APPROXIMATE METHODS

Often the lowest natural frequency is the most important

- If the 1st critical speed of a shaft is above its operating speed range, whirl is avoided
- The 1st mode normally gives the largest displacement for a given excitation

A. Rayleigh's Method

A & B are used to estimate the lowest natural frequency of structures

B. Creating single-degree-of-freedom dynamic models of more complex systems

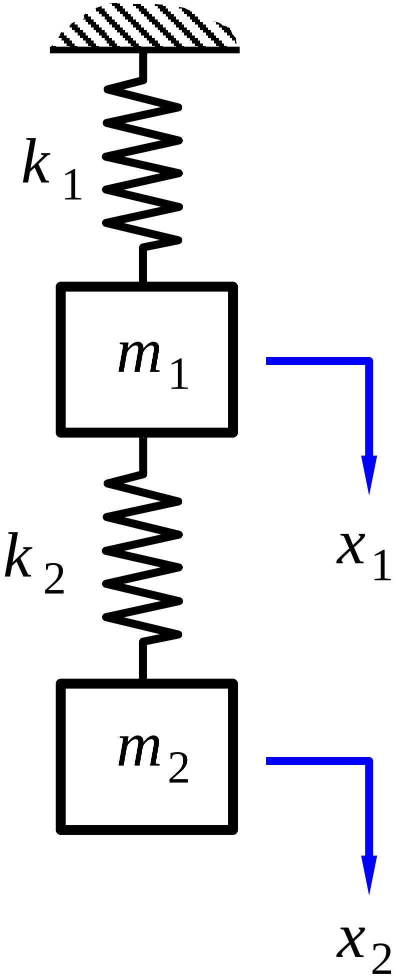
A. RAYLEIGH'S METHOD

For an undamped system in free vibration, energy is conserved, so that

Max. Kinetic Energy = Max. Strain Energy

- These can be found if we know the deflected shape (i.e., the mode shape) of the system
- Normally, we do NOT know the exact mode shape, so we need to make an estimate
- Accuracy depends on making a good guess

Example 1 **Two-degree-of-freedom System**



The **instantaneous** kinetic energy of mass 1 is

$$\tfrac{1}{2} m_1 \dot{x}_1^2$$

If $x_1(t) = X_1 \sin \omega t$ then $\dot{x}_1(t) = (\omega X_1) \cos \omega t$

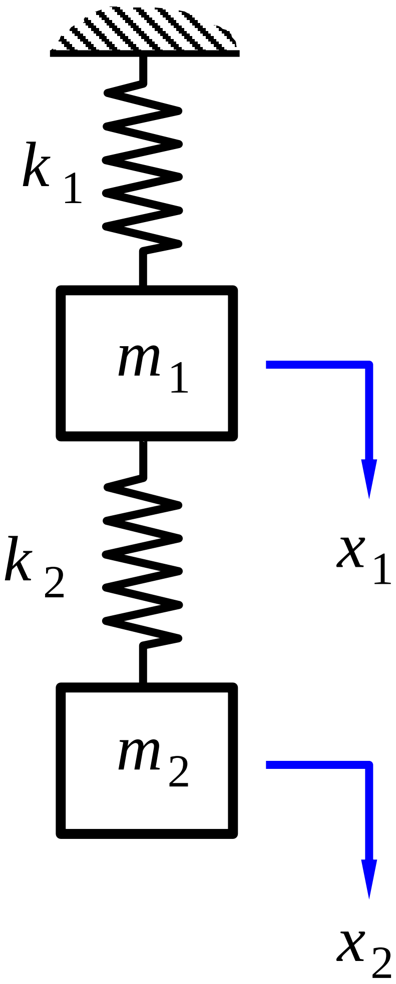
That is, the maximum velocity is (ωX_1)

Therefore, the **maximum** kinetic energy of mass 1 is

$$T_{\mathrm{max}} = \tfrac{1}{2} m_1 (\omega X_1)^2 = \tfrac{1}{2} m_1 \omega^2 X_1^2$$

For both masses

$$\boxed{T_{\mathrm{max}} = \tfrac{1}{2} \omega^2 \left(m_1 X_1^2 + m_2 X_2^2 \right)}$$



The **instantaneous** strain energy of a spring is

$$\tfrac{1}{2} \times \text{Stiffness} \times (\text{Change of length})^2$$

For the top spring, this is $U = \tfrac{1}{2} k_1 x_1^2$

If $x_1(t) = X_1 \sin \omega t$ then $U_{\max} = \tfrac{1}{2} k_1 X_1^2$

For the bottom spring

$$U_{\max} = \tfrac{1}{2} k_2 (X_1 - X_2)^2$$

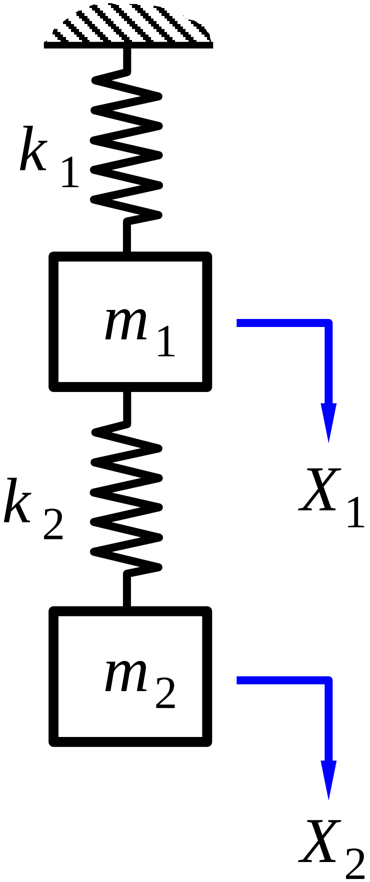
For both springs

$$\boxed{U_{\max} = \tfrac{1}{2} k_1 X_1^2 + \tfrac{1}{2} k_2 (X_1 - X_2)^2}$$

Equating $T_{\max} = U_{\max}$, we get

$$\omega^2 = \frac{k_1 X_1^2 + k_2 (X_1 - X_2)^2}{m_1 X_1^2 + m_2 X_2^2}$$

If we can estimate the mode shape, we will have values of $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ that can be substituted into the equation



We need an educated guess based on experience

For $\boldsymbol{m_1} = \boldsymbol{m_2}$ = 2 kg and $\boldsymbol{k_1} = \boldsymbol{k_2}$ = 200N/m,

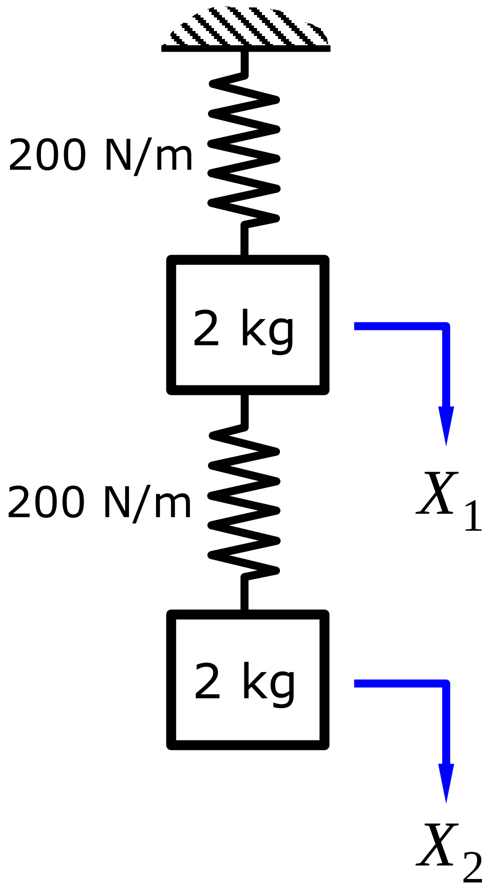
we know that the two masses vibrate in phase and

$$X_2 > X_1$$

Let's guess that $\boldsymbol{X_1}$ = 1 and $\boldsymbol{X_2}$ = 2

This gives a value for ω_n of 1.007 Hz

This is an error of 2.3% (exact value is 0.984 Hz)



A good estimate for the mode shape is the **static deflection shape** due to gravity

Here, the bottom spring supports 2 kg, but the top spring carries the weight of both masses (4 kg)

Therefore the extension of the top spring will be twice the extension of the bottom spring

To find X1 and X2 set a nominal displacement of 1.
X1= 2* Nominal Displacement
X2 = X1 + Nominal Displacement

Hence, $X_1 = 2$ and $X_2 = 3$

This gives $\omega_n = 0.987$ Hz, an error of 0.4%

Exercise

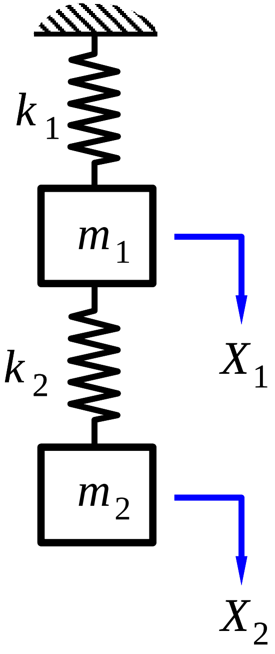
Try using the exact mode shape

$$X_1 = 0.618 \text{ and } X_2 = 1.000$$

You should get the exact answer

In general, $\omega_{\text{Rayleigh}} \geq \omega_{\text{Exact}}$

- **Try several possible mode shapes**
- **Lowest frequency will be most accurate**



For this two-degree-of-freedom system, the lowest natural frequency is predicted by the expression

$$\omega^2 = \frac{k_1 X_1^2 + k_2 \left(X_1 - X_2\right)^2}{m_1 X_1^2 + m_2 X_2^2}$$

This applies **only** to this specific system. Other systems will have **different expressions**.

These can be worked out in the same way

Rayleigh's method for shafts and beams

The expressions for the maximum kinetic and strain energies are

$$T_{\max} = \tfrac{1}{2}\,\omega^2 \int_0^L \rho\, A\left[Y(x)\right]^2 \,\mathrm{d}x$$

$$U_{\max} = \tfrac{1}{2} \int_0^L E\,I \left(\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2}\right)^2 \mathrm{d}x$$

These expressions are on the Formula Sheet

where $Y(x)$ is the mode shape function, which defines the amplitude of vibration of the shaft/beam along its length

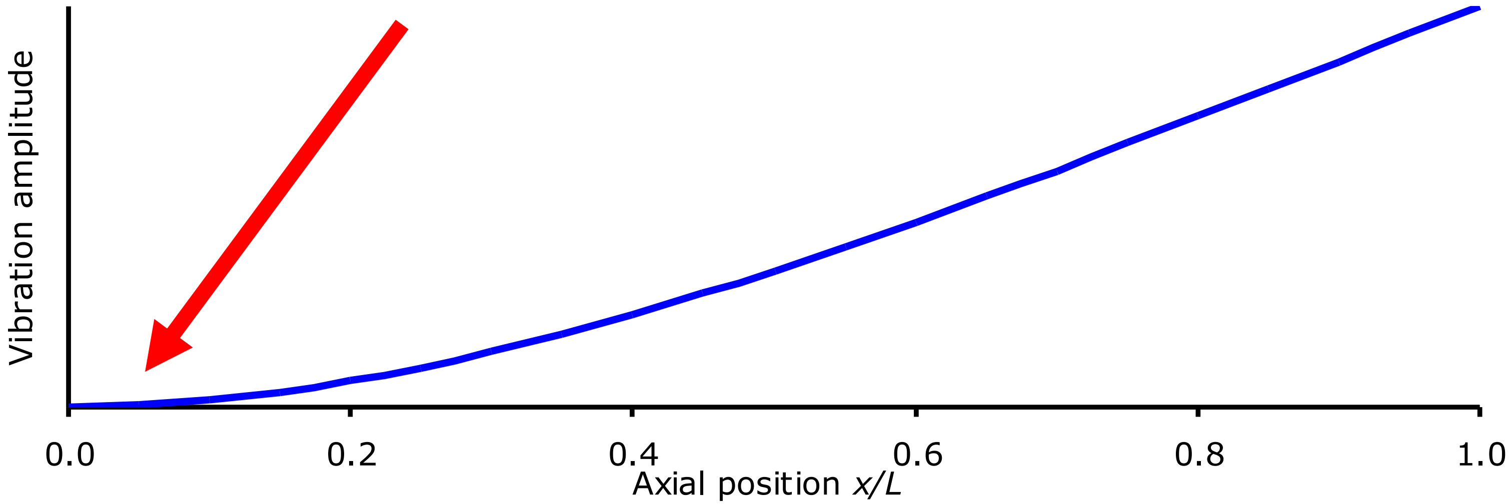
We need to guess $Y(x)$ in order to evaluate the integrals

Example 1: **Uniform Cantilever Beam**

The exact answer is $$\omega_n = \frac{3.52}{L^2}\sqrt{\frac{E\,I}{\rho\,A}}$$

The main criterion for choosing the mode shape is that it should satisfy the displacement and slope conditions at the ends

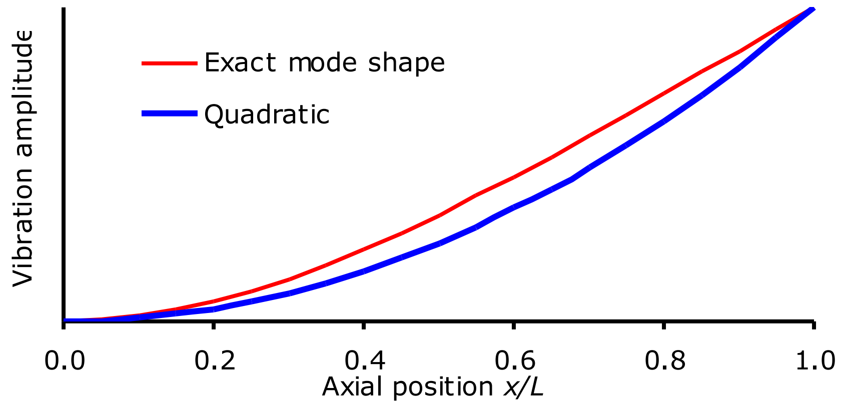
Here $Y = \dfrac{\mathrm{d}Y}{\mathrm{d}x} = 0$ at $x = 0$



$$Y_1(x) = \sin\lambda_1 x - \sinh\lambda_1 x + \sigma_1\left(\cos\lambda_1 x - \cosh\lambda_1 x\right)$$

Choice #1 $Y(x) = C\,x^2$

This satisfies the displacement and slope conditions at the clamped end and gives a shape that is similar to the actual mode shape



Maximum Kinetic Energy

$$T_{\max} = \tfrac{1}{2}\,\omega^2 \int_0^L \rho\,A\,[Y(x)]^2\,\mathrm{d}x$$

Hence

$$T_{\max} = \tfrac{1}{2}\,\omega^2 \int_0^L \rho\,A\left[C\,x^2\right]^2\,\mathrm{d}x$$

$$= \tfrac{1}{2}\,\omega^2\,\rho A \times \frac{C^2\,L^5}{5}$$

$$= \omega^2\,\frac{\rho A C^2\,L^5}{10}$$

Maximum Strain Energy

$$U_{\max} = \tfrac{1}{2}\int_0^L EI\left(\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \qquad \text{where} \quad Y(x) = C\,x^2$$

$$= \tfrac{1}{2}\int_0^L EI\,(2C)^2\,\mathrm{d}x$$

$$= 2\,E\,I\,C^2\,L$$

Equating gives

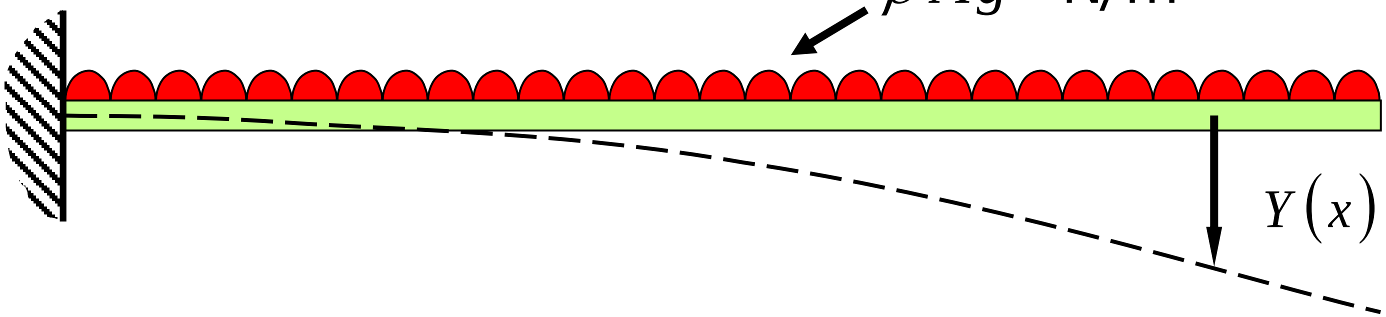
$$\omega^2 = 20\,\frac{E\,I}{\rho\,A\,L^4}$$

Prediction is

$$\omega_n = \frac{4.47}{L^2}\sqrt{\frac{E\,I}{\rho\,A}}$$

Error is 27%

Choice #2 Static deflection due to self-weight

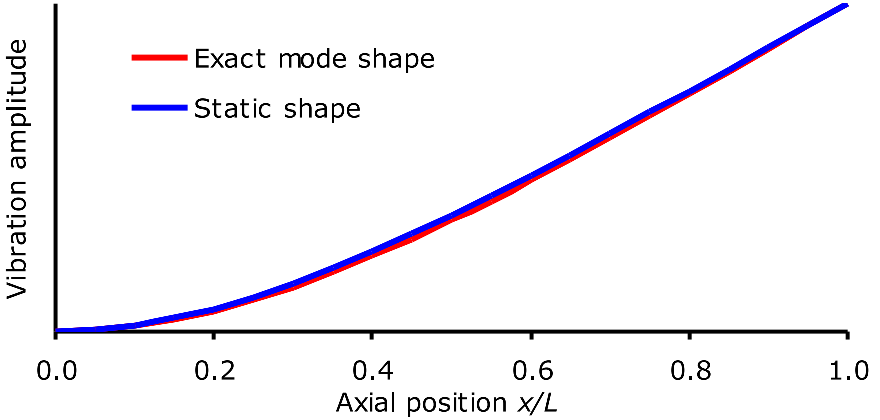


Integrating $$EI\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} = -M(x)$$ **Macaulay's Method**

Leads to $$Y(x) = C\left(x^4 - 4Lx^3 + 6L^2x^2\right)$$

Prediction is $$\omega_n = \frac{3.54}{L^2}\sqrt{\frac{EI}{\rho A}}$$

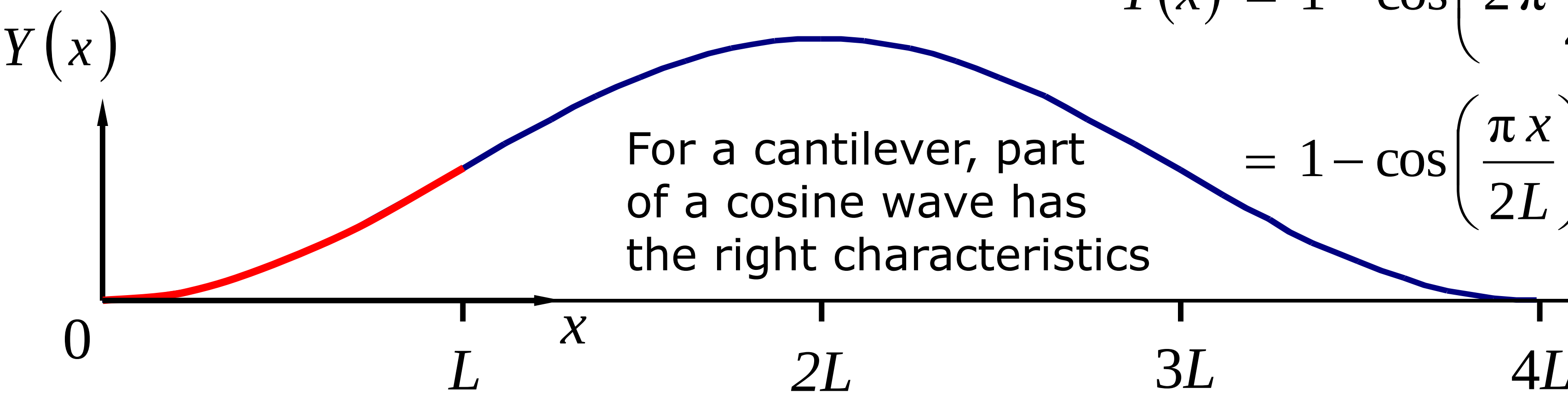
Error < 1%



Choice #3 A trigonometric function

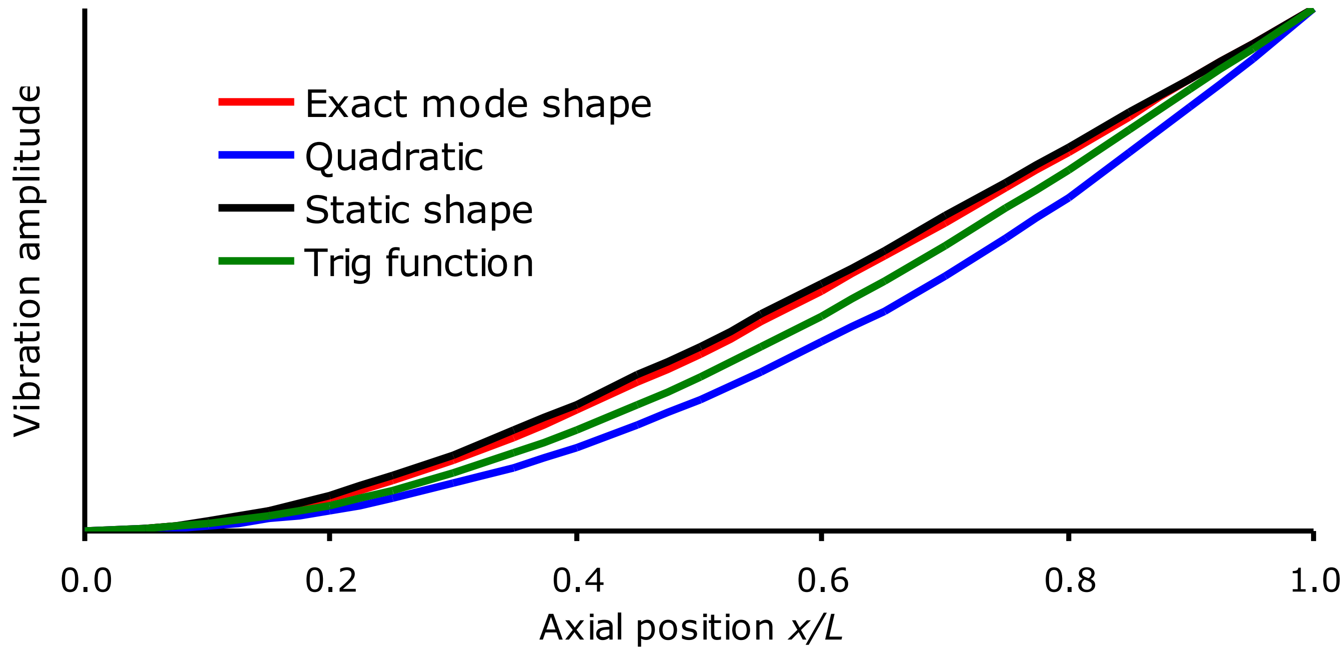
Trigonometric functions are often good approximations to the deflected shapes of beams and can be found by inspection for particular cases

$$Y(x) = 1 - \cos\left(2\pi \frac{x}{4L}\right) = 1 - \cos\left(\frac{\pi x}{2L}\right)$$



Using this, the prediction is $\omega_n = \dfrac{3.66}{L^2}\sqrt{\dfrac{EI}{\rho A}}$ **4% high**

The better the mode shape estimate, the more accurate the prediction



Static deflection shape is best

- Very close to exact mode shape

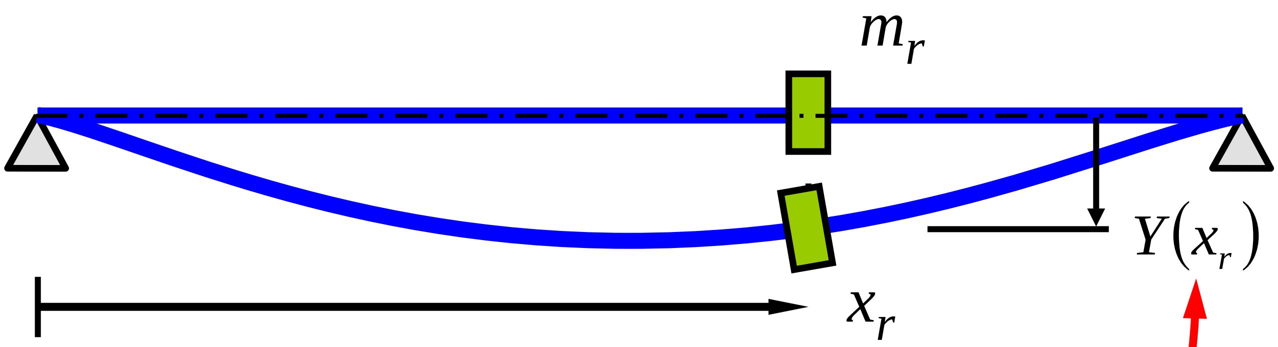
Quadratic function is poor

- Over-estimates strain energy
- Under-estimates kinetic energy

Trig function is a good compromise

- Can be found for different beams by inspection

Example 2: **Beams/shafts with added masses**



Added masses don't change the strain energy, but they do add extra kinetic energy

For mass at $x = x_r$ max. velocity is $\omega \cdot Y(x_r)$

Contribution of **mass #*r*** to the KE is $\frac{1}{2} m_r \omega^2 \left[Y(x_r) \right]^2$

These contributions are added to the kinetic energy of the shaft itself

Total KE is therefore

$$T_{\max} = \boxed{\tfrac{1}{2}\,\omega^2 \int_0^L \rho\, A\left[Y(x)\right]^2\,\mathrm{d}x} + \boxed{\sum_r \tfrac{1}{2}\, m_r\, \omega^2 \left[Y(x_r)\right]^2}$$

KE of beam

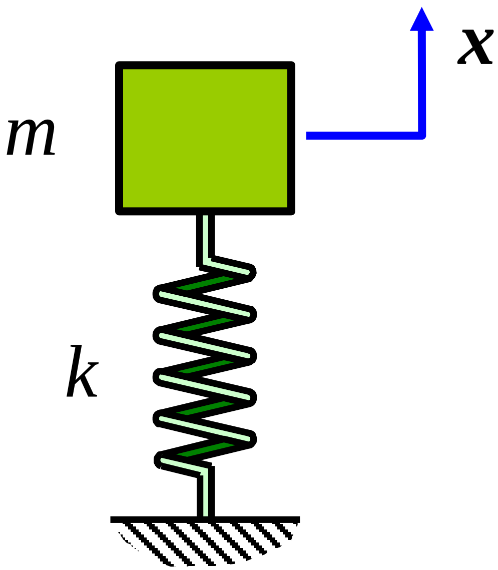
KE of added masses

SUMMARY

Taking Dunkerley & Rayleigh together, we can obtain a band containing the exact frequency

$$\omega_{\text{Exact}} \leq \omega_{\text{Rayleigh}}$$

Equations of Motion vs Energy Methods



The equation of motion is

$$m\ddot{x} + kx = 0 \qquad (1)$$

With no damping, energy is conserved so that at any instant, the sum of the kinetic and strain energies is constant

$$\tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} k x^2 = \text{constant}$$

Differentiation gives the rate of change of energy

$$m\dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} + kx\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

Cancelling the velocity gives Equation (1)

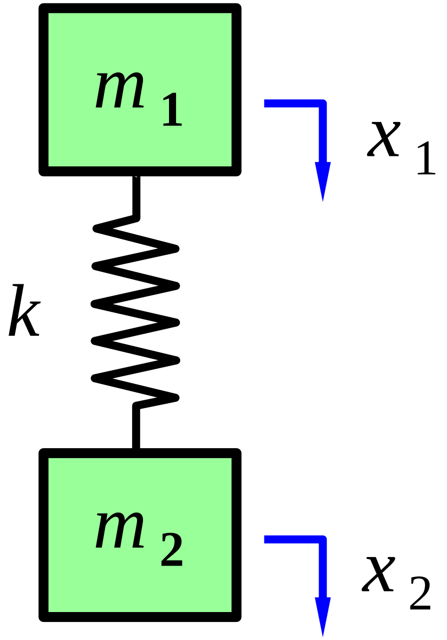
The equation of motion reflects the rates of change of the kinetic and strain energies of the system

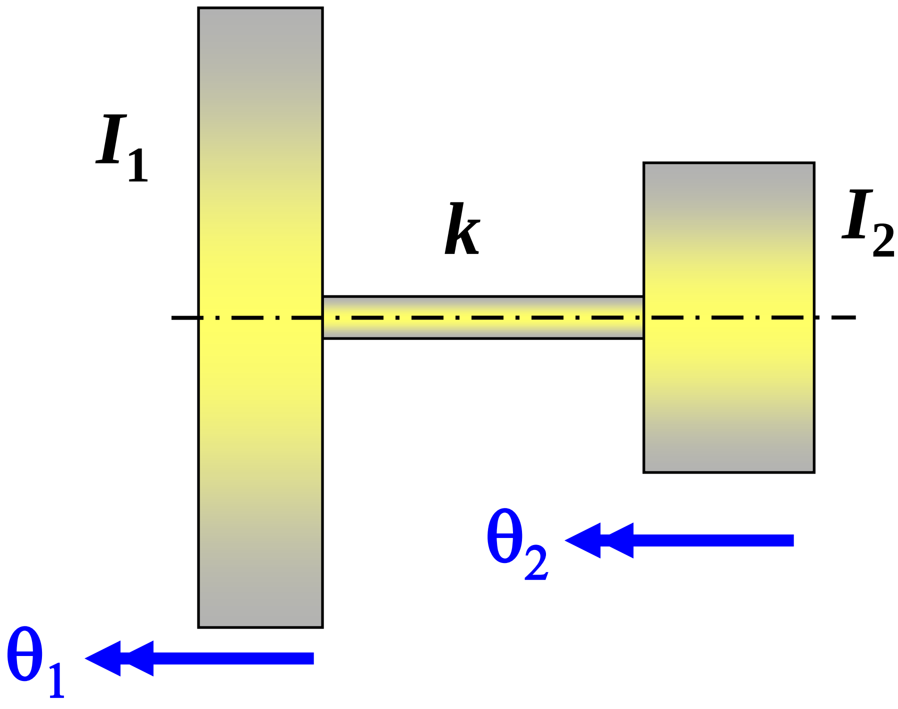
Lagrange’s Equations are based on this and provide a method of deriving the equations of motion of a system (including damping) from energy expressions

Also linked to energy considerations is the concept of **Dynamically Equivalent Systems**

If the total strain and kinetic energies of two different dynamic systems are identical, an exact analogue relationship will exist between the two systems

Dynamically Equivalent Systems





B. Single-degree-of-freedom Dynamic Models of Complex Systems

If only one mode of vibration is of interest, the concept of Dynamically Equivalent Systems provides a method for producing an approximate single-degree-of-freedom model to describe that mode of vibration

We will consider only undamped systems

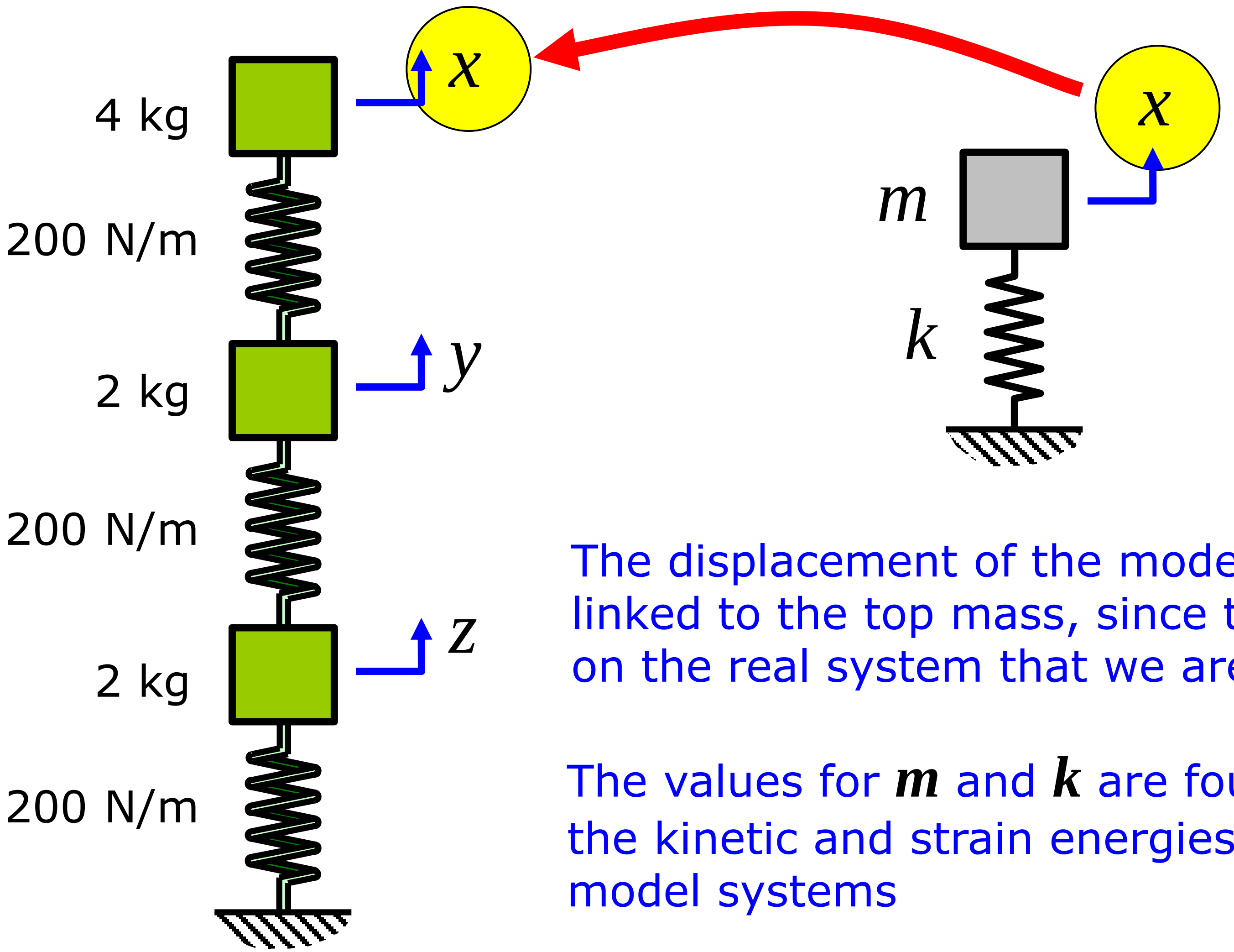
The approximate model consists of a simple mass-spring system, in which **the displacement of the mass represents the displacement of some chosen point on the real structure**

The mass and spring stiffness of the approximate model are chosen so that the maximum strain and kinetic energies of the real and model systems are the same

To do this, we assume that the lowest mode of vibration is dominant and therefore defines the deformation pattern in the structure

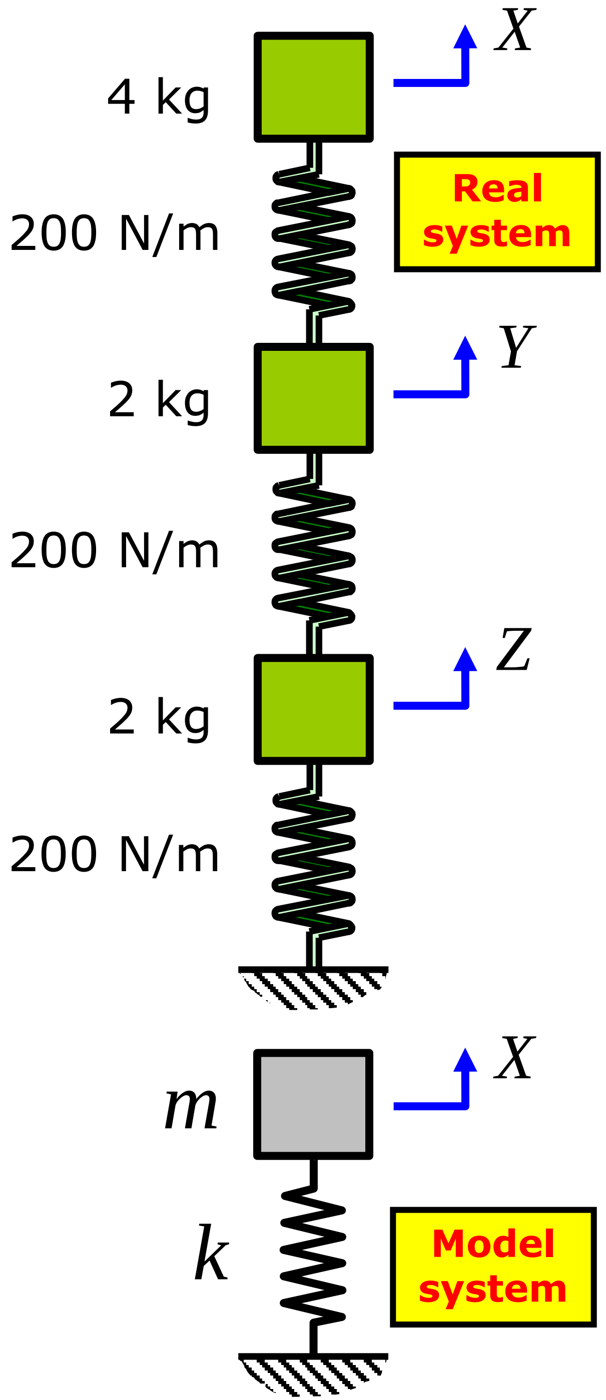
Example 1: **Lumped mass system**

Objective:
To find an approximate single-degree-of-freedom model to analyse the motion of the top mass of a 3-degree-of-freedom system



The displacement of the model mass will be linked to the top mass, since that's the point on the real system that we are interested in

The values for m and k are found by equating the kinetic and strain energies in the real and model systems



Equating max. **kinetic** energies in the real and model systems

$$\tfrac{1}{2}4\left(\omega X\right)^2 + \tfrac{1}{2}2\left(\omega Y\right)^2 + \tfrac{1}{2}2\left(\omega Z\right)^2$$

$$= \tfrac{1}{2}m\left(\omega X\right)^2$$

Hence $$\boxed{m = \frac{4X^2 + 2Y^2 + 2Z^2}{X^2}}$$

Equating max. **strain** energies in the real and model systems

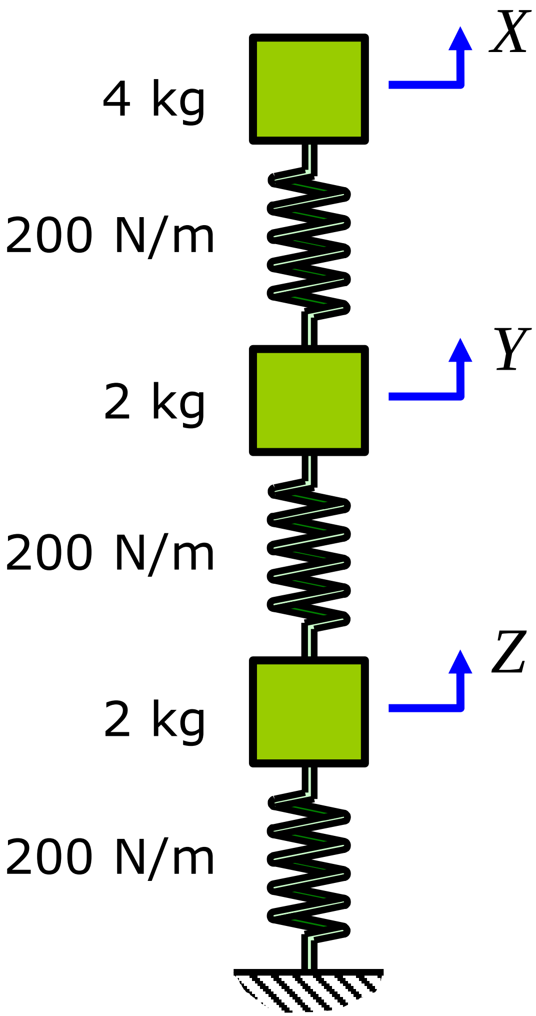
$$\tfrac{1}{2}200\left(X - Y\right)^2 + \tfrac{1}{2}200\left(Y - Z\right)^2 + \tfrac{1}{2}200\left(Z\right)^2$$

$$= \tfrac{1}{2}k\left(X\right)^2$$

Hence $$\boxed{k = \frac{200\left[\left(X - Y\right)^2 + \left(Y - Z\right)^2 + Z^2\right]}{X^2}}$$

$$m = \frac{4X^2 + 2Y^2 + 2Z^2}{X^2} \qquad k = \frac{200\left[(X-Y)^2 + (Y-Z)^2 + Z^2\right]}{X^2}$$

To get values for $\boldsymbol{m}$ and $\boldsymbol{k}$, we need an estimate for the mode shape



Characteristics:

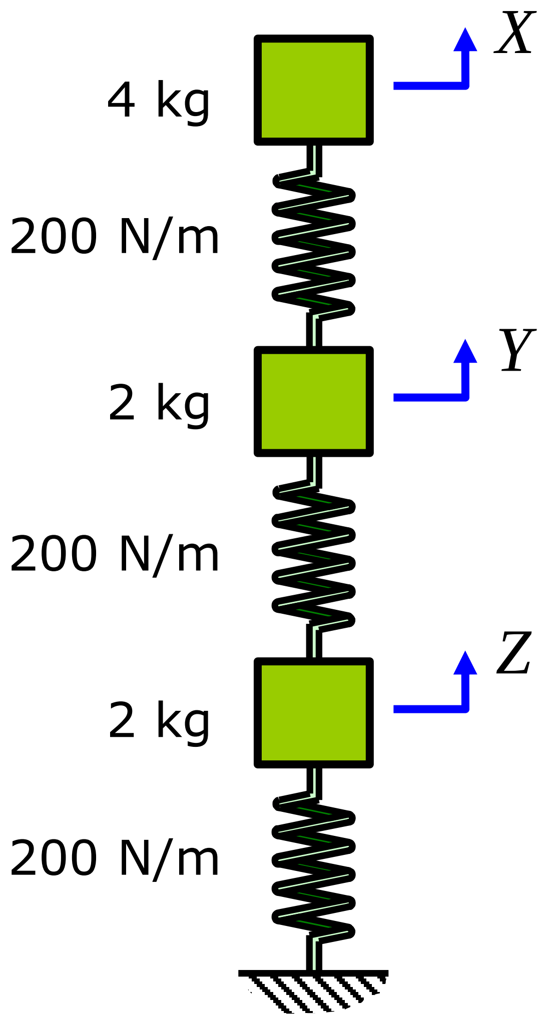
- All masses move in phase with each other
- $\boldsymbol{X > Y > Z}$

Choice #1

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{Bmatrix} 3 \\ 2 \\ 1 \end{Bmatrix}$$

Since all springs have the same stiffness, we might guess that they all deflect by the same amount

$\boldsymbol{m}$ = 5.11 kg and $\boldsymbol{k}$ = 66.7 N/m



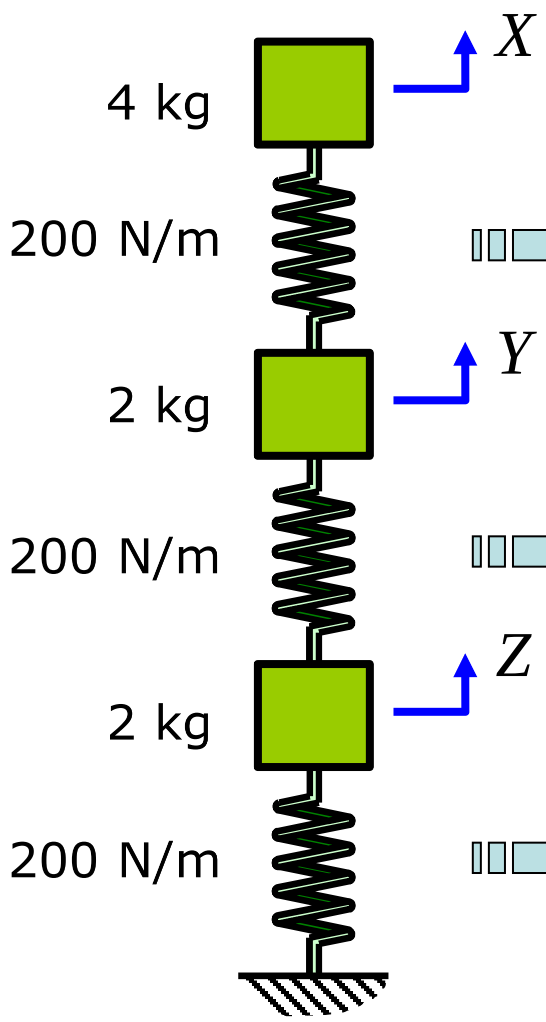
Choice #2

Bottom spring supports 8 kg, whereas top spring only supports 4 kg

So let's guess that the deflection in the bottom spring is twice the others

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{Bmatrix} 4 \\ 3 \\ 2 \end{Bmatrix}$$

$\boldsymbol{m}$ = 5.63 kg and $\boldsymbol{k}$ = 75.0 N/m



Choice #3 Static deflected shape

Deflection = $\dfrac{4g}{200}$ = 1 unit

Deflection = $\dfrac{6g}{200}$ = 1.5 units

Deflection = $\dfrac{8g}{200}$ = 2 units

Hence $\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{Bmatrix} 4.5 \\ 3.5 \\ 2 \end{Bmatrix}$ $\boldsymbol{m}$ = 5.60 kg and $\boldsymbol{k}$ = 71.6 N/m

Q1: Which model is best? **Q2:** How do we know?

Guess #	m (kg)	k (N/m)	
1	5.11	66.7	
2	5.63	75.0	
3	5.60	71.6	

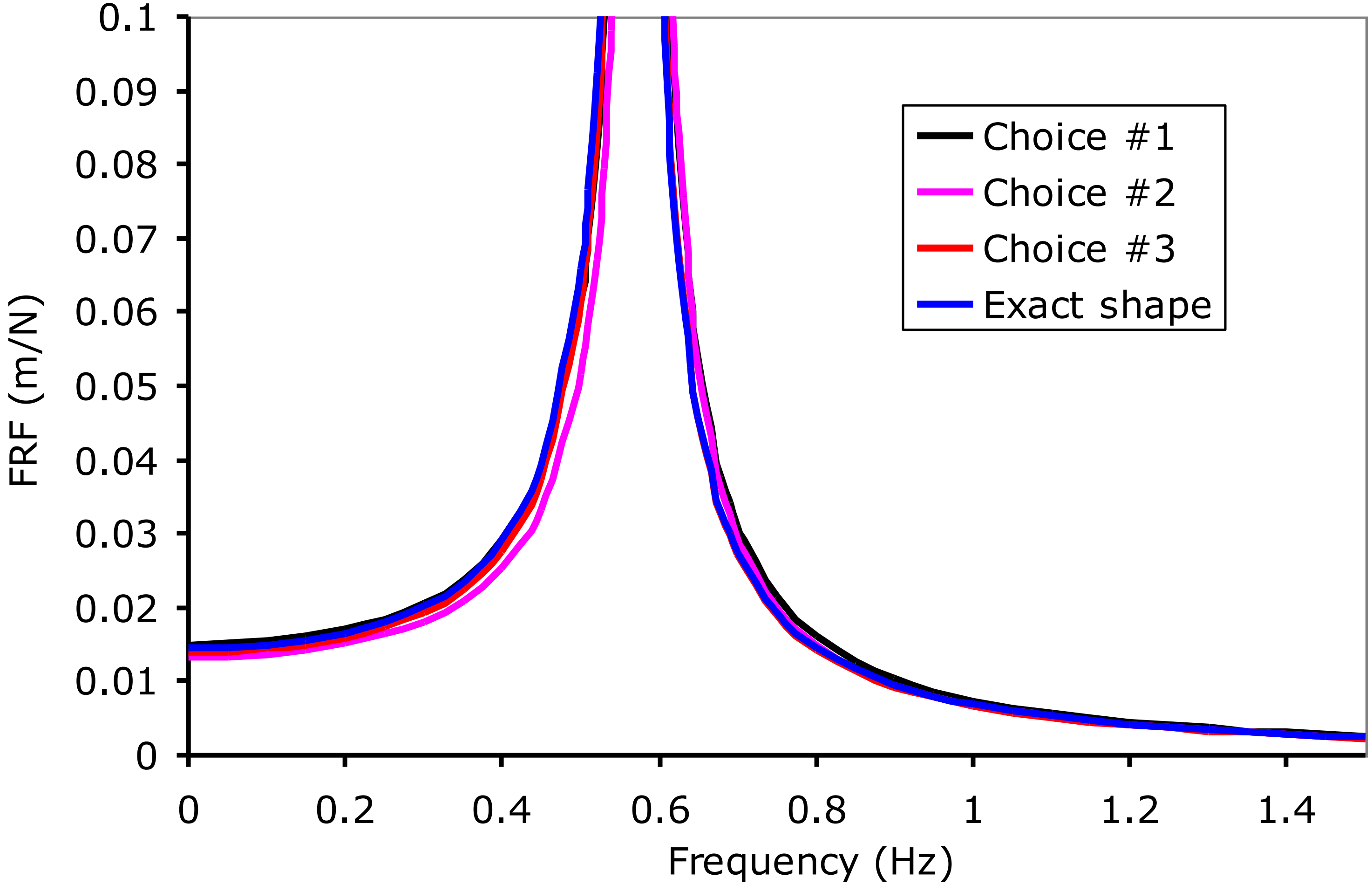
One test is the accuracy of the natural frequency predicted by the model

On this basis, #3 (based on static deflection) is best

Note that the exact mode shape is

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{Bmatrix} 5.02 \\ 3.75 \\ 2.0 \end{Bmatrix} \Rightarrow \quad m = 5.43 \text{ kg} \quad \text{and} \quad k = 68.8 \text{ N/m}$$

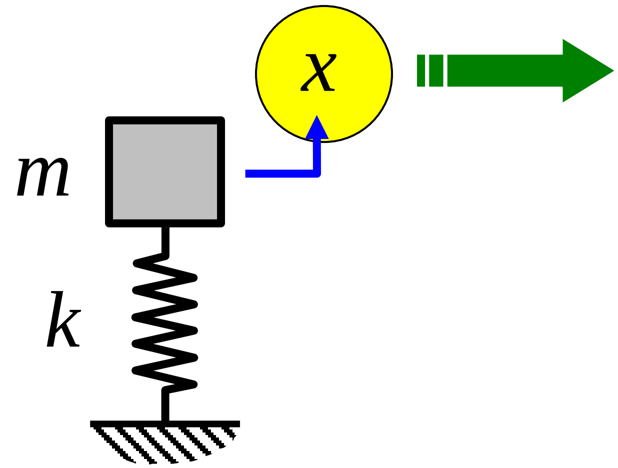
Q1: Which model is best? **Q2:** How do we know?

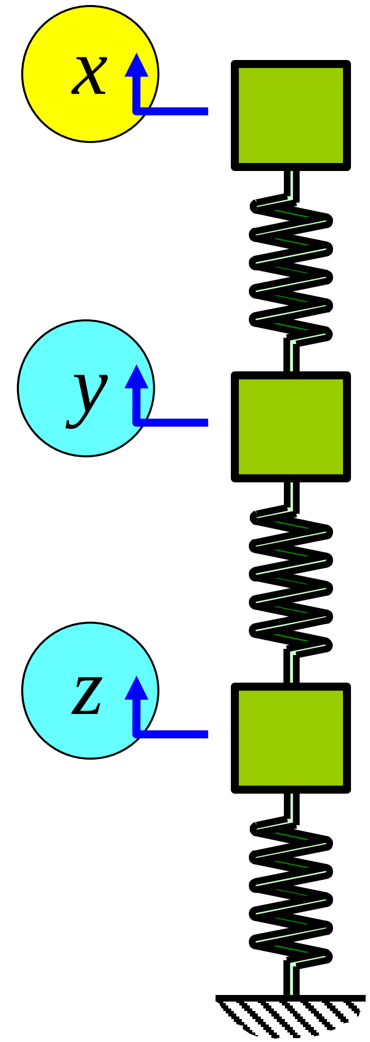


FRFs predicted by all models are similar

The aim of developing the 1DoF model is to use it to predict the response of the real system

Finding $x(t)$ from the model will tell us how the top mass will respond

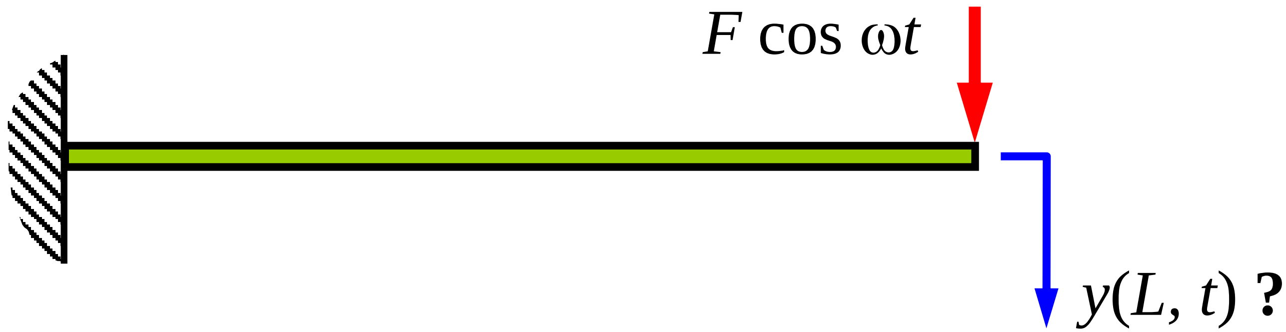




We have assumed that the motion of the real system is defined by the chosen mode shape, so not only does the model predict $x(t)$, it also gives the motion of the other coordinates since they are all linked by the mode shape

In this example, we can get $y(t)$ and $z(t)$, since each is related to $x(t)$ in proportion to the mode shape $\{X:Y:Z\}$

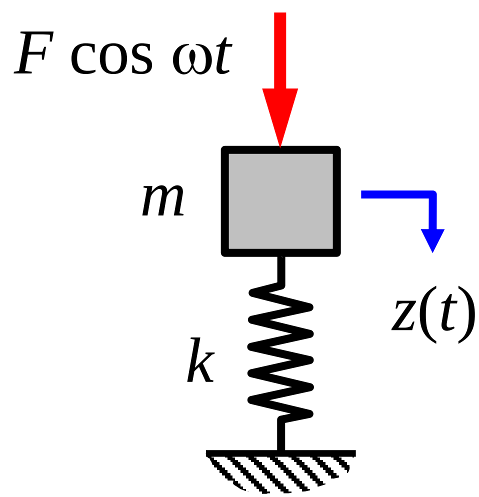
Example 2: **Forced response of a cantilever beam**



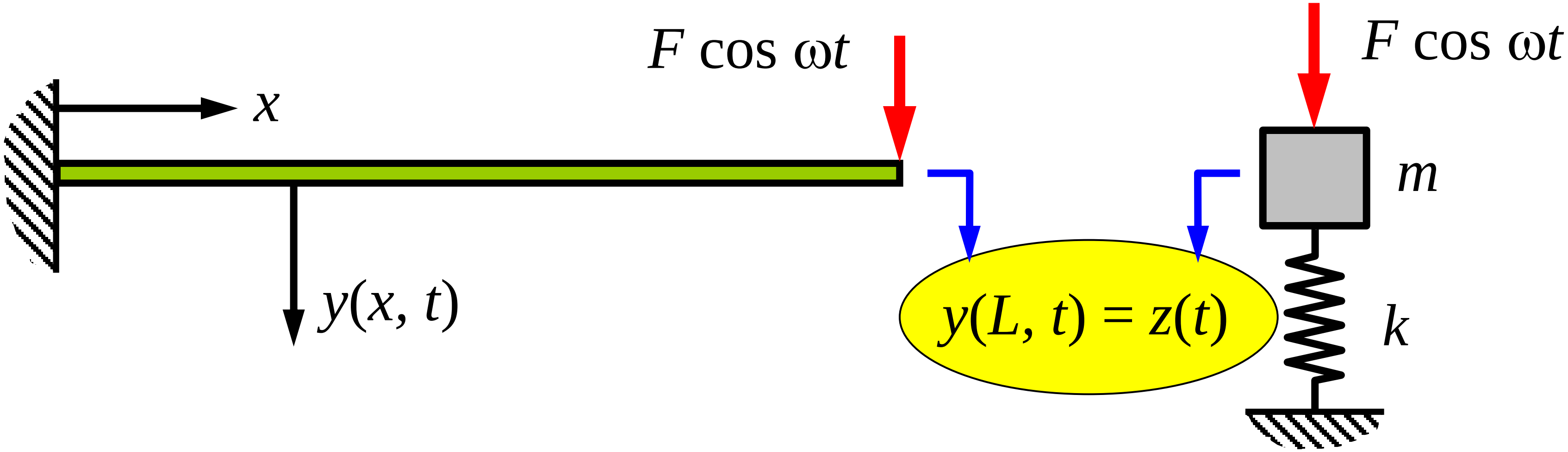
Objective: Use a single-degree-of-freedom model to estimate the steady-state response at the free end of a uniform cantilever beam due to a sinusoidal force with a frequency near the lowest natural frequency of the beam

There are two stages

1. Set up the approximate model
2. Use it to do the steady-state response calculation



Stage 1: Set up the model



We must link the displacement of the model mass to that at the free end of the cantilever

In terms of the chosen displacement variables, $y(L, t) = z(t)$

For steady-state, sinusoidal vibration, the link can be written as

$$Z \cos \omega t = Y(L) \cos \omega t \quad \text{or} \quad Z = Y(L)$$

To proceed, we need to choose an expression for the deflected shape of the cantilever

Choice #1 $Y(x) = C\,x^2$

We must find the value of $\boldsymbol{C}$ that links the two systems

$$Z = Y(L) = C\,L^2$$

Hence $C = \dfrac{Z}{L^2}$ and $\boxed{Y(x) = \dfrac{Z}{L^2}\,x^2}$

This expression for $Y(x)$ is used to calculate the maximum kinetic and strain energies in the beam

Equating each with the equivalent expressions for the single-degree-of-freedom model gives the required mass and stiffness values

Thus $$T_{\max} = \tfrac{1}{2}\,\omega^2 \int_0^L \rho\,A\big[Y(x)\big]^2\,\mathrm{d}x = \tfrac{1}{2}\,m\,\omega^2\,Z^2$$

Hence, the model mass is $\boxed{m = 0.2\,\rho A L}$

Equating the strain energies, gives the model stiffness $\boxed{k = 4\,\frac{EI}{L^3}}$

Applying the natural frequency test to the approximate model,

we find that $\omega_n = \sqrt{\frac{k}{m}} = \frac{4.47}{L^2}\sqrt{\frac{EI}{\rho A}}$

This is the same (poor) result obtained with Rayleigh's Method using this choice for $Y(x)$

For instance, from your previous courses you would have found that the stiffness of a uniform cantilever beam is

$$k = 3\,\frac{EI}{L^3}$$

Choice #2 Static deflected shape, $Y(x) = C\left(x^4 - 4Lx^3 + 6L^2x^2\right)$

Linking this expression at $\boldsymbol{x = L}$ with $\boldsymbol{Z}$, we get

$$Z = Y(L) = C\left[L^4 - 4L^4 + 6L^4\right] = 3CL^4$$

Hence, $C = \dfrac{Z}{3L^4}$ and $\boxed{Y(x) = \dfrac{Z}{3L^4}\left[x^4 - 4Lx^3 + 6L^2x^2\right]}$ (A)

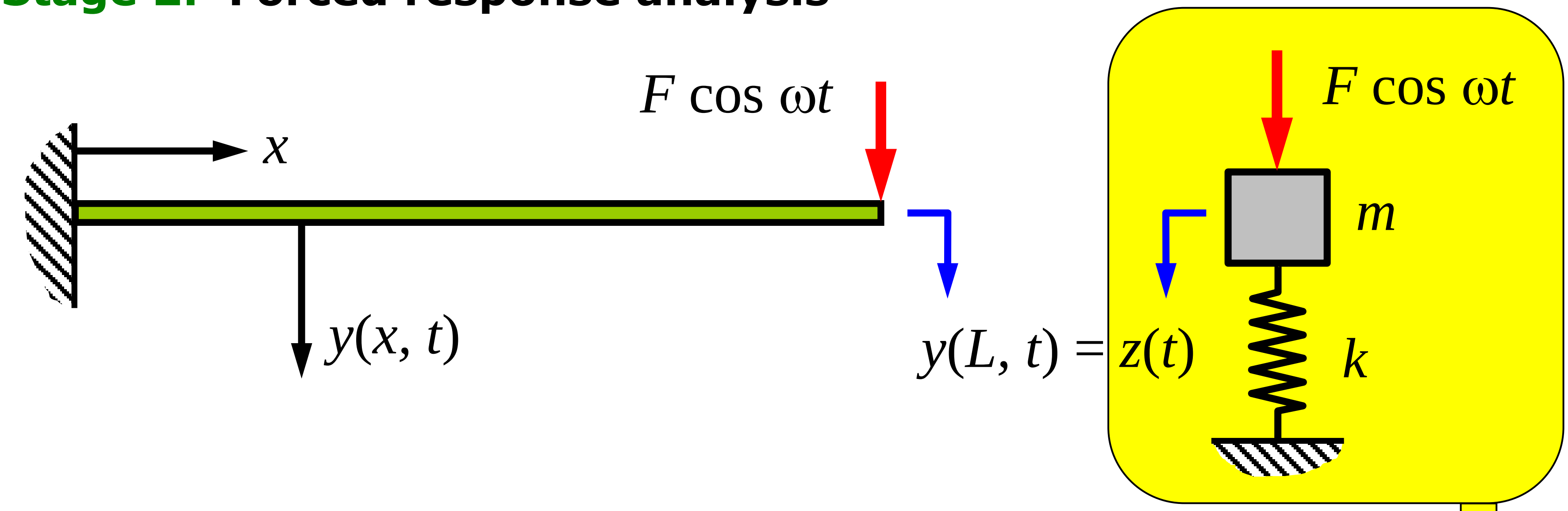
Equating the kinetic and strain energies, the model mass and stiffness values are

$$m = 0.257\,\rho \dot{A} L \quad \text{and} \quad k = 3.20\,\frac{EI}{L^3}$$

The natural frequency test in this case gives $\omega_n = \dfrac{3.54}{L^2}\sqrt{\dfrac{EI}{\rho A}}$

This is lower than the result from the first choice, confirming that the static deflection shape is the better approximation

Stage 2: Forced response analysis

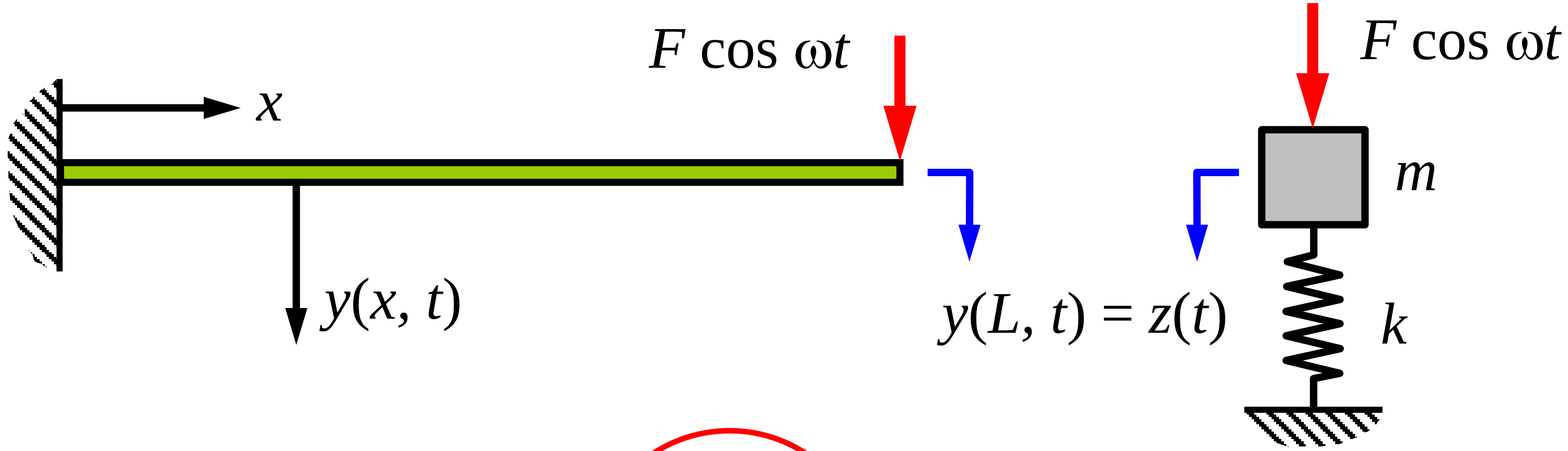


The equation of motion for the model is $m\ddot{z} + kz = F(t)$

Put $F(t) = F\,e^{i\omega t}$ and $z(t) = Z^*\,e^{i\omega t}$ to give $Z^* = \dfrac{F}{(k - m\omega^2)}$

Since there is no damping in this model, $\boldsymbol{Z^*}$ is real

Hence, the response to $F\cos\omega t$ is $z(t) = \dfrac{F}{(k - m\omega^2)}\cos\omega t$



$$z(t) = \frac{F}{\left(k - m\omega^2\right)} \cos \omega t$$

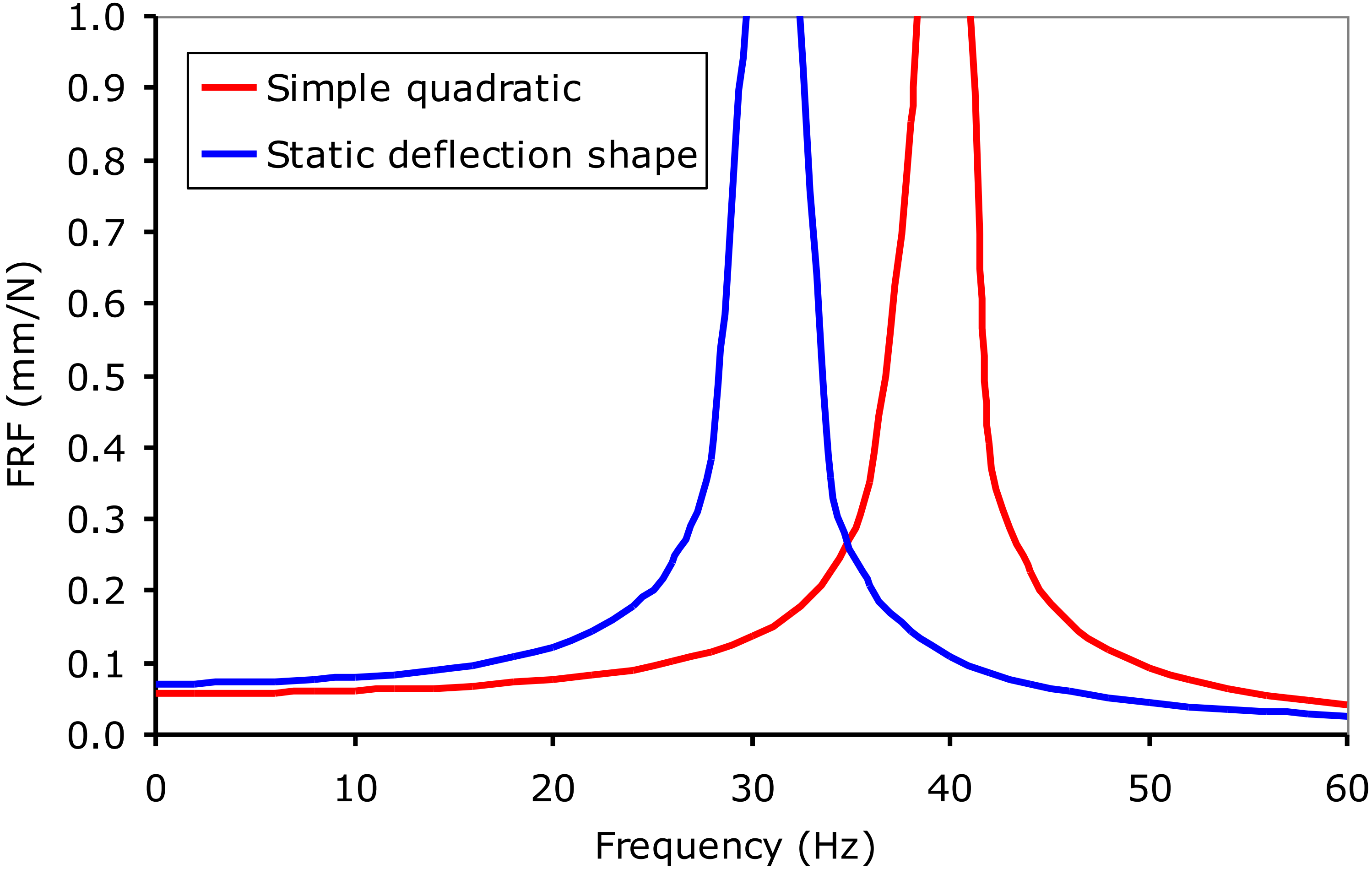
This is the model's prediction of the steady-state response at the free end of the cantilever

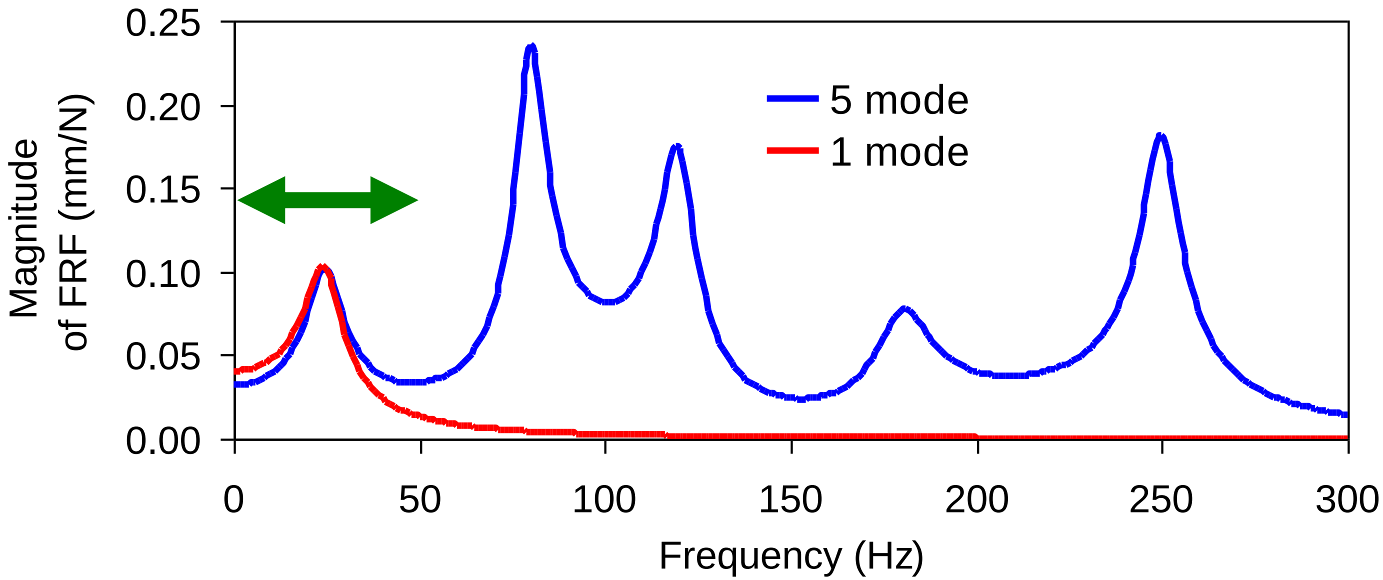
From the assumed deflected shape of the beam (eqn. **A**), we can find the vibration amplitude at **any point** along its length

In the case of Choice #2

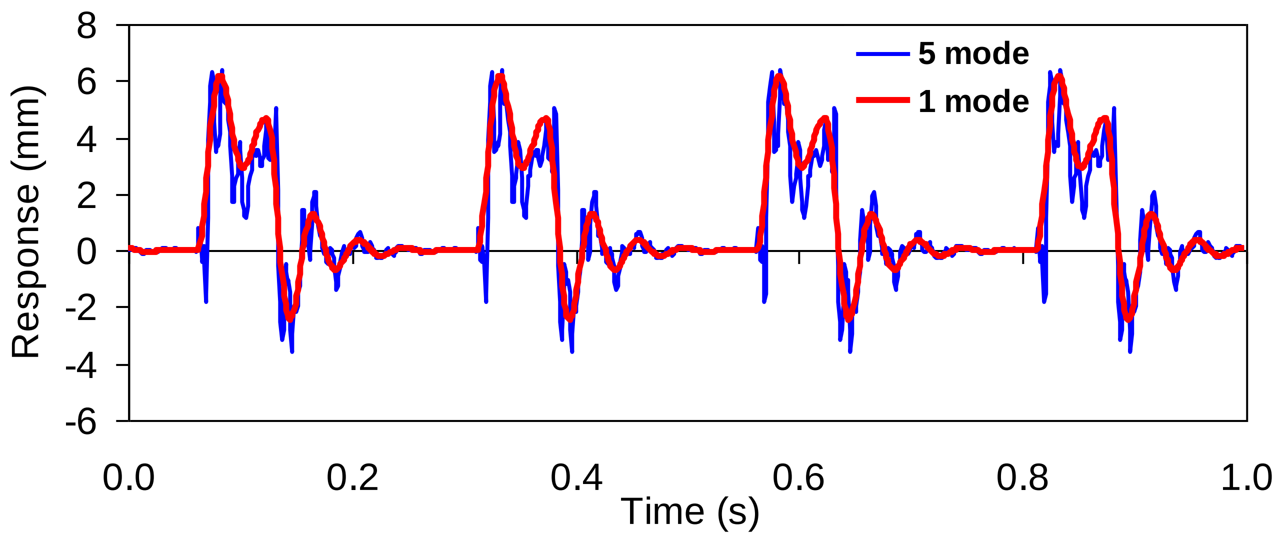
$$y(x,t) = Y(x) \cos \omega t \quad \text{and} \quad Y(x) = \frac{Z}{3L^4}\left[x^4 - 4Lx^3 + 6L^2x^2\right]$$

In this case, there is a significant difference between the two models





For sinusoidal excitation, the 1DoF model works well at low frequencies



It can also give useful predictions for some non-sinusoidal cases http://www.youtube.com/watch?feature=player_embedded&v=pp89tTDxXuI

Videos shown in class for fun time allowing.

To achieve this feat EZH started with a 2D mathematical model. The goal of the model was to understand what motion a quadrocopter would need to perform to throw the pendulum. In other words, what is required for the pendulum to lift off from the quadrocopter and become airborne? This first step allowed to determine (theoretical) feasibility. In addition, it showed the ideal trajectory in terms of positions, speeds, and angles the quadrocopter needed to follow to throw a pendulum. And it offered an insight into the throwing process, including identification of its key design parameters.

The main goal of the next step was to determine how well the theoretic model described reality: How well does the thrown pendulum's motion match the mathematical prediction? Does the pendulum really leave the quadrocopter at the pre-computed time? How does the pendulum behave while airborne? How well do assumptions for catching the pendulum (e.g., completely inelastic collisions, completely rigid pendulum, infinite friction between quadrocopter and pendulum when balancing) hold?

http://www.youtube.com/watch?v=3CR5y8qZf0Y&feature=youtu.be