Department of Mechanical, Materials & Manufacturing Engineering

## DYNAMICS (VIBRATION)

### SHEET 1: SINGLE-DEGREE-OF-FREEDOM SYSTEMS

1. Determine the equivalence spring stiffness, $k_e$, for the mass-spring system in Figure Q1a. Calculate the natural frequency of the system.
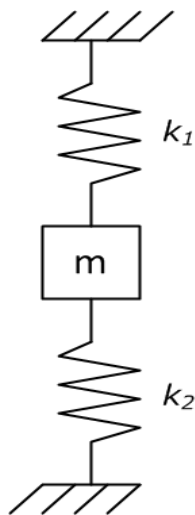


Fig. Q1(a)                                   Fig. Q1(b)

$$k_e = k_1 + k_2; \quad \omega_n = \sqrt{\frac{k_1 + k_2}{m}}$$

2. Draw the Free-Body-Diagram (FBD) of the mass-spring-pulley system in Figure Q2, and determine the natural frequency of the system. Note that the pulley has a uniform mass $M$ with a radius of $r$.
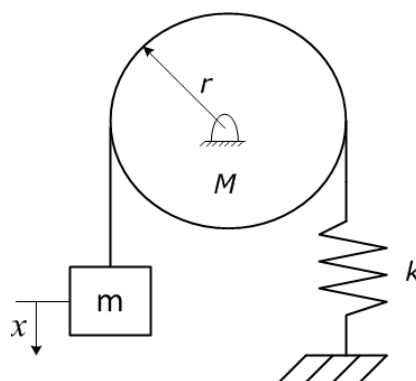


Figure Q2

$$\omega_n = \sqrt{\frac{k}{\frac{M}{2} + m}}$$

3. A uniform stiff rod is restrained by linear and torsional springs as shown in Figure Q3. The stiffness of linear spring is $k$, while the stiffness of torsional spring is $K$. Calculate the natural frequency of the vertical oscillation of the rod.
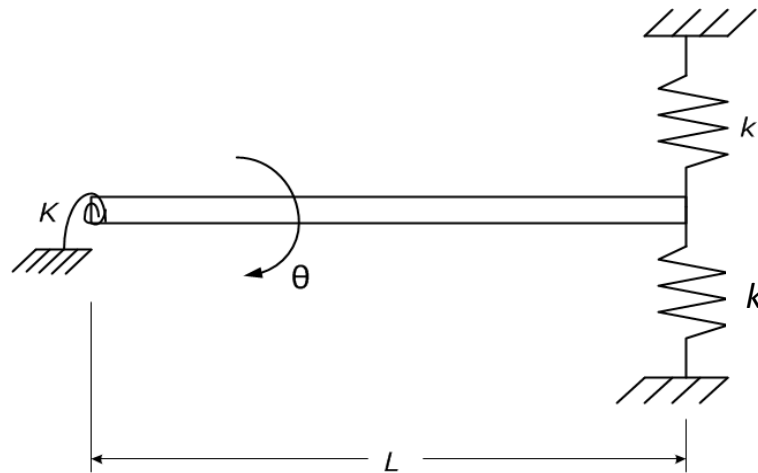


Figure Q3

$$\omega_n = \sqrt{\frac{3K + 6kL^2}{mL^2}}$$

4. For the system given below, draw the correct FBD for each mass, $M$, and the resulting EOM for the system. Then determine the resulting natural frequency for the systems. Assume the beam in Figure Q4(c) is rigid.
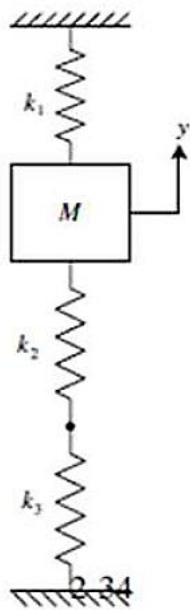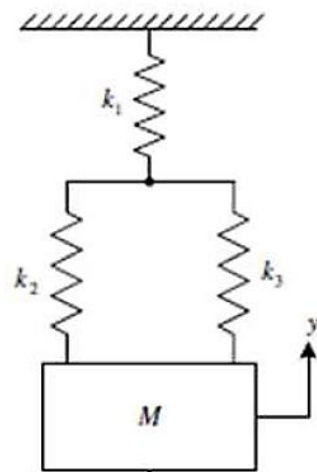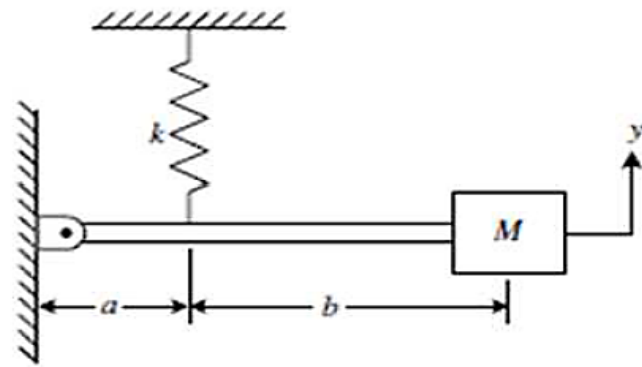


Fig. Q4(a)



Fig. Q4(b)

Fig. Q4(c)

4a) $\omega_n = \sqrt{\dfrac{k1 + \dfrac{k2k3}{k2+k3}}{m}}$

4b) $\omega_n = \sqrt{\dfrac{k1\,(k2+k3)}{(k1+k2+k3)m}}$

4c) $\omega_n = \sqrt{\dfrac{k\left(\dfrac{a}{a+b}\right)^2}{m}}$

5. For the following systems, assume the beams are flexible and massless (i.e. they can be considered as a spring and therefore have a stiffness associated with them). Draw the correct FBD for the system about mass, $M$, then determine the resulting EOM and natural frequency for each system.
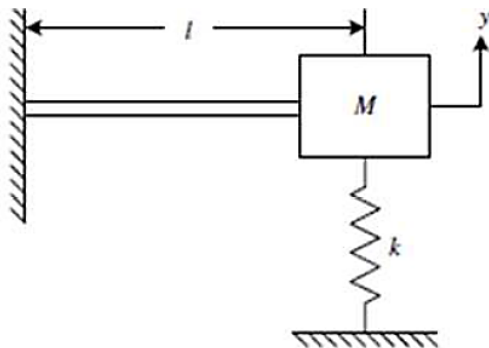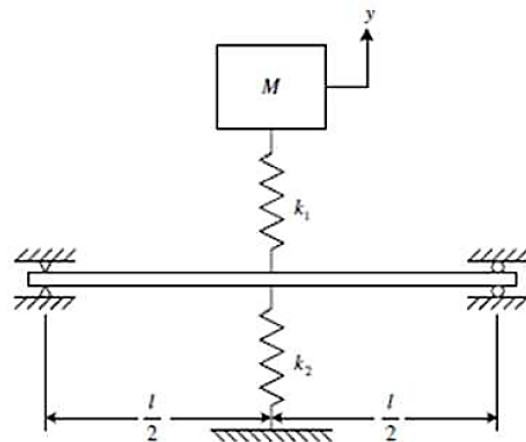


Fig. Q5(a)



Fig. Q5(b)

5a) $\quad \omega_n = \sqrt{\dfrac{\frac{3EI}{l^3}+k}{m}}$

5b) $\quad \omega_n = \sqrt{\dfrac{k_1\left(k_2+\frac{48EI}{l^3}\right)}{\left(k_1+k_2+\frac{48EI}{l^3}\right)m}}$
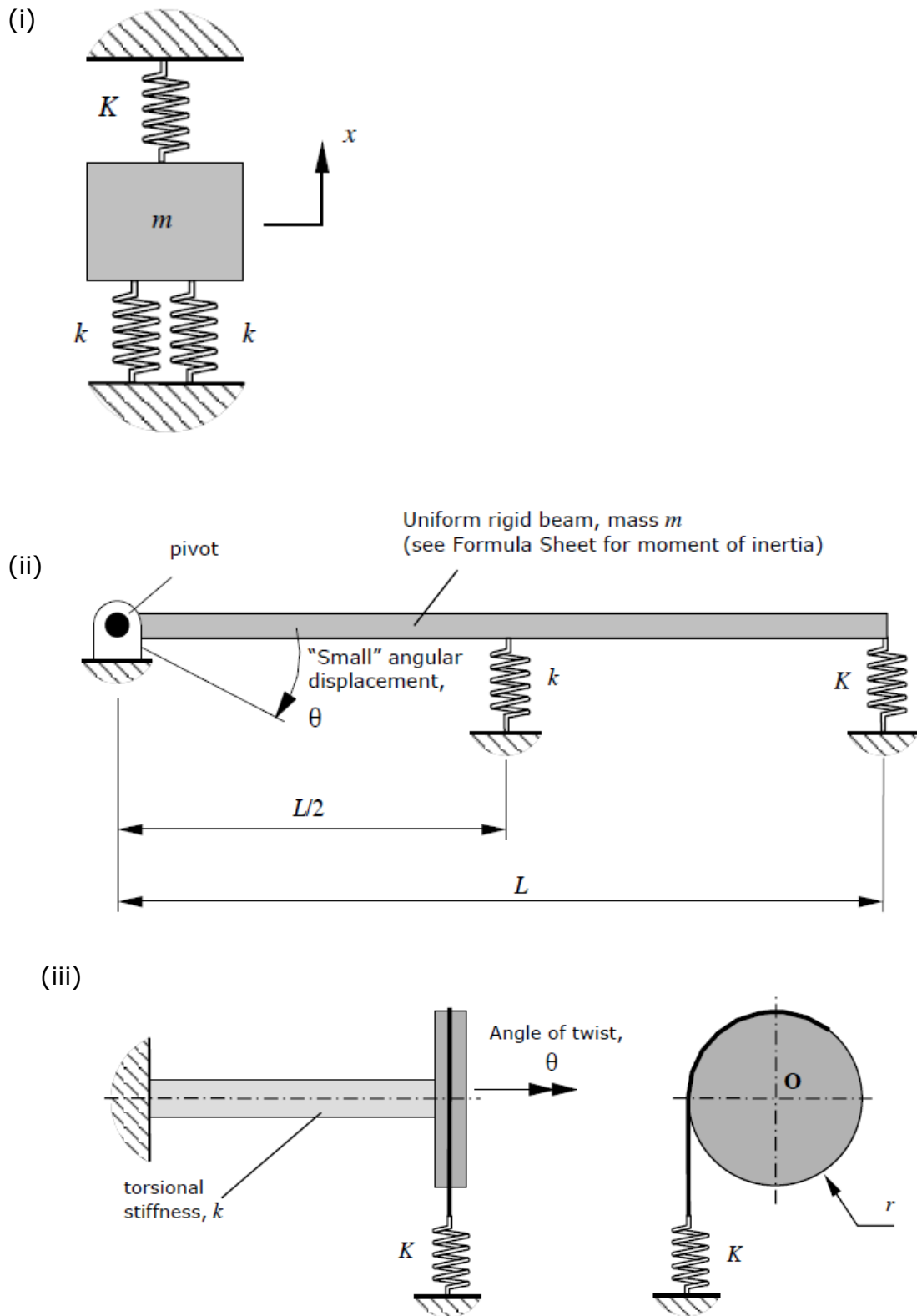
6. Derive the equations of motion and hence find the natural frequencies for the vibrating systems shown in Figure Q6 (overleaf). Assume that all displacements are small. For system (iii), you should assume that the spring is pre-loaded in tension so that it never goes slack.

$(i)\ m\ddot{x} + (2k + K)x = 0 \qquad \omega_n = \sqrt{\dfrac{2k + K}{m}}$

$(ii)\ \dfrac{m}{3}\ddot{\theta} + \left(K + \dfrac{k}{4}\right)\theta = 0 \qquad \omega_n = \sqrt{\dfrac{3(4K + k)}{4m}}$

$(iii)\ I_0\ddot{\theta} + \left(k + K r^2\right)\theta = 0 \qquad \omega_n = \sqrt{\dfrac{k + K r^2}{I_0}}$

## Figure Q6

**(i)**



**(ii)**

Uniform rigid beam, mass $m$
(see Formula Sheet for moment of inertia)

pivot

"Small" angular displacement, $\theta$

$k$

$K$

$L/2$

$L$



**(iii)**

Angle of twist, $\theta$

torsional stiffness, $k$

$K$

$K$

O

$r$

7. A wheel (radius $r$, mass $m$, moment of inertia about its centre $I$ ) can roll without slipping on a horizontal plane. It is restrained by a horizontal spring (stiffness $k$) attached at one end to the centre of the wheel and at the other end to a rigid vertical wall, as in Figure Q7. Derive the equation of motion and hence find the natural frequency for the system.

What would the natural frequency be if there was no friction between the wheel and the plane?

$$\omega_n = \sqrt{\frac{k}{m + \dfrac{I}{r^2}}} \; ; \quad \omega_n = \sqrt{\frac{k}{m}}$$
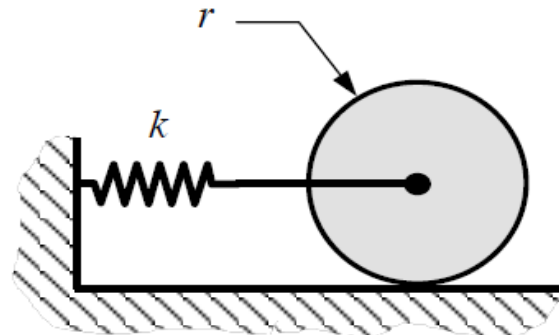


Figure Q7

8. In two of the examples from the lectures, a block is supported at its centre by a cantilever beam acting as a massless spring. The beam is 150 mm long, has a circular cross-section of diameter 16 mm and is made of steel (take Young's modulus, $E$ = 200 GN/m2 and the shear modulus, $G$ = 82.5 GN/m2). When tested, it is found that the natural frequencies for bending and torsional vibration of the beam are 15 Hz and 23 Hz respectively.
Consider the two modes of vibration separately and calculate the mass of the block and its moment of inertia about the beam axis.

*64.4 kg, 0.165 kg m²*

9. A mass $m$ hung on the end of a spring of stiffness $k$ produces a static deflection $x_0$ to reach its equilibrium position. By considering the equation for static equilibrium, show that the natural frequency of the mass-spring system can be expressed as $\omega_n = \sqrt{\dfrac{g}{x_0}}$

Note: *this question is much easier than you may think!*