FREE VIBRATION OF SINGLE DEGREE-OF-FREEDOM SYSTEMS

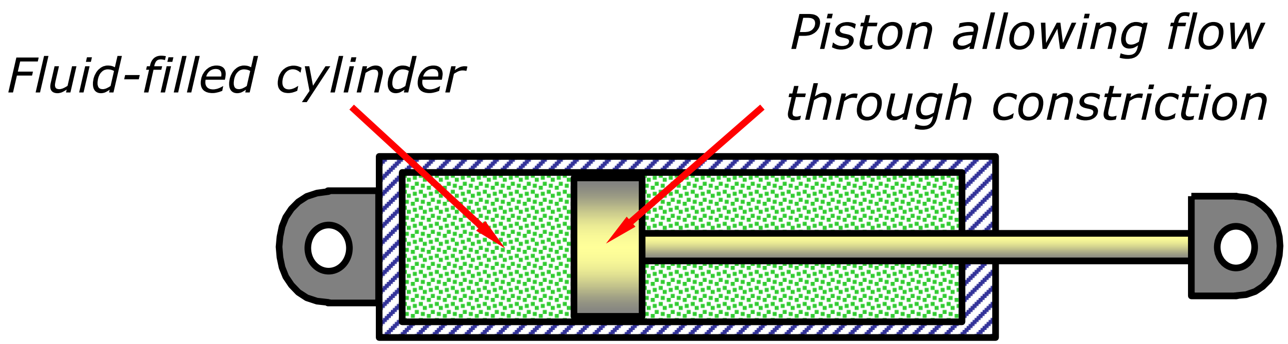
This section will analyse the response of single degree-of-freedom systems to external excitation that is removed when time starts (t=0)

This takes the form either of **applied forces** and/or **moments** or of **imposed displacement** on part of the system.

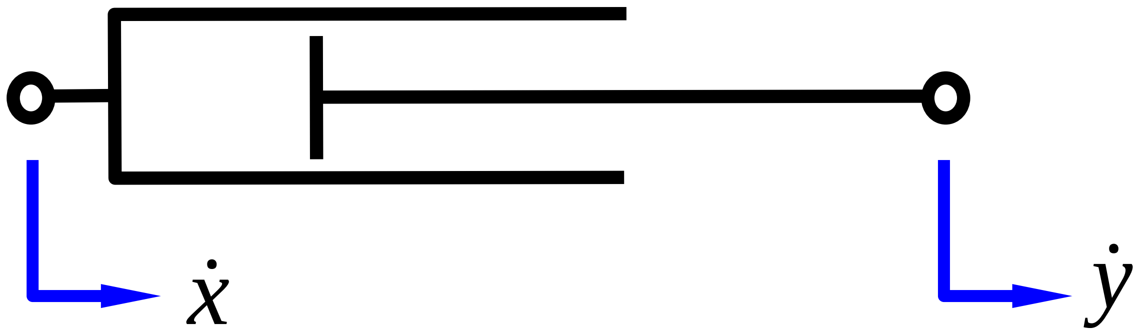
Damping

Damping is a phenomenon of energy dissipation in a vibrating structure

We will consider one theoretical damping model, called ***viscous damping*** and will only consider discrete dampers



Diagrammatic form



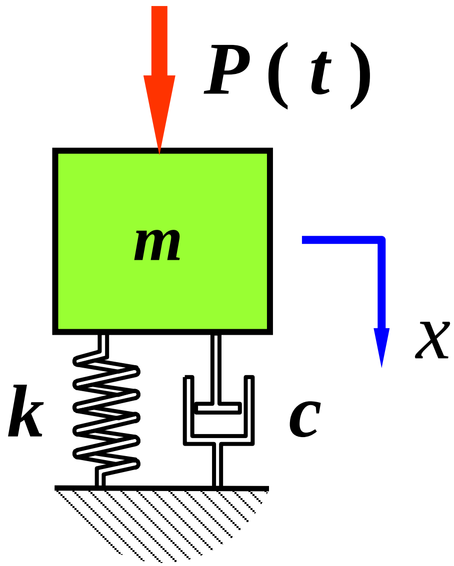
Assumption **Damper force is proportional to the relative velocity and acts in a direction to oppose the motion**

Damper force is $c\left(\dot{x}-\dot{y}\right)$

c is the ***damping coefficient*** which has units of N / (m/s) or Ns/m

Example 1 Mass-Spring-Damper System

STEP 1: Dynamic model

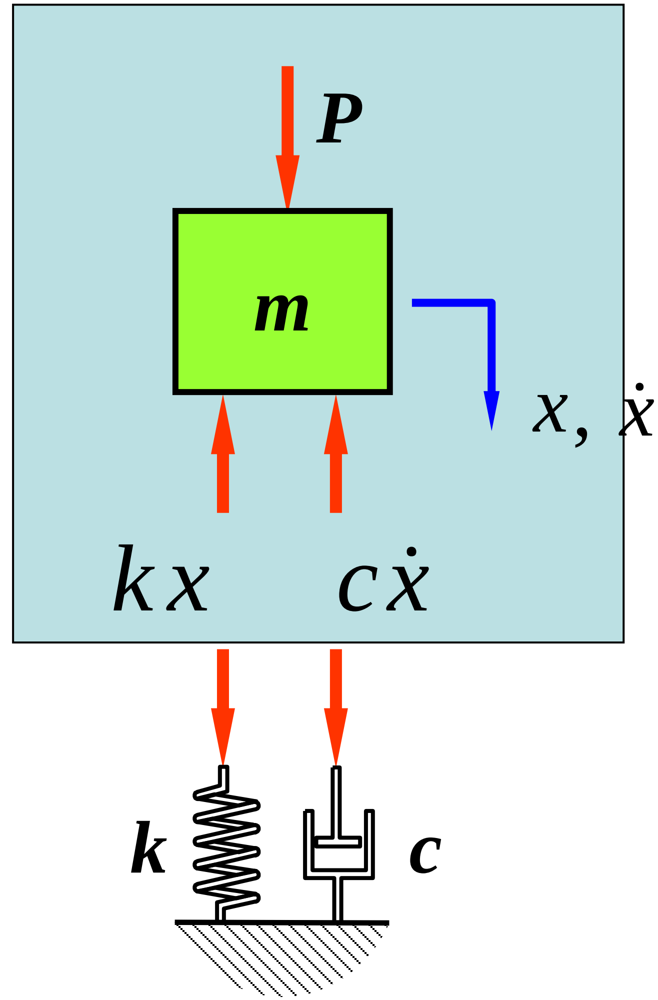


(i) Remove spring & damper

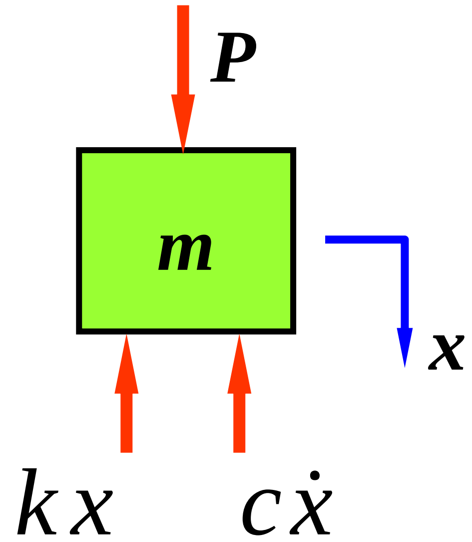
(ii) Add the motion coordinate

(iii) Add the forces

STEP 2: Free Body Diagram



STEP 3: Equation of motion



x

$\dot{x}$

$\ddot{x}$

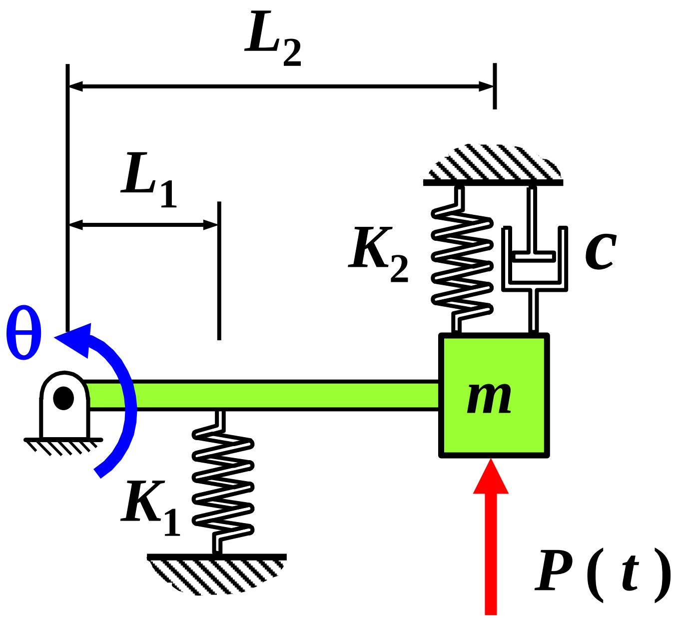
$$P - kx - c\dot{x} = m\ddot{x}$$

or

$$m\ddot{x} + c\dot{x} + kx = P(t)$$

Example 2 Rocker System (used for a previous example)

STEP 1: Dynamic model

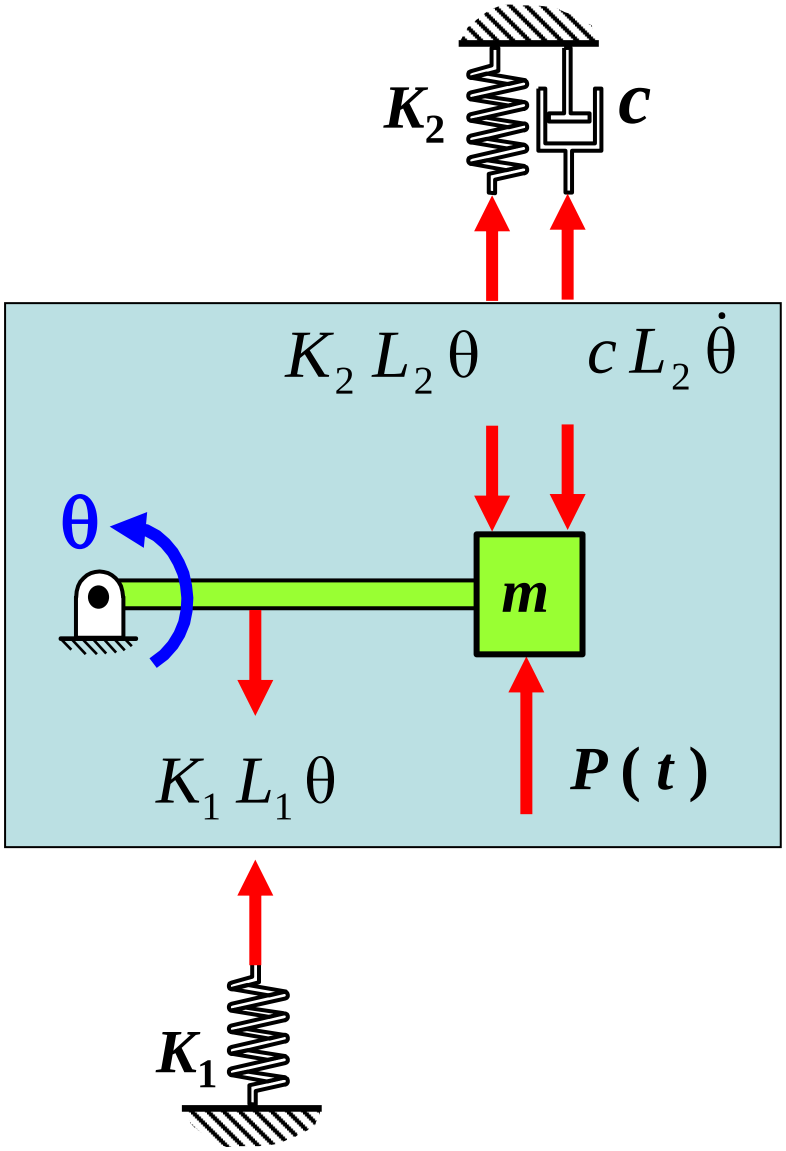


(i) Remove springs & damper

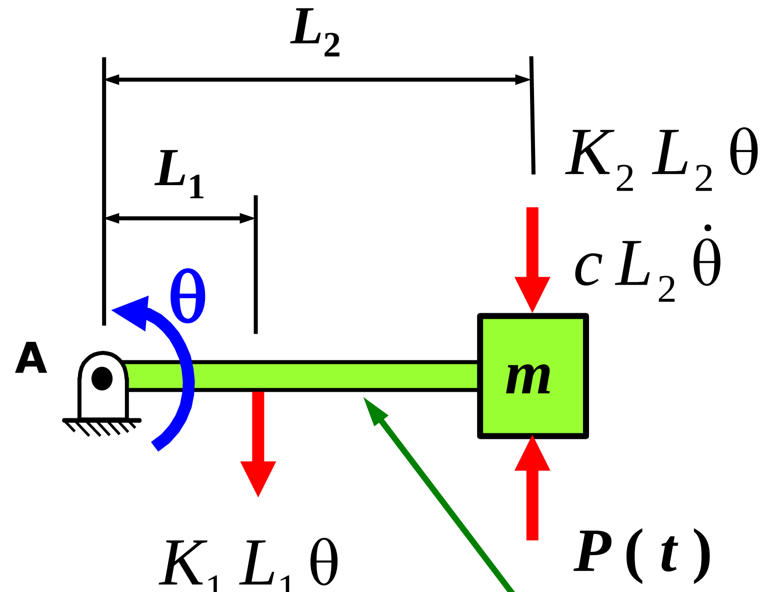
(ii) Add the motion coordinate

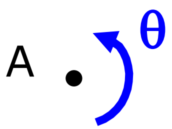
(iii) Add the forces

STEP 2: Free Body Diagram



STEP 3 Equation of motion





Take moments of the forces about the fixed pivot, A

$$-K_1 L_1 \theta . L_1 - K_2 L_2 \theta . L_2 - c L_2 \dot{\theta} . L_2 + P . L_2 = I_A \ddot{\theta}$$

$$I_A = m L_2^2$$

$$m L_2^2 \ddot{\theta} + c L_2^2 \dot{\theta} + \left(K_1 L_1^2 + K_2 L_2^2 \right) \theta = P L_2$$

Example 3   Single-axle caravan

Assumptions

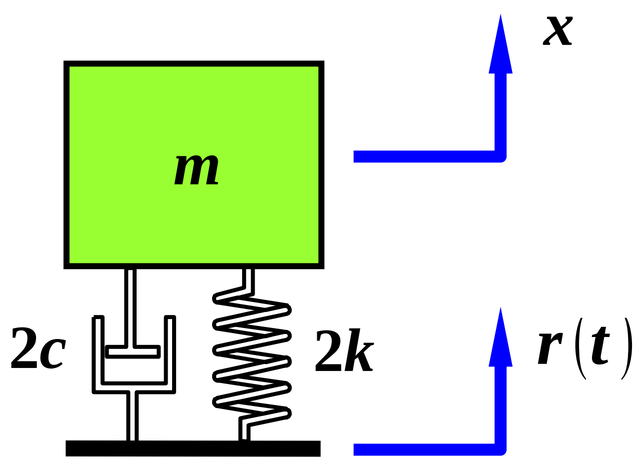
- tyres are very stiff compared to suspension springs
- tyres stay in contact with the road
- caravan acts as a rigid mass
- body motion is vertical translation

STEP 1: Dynamic model

Body

Springs & Dampers

Axle



An example of **Displacement excitation**

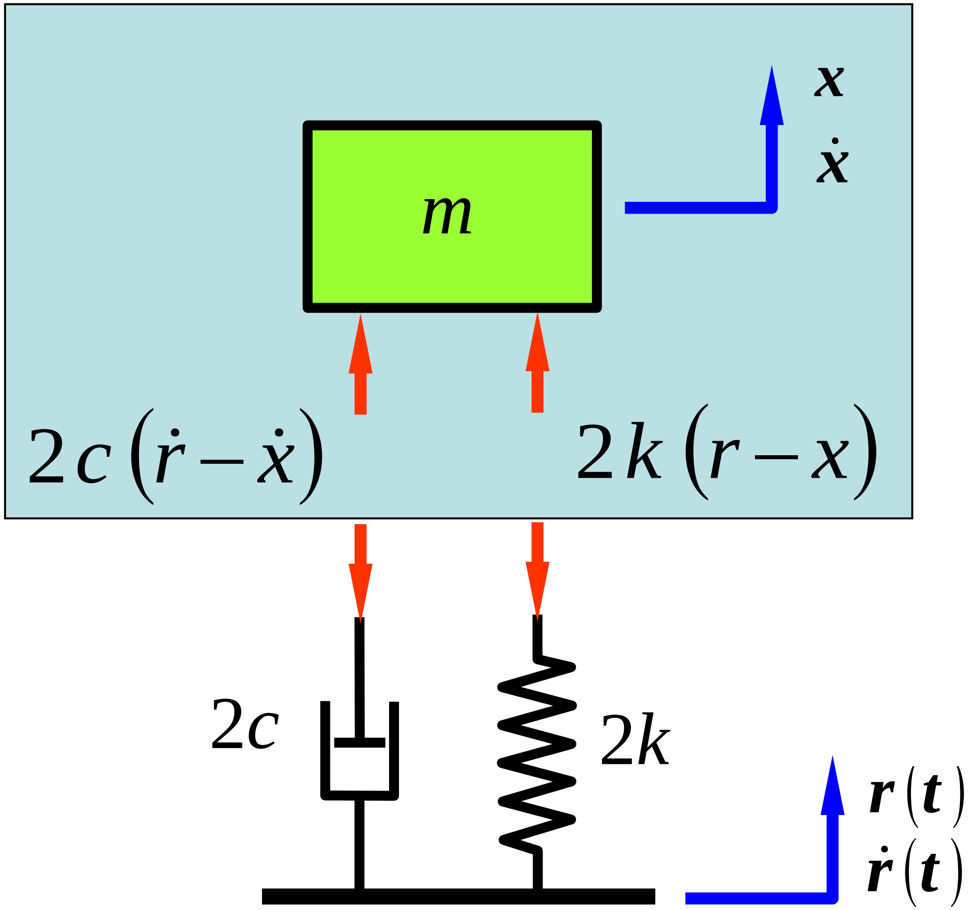
Displacement $r(t)$ is defined exactly by the road profile and vehicle speed

STEP 2 Free Body Diagram

(i) Remove springs & dampers

(ii) Add the motion coordinates

(iii) Add the forces



(a) Springs

Spring force* = *Stiffness* x *Change of length

What is the **change of length** of the spring? ⇨ $r-x$

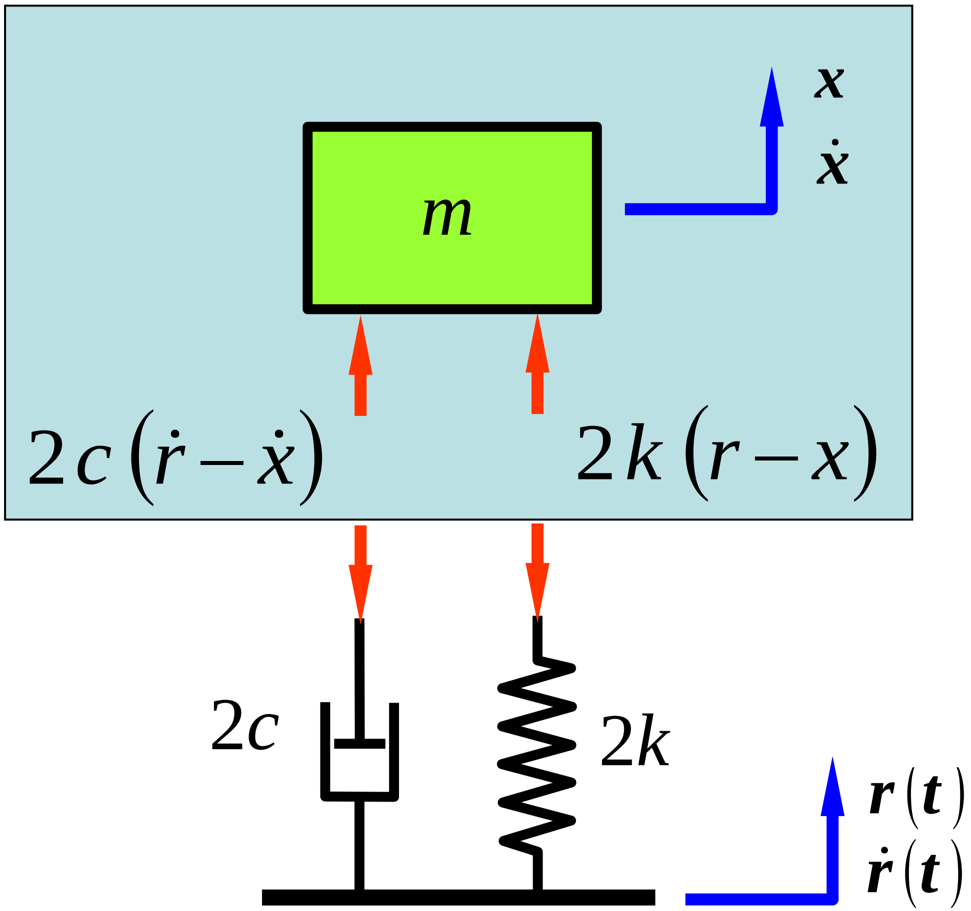
Is the spring in **tension** or **compression**? ⇨ **compression**

STEP 2 Free Body Diagram

(i) Remove springs & dampers

(ii) Add the motion coordinates

(iii) Add the forces



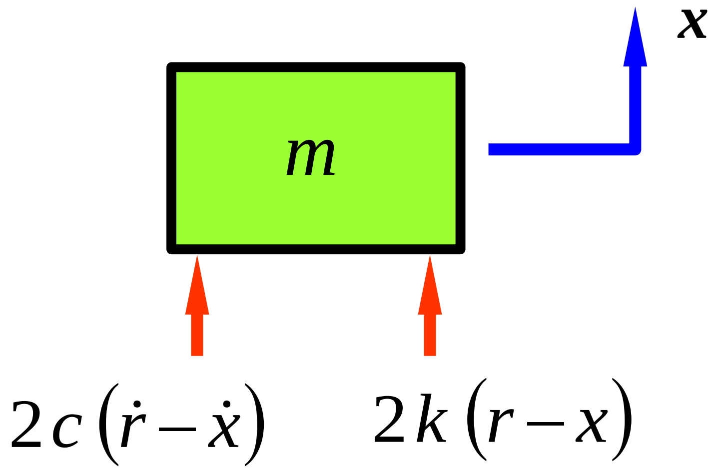
(b) Dampers

***Damper force* = *Damping coefficient* x Relative velocity**

What is the **relative velocity** between the ends? ⇨ $\dot{r}-\dot{x}$

Is the damper **extending** or **compressing**? ⇨ **compressing**

STEP 3: Equation of motion



$$\uparrow x \qquad +2k(r-x) + 2c(\dot{r}-\dot{x}) = m\ddot{x}$$

or

$$m\ddot{x} + 2c\dot{x} + 2kx = 2c\dot{r}(t) + 2kr(t)$$

Summary so far

Mass-spring-damper system	$m\ddot{x} + c\dot{x} + kx = P(t)$
Rocker system	$mL_2^2\ddot{\theta} + cL_2^2\dot{\theta} + \left(K_1L_1^2 + K_2L_2^2\right)\theta = L_2P(t)$
Single-axle caravan	$m\ddot{x} + 2c\dot{x} + 2kx = 2c\dot{r}(t) + 2kr(t)$

All are second-order ODEs with constant coefficients

All linear, single-degree-of-freedom systems have this form, which can be written generically as:

$$\boldsymbol{M\ddot{z} + C\dot{z} + Kz = F(t)} \qquad (1)$$

$\boldsymbol{z}$	is the response coordinate
$\boldsymbol{M}$	is the Mass coefficient of acceleration $\boldsymbol{\ddot{z}}$
$\boldsymbol{C}$	is the Damping coefficient of velocity $\boldsymbol{\dot{z}}$
$\boldsymbol{K}$	is the Stiffness coefficient of displacement $\boldsymbol{z}$
$\boldsymbol{F(t)}$	is the excitation function (independent of $\boldsymbol{z}$)

Remember that **every term in the expressions for the coefficients** $\boldsymbol{M}$**,** $\boldsymbol{C}$ **and** $\boldsymbol{K}$ **must be positive** and that **any negative sign means that** ***your equation is definitely wrong***

The solution to the equation of motion depends on the nature of the excitation function and on the amount of damping in the system.

There are 3 types of response we will consider here

A: "FREE" VIBRATION – I.e. no external forces

Case (i)	**Zero damping**
Case (ii)	**High damping**
Case (iii)	**Critical damping**
Case (iv)	**Light damping**

B: FORCED VIBRATION – RESPONSE TO SINUSOIDAL EXCITATION

C: FORCED VIBRATION – RESPONSE TO PERIODIC EXCITATION

You must be able to recognise the various cases so that you can apply the appropriate solution procedure

A: "FREE" VIBRATION

"Free" vibration means that there is no external applied force or moment acting on the structure

$$z(t) = A\cos\lambda t = A\,\mathrm{e}^{\lambda t}$$

$$\dot{z}(t) = \lambda A\,\mathrm{e}^{\lambda t}$$

$$\ddot{z}(t) = \lambda^2 A\,\mathrm{e}^{\lambda t}$$

For $F(t) = 0$, the general system response solution is

Substituting into the equation of motion gives,

$$M\lambda^2 A\mathrm{e}^{\lambda t} + C\lambda A\mathrm{e}^{\lambda t} + K A\mathrm{e}^{\lambda t} = 0$$

For a non-trivial solution, $M\lambda^2 + C\lambda + K = 0$

so that

$$\lambda_{1,2} = \frac{-C \pm \sqrt{C^2 - 4KM}}{2M} \qquad (2)$$

The complete solution for position as a function of time is then

$$z(t) = A_1\,\mathrm{e}^{\lambda_1 t} + A_2\,\mathrm{e}^{\lambda_2 t} \qquad (3)$$

$$z(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \qquad (3)$$

The integration constants, $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$, are found from the "initial conditions" specified in the problem.

Usually these are given to you in numerical or plot form.

I.e. at time equals zero the mass is lifted up by 0.2 m and released from rest.

Therefore at $t = 0, \quad z(0) = 0.2, \text{ and } \dot{z}(0) = 0$

This gives you 2 equations and 2 unknowns to solve for A_1 and A_2.

$$z(0) = 0.2 = A_1 e^{\lambda_1 * 0} + A_2 e^{\lambda_2 * 0}$$

$$0.2 = A_1 + A_2$$

$$\dot{z}(0) = 0 = A_1 \lambda_1 e^{\lambda_1 * 0} + A_2 \lambda_2 e^{\lambda_2 * 0}$$

$$0 = A_1 \lambda_1 + A_2 \lambda_2$$

$$\lambda_{1,2} = \frac{-C \pm \sqrt{C^2 - 4KM}}{2M} \qquad (2)$$

It can be seen from equation (2) that the roots $\lambda_{1,2}$ can be either **real** or **complex**, depending on the amount of damping present

There are **FOUR CASES** to consider

Case (i)	**Zero damping**
Case (ii)	**High damping**
Case (iii)	**Critical damping**
Case (iv)	**Light damping**

Case (i) Zero Damping

For zero damping, the system will oscillate with simple harmonic motion, although the sinusoidal waveform is not obvious from equation (3)

$$z(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \quad (3)$$

What we need to do is look at λ

$$\lambda_{1,2} = \frac{-C \pm \sqrt{C^2 - 4KM}}{2M} \quad (2)$$

For $\boldsymbol{C = 0}$,

$$\lambda_{1,2} = \pm \frac{\sqrt{-4KM}}{2M} = \pm \mathbf{i} \sqrt{\frac{K}{M}}$$

The term $\sqrt{\frac{K}{M}}$ is called the ***undamped natural frequency*** and is given the symbol $\boldsymbol{\omega_n}$

Returning to the general case, equation (3) becomes

$$z(t) = A_1 \, \mathrm{e}^{\mathbf{i}\,\omega_n t} + A_2 \, \mathrm{e}^{-\mathbf{i}\,\omega_n t}$$

This still doesn't look much like a sinusoidal waveform. However,

$$\mathrm{e}^{\mathbf{i}\,\omega_n t} = \cos\omega_n t + \mathbf{i}\sin\omega_n t \quad \text{and} \quad \mathrm{e}^{-\mathbf{i}\,\omega_n t} = \cos\omega_n t - \mathbf{i}\sin\omega_n t$$

$$z(t) = A_1 \cos\omega_n t + A_1 i \sin\omega_n t + A_2 \cos\omega_n t - A_2 i \sin\omega_n t$$

Therefore $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are a complex conjugate pair, and

$$z(t) = B\cos\omega_n t + C\sin\omega_n t \qquad (4)$$

As before you can now solve for *B* and *C* using known conditions for the system

Case (ii) High Damping

The ***damping ratio***, γ, is

$$\gamma = \frac{C}{\text{critical damping}} = \frac{C}{C_{cr}} = \frac{C}{2\sqrt{KM}}$$

Damping is said to be "high" if $\gamma > 1$

(sometimes refered to as $C^2 > 4KM$)

In this case, the two roots, $\lambda_{1,\,2}$ are both **REAL** and **NEGATIVE**

The response is given by equation (3) and is the sum of two decaying exponential functions

The constants A_1 and A_2 are found from the initial conditions as usual

Case (iii) Critical damping

Damping is said to be if $\gamma = 1$ ($C^2 = 4KM$)

Thus $$C_{crit} = 2\sqrt{KM} \qquad (5)$$

From equation (2) it will be seen that

$$\lambda_1 = \lambda_2 = -\frac{C_{crit}}{2M} \equiv -\omega_n$$

To maintain distinct parts to the solution, the response is given by

$$z(t) = A_1 e^{-\omega_n t} + A_2 \boldsymbol{t} e^{-\omega_n t} \qquad (6)$$

*Note the **"t"** in the second term*

Case (iv) Light Damping

Most engineering structures have damping levels much less than critical

Damping is "light" when $\gamma < 1$ ($C^2 < 4KM$)

The roots of equation (2) are a complex conjugate pair

$$\lambda_{1,2} = -\frac{C}{2M} \pm \mathbf{i}\,\frac{\sqrt{4KM - C^2}}{2M} \qquad (7)$$

The ***damping ratio***, $\boldsymbol{\gamma}$, is

$$\gamma = \frac{C}{\text{critical damping}} = \frac{C}{2\sqrt{KM}}$$

Using the undamped natural frequency, $\boldsymbol{\omega_n}$, equation (7) becomes

$$\lambda_{1,2} = -\gamma\,\omega_n \pm \mathbf{i}\,\omega_n\sqrt{1-\gamma^2} \qquad (8)$$

Equation (3) gives

$$z(t) = A_1 \, e^{\left(-\gamma\omega_n + \mathbf{i}\omega_n\sqrt{1-\gamma^2}\right)t} + A_2 \, e^{\left(-\gamma\omega_n - \mathbf{i}\omega_n\sqrt{1-\gamma^2}\right)t} \qquad (9)$$

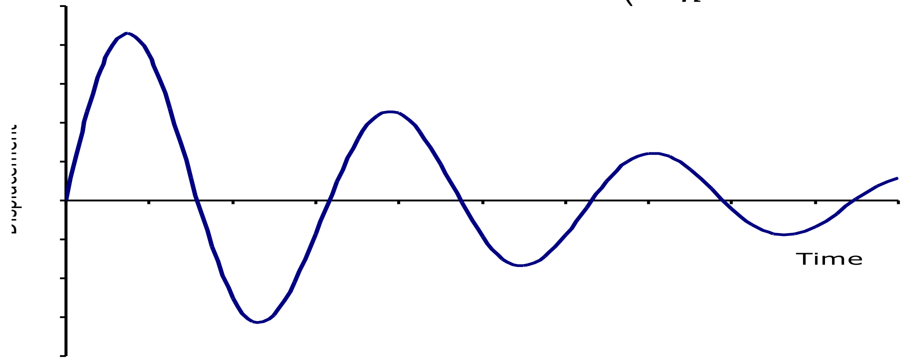
Using of the complex exponential identities and the fact that $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are a complex conjugate pair, equation (9) becomes

$$z(t) = e^{-\gamma\omega_n t}\left[B_1 \cos \omega_n \sqrt{1-\gamma^2}\, t + B_2 \sin \omega_n \sqrt{1-\gamma^2}\, t\right] \qquad (10)$$

Equation (10) describes a sinusoidal waveform (indicated by the terms in the square brackets) with an exponentially decaying term that will cause the amplitude of the sinusoid to decrease

An alternative to equation (10) is

$$z(t) = C_0 \, e^{-\gamma\omega_n t} \cos\left(\omega_n \sqrt{1-\gamma^2}\, t - \psi\right) \qquad (11)$$



Equation (3) gives

$$z(t) = A_1\, e^{\left(-\gamma\omega_n + \mathbf{i}\omega_n\sqrt{1-\gamma^2}\right)t} + A_2\, e^{\left(-\gamma\omega_n - \mathbf{i}\omega_n\sqrt{1-\gamma^2}\right)t} \qquad (9)$$

Using of the complex exponential identities and the fact that $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are a complex conjugate pair, equation (9) becomes

$$z(t) = e^{-\gamma\omega_n t}\left[B_1 \cos\omega_n\sqrt{1-\gamma^2}\, t + B_2 \sin\omega_n\sqrt{1-\gamma^2}\, t\right] \qquad (10)$$

Equation (10) describes a sinusoidal waveform (indicated by the terms in the square brackets) with an exponentially decaying term that will cause the amplitude of the sinusoid to decrease

The frequency of vibration is $\Omega_n = \omega_n\sqrt{1-\gamma^2}$

This is known as the ***damped*** ***natural frequency*** and is less than the undamped natural frequency, $\boldsymbol{\omega_n}$. It is sometimes given the variable name $\boldsymbol{\omega_d}$.

To determine the free response of any system all you need to do is know what damping level it contains and choose the corresponding equation to solve.

Case (i) Zero Damping $C = 0$

$$z(t) = B \cos \omega_n t + C \sin \omega_n t \qquad (4)$$

Case (ii) High Damping $\gamma > 1$ $(C^2 > 4KM)$

$$z(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \qquad (3) \quad \text{where} \quad \lambda_{1,2} = \frac{-C \pm \sqrt{C^2 - 4KM}}{2M}$$

Case (iii) Critical damping $\gamma = 1$ $(C^2 = 4KM)$

$$z(t) = A_1 e^{-\omega_n t} + A_2 \boldsymbol{t} e^{-\omega_n t} \qquad (6)$$

Case (iv) Light Damping $\gamma < 1$ $(C^2 < 4KM)$

$$z(t) = e^{-\gamma \omega_n t} \left[B_1 \cos \omega_n \sqrt{1-\gamma^2}\, t + B_2 \sin \omega_n \sqrt{1-\gamma^2}\, t \right] \quad (10)$$

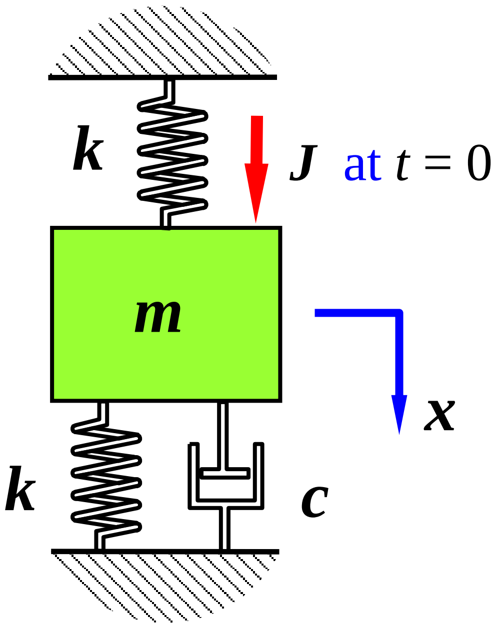
OR

$$z(t) = C_0 e^{-\gamma \omega_n t} \cos\left(\omega_n \sqrt{1-\gamma^2}\, t - \psi \right) \quad (11)$$

where $\gamma = \dfrac{C}{2\sqrt{KM}}$

Worked Example

STEP 1: Dynamic model

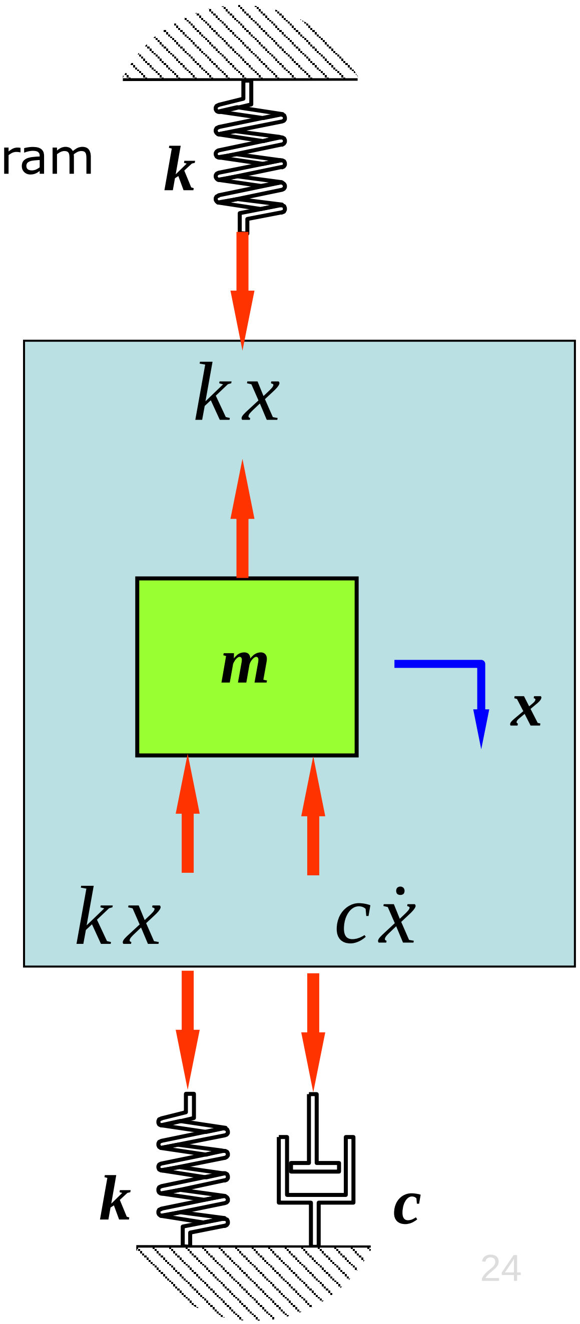


When at rest in equilibrium, the mass receives an impulse of 5 Ns applied at time, t = 0

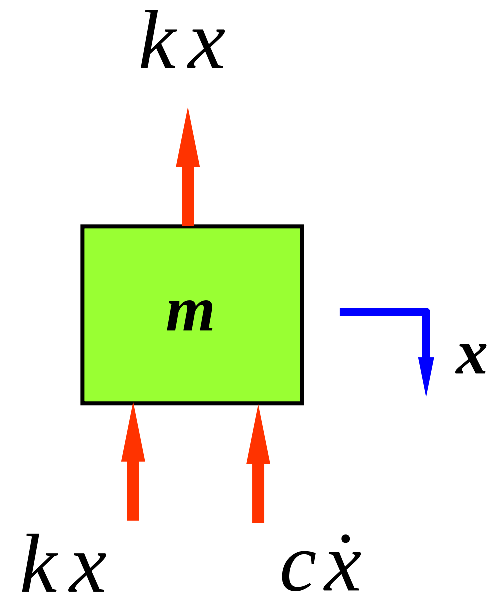
Find the response for $t > 0$

Data: k = 500 N/m
c = 20 Ns/m
m = 10 kg

STEP 2:
Free Body Diagram for $t > 0$



STEP 3: Equation of motion



$\downarrow x \qquad -2kx - c\dot{x} = m\ddot{x}$

or $\quad m\ddot{x} + c\dot{x} + 2kx = 0$

c.f. $\quad M\ddot{z} + C\dot{z} + Kz = 0$

$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{2k}{m}} = 10 \text{ rad/s}$$

$$\gamma = \frac{C}{2\sqrt{KM}} = \frac{c}{2\sqrt{2km}} = 0.1$$

∴ "light" damping

From (10) $$x(t) = e^{-\gamma\omega_n t}\left[B_1 \cos \Omega_n t + B_2 \sin \Omega_n t\right]$$

where $$\Omega_n = \omega_n \sqrt{1-\gamma^2}$$

You now have one equation (10), with two unknowns B_1 and B_2. You therefore need to look at initial conditions to solve for the unknowns.

Initial conditions: The system starts at rest

$$\text{at} \quad t = 0 \quad x = 0 \quad \therefore B_1 = 0$$

Hence $$x(t) = B_2\, e^{-\gamma\omega_n t} \sin \Omega_n t$$

Initial velocity: You are given the impulse J = 5 Ns

The velocity immediately after the impulse, $\dot{x}_0$, is given by

Impulse = Change in momentum

$$J = m(\dot{x}_0 - 0)$$

$$\therefore \dot{x}_0 = \frac{J}{m}$$

Differentiate $x(t) = B_2 \, e^{-\gamma \omega_n t} \sin \Omega_n t$ to give

$$\dot{x} = B_2 \left[\Omega_n \, e^{-\gamma \omega_n t} \cos \Omega_n t - \gamma \omega_n \, e^{-\gamma \omega_n t} \sin \Omega_n t \right]$$

$$\dot{x} = \frac{J}{m} \quad \text{at} \quad t = 0 \qquad \therefore \frac{J}{m} = B_2 \left[\Omega_n - 0 \right]$$

Hence $$B_2 = \frac{J}{m \Omega_n}$$

and $$x(t) = \frac{J}{m \Omega_n} e^{-\gamma \omega_n t} \sin \Omega_n t$$

Substituting the numerical values gives

$$x(t) = 0.0505 \, e^{-t} \sin 9.9 \, t \quad [\mathrm{m}]$$

Estimating Damping

$$z(t) = e^{-\gamma \omega_n t}\left[B_1 \cos \Omega_n t + B_2 \sin \Omega_n t\right] \qquad (10)$$

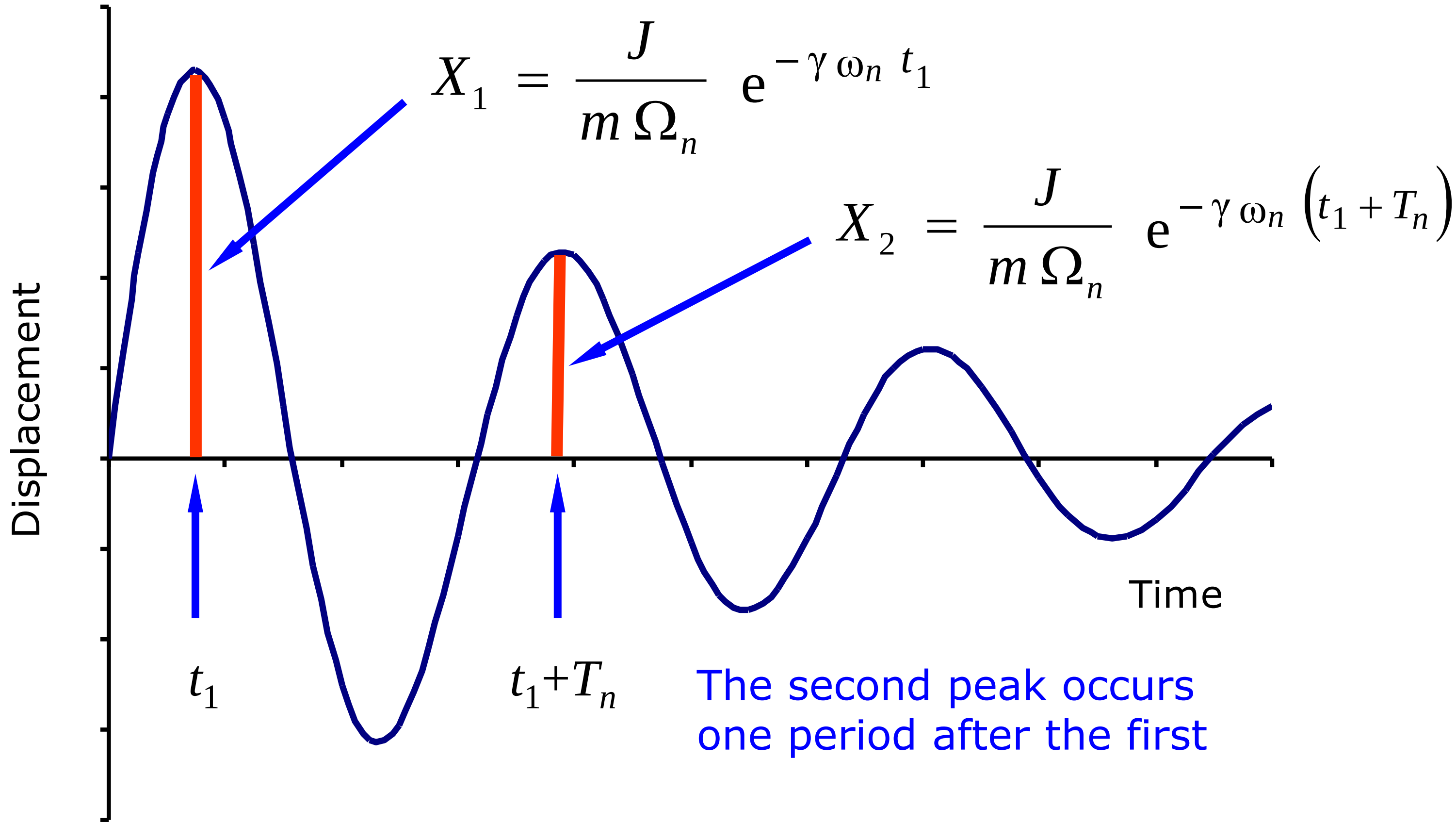
$$z(t) = C_0\, e^{-\gamma \omega_n t} \cos\left(\Omega_n t - \psi\right) \qquad (11)$$

where $\Omega_n = \omega_n \sqrt{1-\gamma^2}$

Equation (10) or (11) shows that the rate of decay of the free vibration of a structure depends directly on the damping ratio and this gives a method of measuring damping

In the previous worked example, suppose we didn't know the damping value, but had done an experiment to measure the transient displacement caused by the impulse

In the worked example, $$x(t) = \frac{J}{m\,\Omega_n}\, e^{-\gamma\,\omega_n t} \sin \Omega_n t$$



The ratio of the amplitudes is $$\frac{X_1}{X_2} = e^{\gamma\,\omega_n\, T_n}$$

The ratio of the amplitudes is $\dfrac{X_1}{X_2} = e^{\gamma \omega_n T_n}$

Period of the damped vibration $T_n = \dfrac{2\pi}{\Omega_n} = \dfrac{2\pi}{\omega_n \sqrt{1 - \gamma^2}}$

$$\therefore \frac{X_1}{X_2} = e^{\gamma(2\pi)}$$

Taking logs, $\ln\left(\dfrac{X_1}{X_2}\right) = 2\pi\gamma$

In this example, $X_1 = 0.0431\,\text{m}$ and $X_2 = 0.0229\,\text{m}$

Hence, $\gamma = 0.101$ and $c = 20.1\ \text{Ns/m}$

Note that the ratio of **any** two successive peaks is a constant

$$\frac{X_1}{X_2} = \frac{X_2}{X_3} = \frac{X_3}{X_4} = \cdots$$