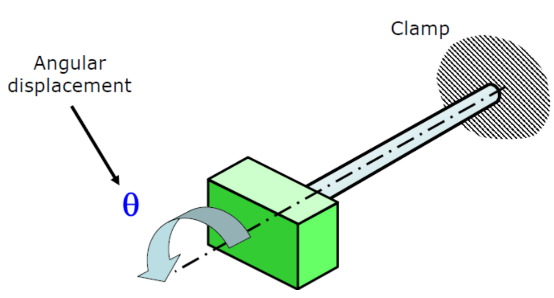
$$m\ddot{x} + k_B x = 0$$

$$\omega_{nB} = \sqrt{\frac{k_B}{m}} \longrightarrow m = \frac{k_B}{(\omega_{nB})^2}$$

The bending stiffness coefficient is $k_B = \frac{3EI}{L^3} = \frac{3E\pi D^4}{64L^3} = 571.9\ kN$

$$m = \frac{k_B}{(\omega_{nB})^2} = \frac{571900}{(2\pi 15)^2} = 64.4\ kg$$

<u>Beam in torsion</u>



Equation of motion:

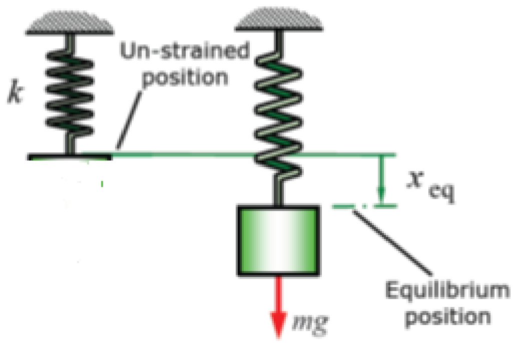
$$I\ddot{\theta} + k_T\theta = 0$$

$$\omega_{nB} = \sqrt{\frac{k_T}{I}} \qquad I = \frac{k_T}{(\omega_{nT})^2} \qquad k_T = \frac{GJ}{L} = \frac{G\pi D^4}{32L}$$

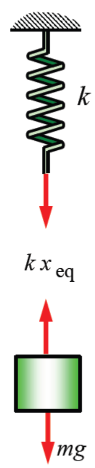
$$I = \frac{k_T}{(\omega_{nT})^2}$$

Q9

1- In the first figure to the left the spring I s shown without the mass element, no extensions, in the figure to the right the mass element is attached to the spring and the spring is stretched downward a distance $x_e = x_0$ called the static deflection.



2- The FBD is shown below:



$x_e = x_0$

Static equilibrium shows that $kx_0 = mg$ → $mg - kx_0 = 0$ (This is not an equation of motion)

$$\frac{k}{m} = \frac{g}{x_0}$$

We know that: $\omega_n = \sqrt{\frac{k}{m}}$ From which $\omega_n = \sqrt{\frac{g}{x_0}}$