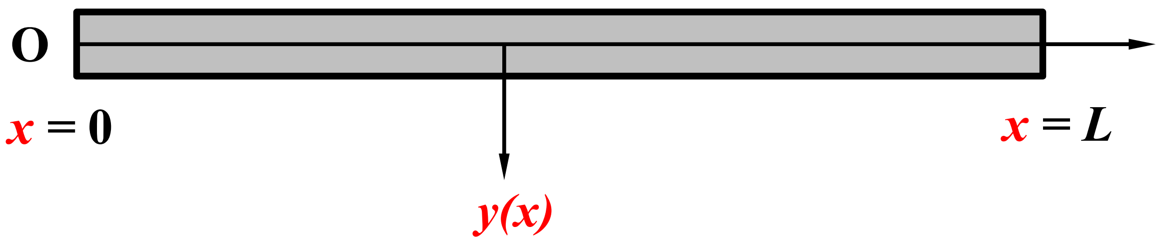
Overview of Beam Vibration



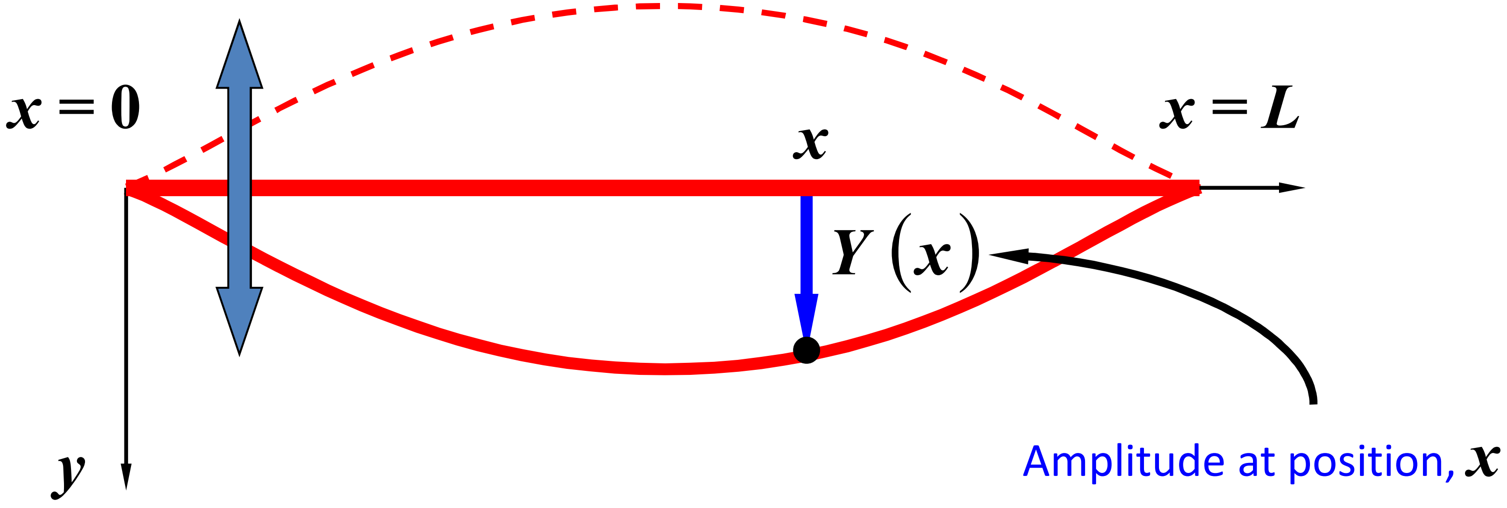
- Given a generalized beam we wish to solve for
  - Natural Frequency ω_{nr}
    - Where r is the frequency number (1, 2, 3, …)
  - Mode shapes associated with specific values of ω_{nr}
    - Essentially we are looking for the vertical displacement, y, for any given point along the beam, x

- From previous experience we know then that we need to find a generalized equation

$$[Z]\{C\} = \{0\}$$

- Where $\det[Z]=0$ will give us ω_{nr}
- Solving the solution vector $\{C\}$ at ω_{nr} will define the mode shapes
- To do this you need a generalized equation for vertical displacement, y, as a function of distance along the beam, x, and time, t.

- For free vibration at a natural frequency, the motion of each point on the beam will be sinusoidal, but the amplitude of vibration will vary along the length



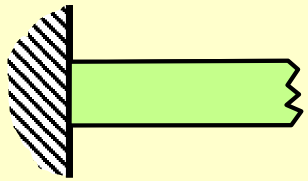
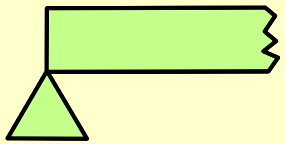
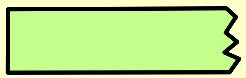
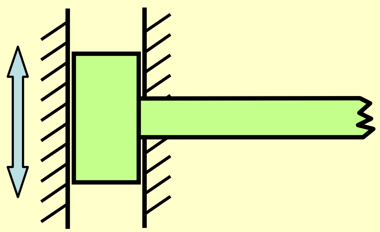
- Substitution of $y(x,t) = Y(x)\cos\omega t$ into $EI\frac{\partial^4 y}{\partial x^4} = -\rho A \frac{\partial^2 y}{\partial t^2}$

$$Y(x) = C_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \sinh\lambda x + C_4 \cosh\lambda x \qquad (7)$$

$$Y(x) = C_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \sinh\lambda x + C_4 \cosh\lambda x$$

- This results in a generalized equation for displacement of *y* at any given point along the beam, *x*, for a given frequency of vibration (contained in λ)

- **HOWEVER**, this contains 4 unknowns (C_1, C_2, C_3 and C_4) and you will therefore need a minimum of 4 equations to solve for them
  - Boundary conditions must be used!!!

Descriptive terms	Diagrammatic	Boundary conditions
Built-in clamped encastré		$y = 0 \quad \frac{\partial y}{\partial x} = 0$
Simple support hinged pinned		$y = 0$ $M = 0 \quad \therefore \quad \frac{\partial^2 y}{\partial x^2} = 0$
Free		$M = 0 \quad \therefore \quad \frac{\partial^2 y}{\partial x^2} = 0$ $S = 0 \quad \therefore \quad \frac{\partial^3 y}{\partial x^3} = 0$
Massless slider		$\frac{\partial y}{\partial x} = 0$ $S = 0 \quad \therefore \quad \frac{\partial^3 y}{\partial x^3} = 0$

You will therefore need to partially differentiate (7)

$$Y(x) = C_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \sinh\lambda x + C_4 \cosh\lambda x \quad \textbf{(7a)}$$

several times with depending on what boundary conditions you have

$$\frac{dY}{dX} = C_1 \lambda\cos\lambda x - C_2 \lambda\sin\lambda x + C_3 \lambda\cosh\lambda x + C_4 \lambda\sinh\lambda x \quad \textbf{(7b)}$$

$$\frac{d^2Y}{dx^2} = -C_1 \lambda^2 \sin\lambda x - C_2 \lambda^2 \sin\lambda x + C_3 \lambda^2 \sinh\lambda x + C_4 \lambda^2 \cosh\lambda x \quad \textbf{(7c)}$$

$$\frac{d^3Y}{dx^3} = -C_1 \lambda^3 \cos\lambda x - C_2 \lambda^3 \sin\lambda x + C_3 \lambda^3 \cosh\lambda x + C_4 \lambda^3 \sinh\lambda x \quad \textbf{(7d)}$$

Example 3 Cantilever (Clamped-pinned) Beam

1. Boundary conditions

The boundary conditions are

Clamped end at $x = 0$, $Y = 0$ and $\frac{dY}{dx} = 0$

Pinned end at $x = L$, $Y = 0$ and $\frac{d^2Y}{dx^2} = 0$

Using these conditions with the previous equations results into the previous 4 equations

Hence, at $\boldsymbol{x = 0}$

$$Y(0)_{x=0} = C_1 \times 0 \;+ C_2 \times 1 \;+ C_3 \times 0 \;+ C_4 \times 1 \quad \textbf{(7a)}$$

$$= C_2 + C_4 = 0$$

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)_{x=0} = \lambda C_1 \times 0 \;- \lambda C_2 \times 1 + \lambda C_3 \times 0 \;+ \lambda C_4 \times 1 \quad \textbf{(7b)}$$

$$= -\lambda C_2 + \lambda C_4 = 0$$

and at $\boldsymbol{x = L}$

(7a)

$$Y(x)_{x=L} = C_1 \sin\lambda L + C_2 \cos\lambda L + C_3 \sinh\lambda L + C_4 \cosh\lambda L = 0$$

(7c)

$$\left(\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}\right)_{x=L} = -\lambda^2 C_1 \sin\lambda L - \lambda^2 C_2 \cos\lambda L + \lambda^2 C_3 \sinh\lambda L + \lambda^2 C_4 \cosh\lambda L = 0$$

YOU NOW HAVE 4 EQUATIONS WITH 4 UKNOWNS!!!!

2. Assemble into matrix form

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -\lambda & 0 & \lambda \\ \sin\lambda L & \cos\lambda L & \sinh\lambda L & \cosh\lambda L \\ -\lambda^2\sin\lambda L & -\lambda^2\cos\lambda L & \lambda^2\sinh\lambda L & \lambda^2\cosh\lambda L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \begin{matrix} (7a) \\ (7b) \\ (7a) \\ (7c) \end{matrix}$$

$$[Z]\{C\} = \{0\}$$

3. Solving $\det[Z]=0$ gives the **Frequency Equation** and its roots will give ω_{nr} contained in λ_r

- This is complicated so we have given you the resulting Frequency Equation for a number of different beam types on **page 5** of your notes

$$\tan\lambda L - \tanh\lambda L = 0$$

$$\tan\lambda_r L - \tanh\lambda_r L = 0$$

- But this is still difficult to solve, so we give you the numerical solutions

Numerical values of roots $\lambda_r L$ of frequency equations

r	1	2	3	4	5	>5
Pinned-pinned	π	2π	3π	4π	5π	$r\pi$
Clamped-clamped & free-free	4.730	7.853	10.996	14.137	17.279	$\approx (r + 0.5)\pi$
Clamped-pinned & free-pinned	3.927	7.069	10.210	13.351	16.493	$\approx (r + 0.25)\pi$
Clamped-free	1.875	4.694	7.855	10.996	14.137	$\approx (r - 0.5)\pi$

Selecting the values of $\lambda_r L$ from the above table for the beam of interest, the natural frequencies can be found from equation (5). That is:

$$\omega_{nr} = \frac{(\lambda_r L)^2}{L^2} \sqrt{\frac{E\,I}{\rho A}}$$

- To solve for the mode shapes at a given natural frequency, ω_{nr} *with r*=1,2,3,… , remember that you have 4 equations with 4 unknowns (C_1, C_2, C_3 and C_4)

$$C_2 + C_4 = 0$$

$$-\lambda_r C_2 + \lambda_r C_4 = 0$$

$$C_1 \sin\lambda_r L + C_2 \cos\lambda_r L + C_3 \sinh\lambda_r L + C_4 \cosh\lambda_r L = 0$$

$$-\lambda_r^2 C_1 \sin\lambda_r L - \lambda_r^2 C_2 \cos\lambda_r L + \lambda_r^2 C_3 \sinh\lambda_r L + \lambda_r^2 C_4 \cosh\lambda_r L = 0$$

- You also have the table for numerical values of $\lambda_r L$
- Finally you have the equations to relate these to λ_r

$$\omega_{nr} = \frac{(\lambda_r L)^2}{L^2} \sqrt{\frac{EI}{\rho A}} \Rightarrow \lambda_r^4 = \frac{\rho A \omega_{nr}^2}{EI}$$

- You should be able to solve these for the constants C_1, C_2, C_3 and C_4 at given natural frequencies (r=1,2,3,…)

- Your amplitude of displacement for any given point along the beam, $Y(x)$, at a given frequency is then back to the general equation (7) from before

$$Y(x) = C_1 \sin\lambda_r x + C_2 \cos\lambda_r x + C_3 \sinh\lambda_r x + C_4 \cosh\lambda_r x$$

- Solving this at various points along the beam will then give you the mode shape of the beam at that frequency