MM2DYN Dynamics: FORMULA SHEET

The symbols have the meanings defined during the course of the module

Velocity and Acceleration Relations on a Rigid Body in Planar Motion

$\underline{\boldsymbol{v}}_{\mathrm{B}} = \underline{\boldsymbol{v}}_{\mathrm{A}} + \underline{\boldsymbol{v}}_{\mathrm{BA}}$ with $v_{\mathrm{BA}} = \omega . AB$

$\underline{\boldsymbol{a}}_{\mathrm{B}} = \underline{\boldsymbol{a}}_{\mathrm{A}} + \underline{\boldsymbol{a}}^{\mathrm{n}}_{\mathrm{BA}} + \underline{\boldsymbol{a}}^{\mathrm{t}}_{\mathrm{BA}}$ with $a^{\mathrm{n}}_{\mathrm{BA}} = \omega^2 . AB$ and $a^{\mathrm{t}}_{\mathrm{BA}} = \alpha . AB$

Moment of Inertia (all the following moments of inertia for an axis through the centre of mass)

Point mass at distance *a* from axis	ma^2
Thin ring, about axis in plane of ring	$\frac{mr^2}{2}$
Thin ring, about axis perpendicular to plane of ring	mr^2
Thin disc, about axis in plane of disc	$\frac{mr^2}{4}$
Thin disc, about axis perpendicular to plane of disc	$\frac{mr^2}{2}$
Solid cylinder, about axis of revolution	$\frac{mr^2}{2}$
Solid cylinder, about axis perpendicular to axis of revolution	$\frac{m}{12}(3r^2 + h^2)$
Hollow cylinder, about axis of revolution	$\frac{m}{2}\left(r_i^2 + r_o^2\right)$
Hollow cylinder, about axis perpendicular to axis of revolution	$\frac{m}{12}\left[3\left(r_i^2 + r_o^2\right) + h^2\right]$
Rod (bar), about axis perpendicular to the rod	$\frac{ml^2}{12}$
Rod (bar), with axis perpendicular through the end	$\frac{ml^2}{3}$
Rectangular lamina, about axis parallel to side of length l_x	$\frac{ml_y^2}{12}$
Rectangular lamina, about axis parallel to side of length l_y	$\frac{ml_x^2}{12}$
Rectangular lamina, about axis perpendicular to plane of lamina	$\frac{m}{12}\left(l_x^2 + l_y^2\right)$
Solid cuboid, about axis parallel to side of length l_x	$\frac{m}{12}\left(l_y^2 + l_z^2\right)$
Solid cuboid, about axis parallel to side of length l_y	$\frac{m}{12}\left(l_x^2 + l_z^2\right)$
Solid cuboid, about axis parallel to side of length l_z	$\frac{m}{12}\left(l_x^2 + l_y^2\right)$
Solid sphere, about any axis through centre	$\frac{2mr^2}{5}$
Perpendicular Axes Rule:	$J_z = J_x + J_y$
Parallel Axes Rule:	$J_A = J_G + md^2$

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Free vibration of single degree of freedom systems

$$\gamma = \frac{C}{2\sqrt{KM}}$$

For zero damping, C=0 or $\gamma = 0$

$$z(t) = B\cos\omega_n t + C\sin\omega_n t$$

For high damping, $C > 2\sqrt{KM}$ or $\gamma > 1$

$$z(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \quad \text{where} \quad \lambda_{1,2} = \frac{-C \pm \sqrt{C^2 - 4KM}}{2M}$$

For critical damping, $C = C_{\text{crit}} = 2\sqrt{KM}$ or $\gamma = 1$

$$z(t) = A_1 e^{-\omega_n t} + A_2 \boldsymbol{t}\, e^{-\omega_n t}$$

For light damping, $C < 2\sqrt{KM}$ or $\gamma < 1$

$$z(t) = C_0\, e^{-\gamma\omega_n t}\cos(\Omega_n t - \psi)$$

or you can use

$$z(t)_{transient} = e^{-\gamma\omega_n t}[B_1\cos(\Omega_n t) + B_2\sin(\Omega_n t)]$$

$$\dot{z}(t)_{transient} = B_1 e^{-\gamma\omega_n t}[-\Omega_n\sin(\Omega_n t) - \gamma\omega_n\cos(\Omega_n t)] + B_2 e^{-\gamma\omega_n t}[\Omega_n\cos(\Omega_n t) - \gamma\omega_n\sin(\Omega_n t)]$$

where the damped natural frequency is $\Omega_n = \omega_d = \omega_n\sqrt{1-\gamma^2}$

Forced vibration due to harmonic excitation

$$z(t)_{SS} = |Z^*|\cos(\omega t + \alpha)$$

Frequency Response Function = $H(\omega)$ = Response per unit applied force

For a simple Single-Degree-of-Freedom mass-spring-damper system with input force of amplitude (F) and frequency (ω)

$$H(\omega) = \frac{Z^*}{F} = \frac{1}{(K - M\omega^2) + \mathbf{i}\,\omega C} = \frac{1}{K}\,\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + \mathbf{i}\,2\gamma\frac{\omega}{\omega_n}}$$

$$|Z^*| = \frac{F}{\sqrt{(K - M\omega^2)^2 + \omega^2 C^2}} = \frac{F}{K\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\gamma^2\frac{\omega^2}{\omega_n^2}}}$$

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$$\alpha = \tan^{-1}\left(\frac{-\omega C}{K - M\omega^2}\right) = \tan^{-1}\left(\frac{-2\gamma\dfrac{\omega}{\omega_n}}{1-\dfrac{\omega^2}{\omega_n^2}}\right)$$

For a simple Single-Degree-of-Freedom system with input displacement of amplitude(R) and frequency (ω) through a spring (K_r) and damper (C_r)

$$H(\omega) = \frac{Z^*}{R} = \frac{K_r + \mathbf{i}\,C_r\,\omega}{\left(K - M\omega^2\right) + \mathbf{i}\,C\omega} = \frac{1}{K}\frac{K_r + \mathbf{i}\,C_r\,\omega}{\left(1-\dfrac{\omega^2}{\omega_n^2}\right) + \mathbf{i}\,2\gamma\dfrac{\omega}{\omega_n}}$$

$$\left|Z^*\right| = \frac{\sqrt{K_r^2 + C_r^2\omega^2}\;R}{\sqrt{\left(K - M\omega^2\right)^2 + C^2\omega^2}} = \frac{\sqrt{K_r^2 + C_r^2\omega^2}\;R}{K\sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2 + 4\gamma^2\dfrac{\omega^2}{\omega_n^2}}}$$

$$\alpha = \tan^{-1}\left(\frac{C_r\omega\left(K - M\omega^2\right) - K_r C\omega}{K_r\left(K - M\omega^2\right) + C_r C\omega^2}\right)$$

Flexural vibration of beams/shafts

$$Y(x) = C_1\sin(\lambda x) + C_2\cos(\lambda x) + C_3\sinh(\lambda x) + C_4\cosh(\lambda x)$$

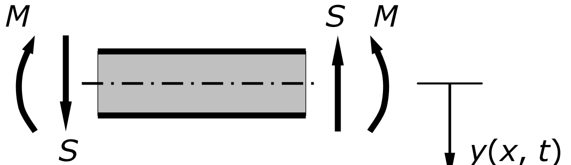
$$\frac{\partial Y}{\partial X} = \lambda C_1\cos(\lambda x) - \lambda C_2\sin(\lambda x) + \lambda C_3\cosh(\lambda x) + \lambda C_4\sinh(\lambda x)$$

$$\frac{\partial^2 Y}{\partial X^2} = -\lambda^2 C_1\sin(\lambda x) - \lambda^2 C_2\cos(\lambda x) + \lambda^2 C_3\sinh(\lambda x) + \lambda^2 C_4\cosh(\lambda x)$$

$$\frac{\partial^3 Y}{\partial X^3} = -\lambda^3 C_1\cos(\lambda x) + \lambda^3 C_2\sin(\lambda x) + \lambda^3 C_3\cosh(\lambda x) + \lambda^3 C_4\sinh(\lambda x)$$

where $\lambda^4 = \omega^2\dfrac{\rho A}{EI}$

$$EI\frac{\partial^4 y}{\partial x^4} = -\rho A\frac{\partial^2 y}{\partial t^2}$$



Sign conventions

Bending moment, $M = -EI\dfrac{\partial^2 y}{\partial x^2}$

Shear force, $S = EI\dfrac{\partial^3 y}{\partial x^3}$

Natural frequencies of a simply-supported beam, $\omega_r = \left(\dfrac{\lambda_r L}{L}\right)^2\sqrt{\dfrac{EI}{\rho A}}$

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Maximum kinetic and strain energies for a beam/shaft

$$T_{MAX} = \frac{1}{2}\,\omega^2 \int_0^L \rho A\,[Y(x)]^2\,\mathrm{d}x$$

$$U_{MAX} = \frac{1}{2}\int_0^L EI\left(\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2}\right)^2 \mathrm{d}x$$

Vibration isolation

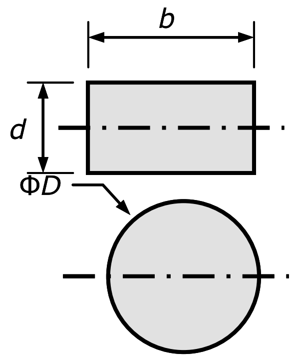
Transmissibility for a Single-Degree-of-Freedom mass-spring-damper system with input force or a Single-Degree-of-Freedom mass-spring-damper system with input displacement through the spring and damper.

$$T_D = T_F = \sqrt{\frac{k^2 + c^2\omega^2}{\left(k - m\omega^2\right)^2 + c^2\omega^2}} = \sqrt{\frac{1 + 4\gamma^2 \dfrac{\omega^2}{\omega_n^2}}{\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2 + 4\gamma^2 \dfrac{\omega^2}{\omega_n^2}}}$$

For a simple mass-spring model, e.g. no damping, $\omega > \omega_n$, $T=T_{max}$ and $\omega = \omega_{MIN}$.

$$k_{max} = m\omega_n^2 = m\left(\frac{T_{max}\omega_{min}^2}{1 + T_{max}}\right)$$

Second moments of area



Rectangular cross-section about neutral axis $\dfrac{bd^3}{12}$

Circular cross-section about a diameter $\dfrac{\pi D^4}{64}$

Polar second moment for circular cross-section $\dfrac{\pi D^4}{32}$

Stiffnesses

Torsional stiffness of uniform circular shaft $\dfrac{G\pi D^4}{32\,L}$

Lateral stiffness of a simply-supported beam at a point distance of *x* from one end

$$\frac{3\,EIL}{x^2(L-x)^2}$$

Lateral stiffness at the free end of a uniform cantilever beam $\dfrac{3\,EI}{L^3}$

Fourier series

$$p(t) = A_0 + \sum_{j=1}^{\infty}[A_j\cos(j\omega_0 t) + B_j\sin(j\omega_0 t)]$$

where $\omega_0 = \dfrac{2\pi}{T}$ and T = repetition period

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Quadratic Equation

For $Ax^2 + Bx + C = 0$ $\qquad x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

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TABLE OF LAPLACE TRANSFORM PAIRS

Entry	Time domain $f(t)$	Laplace domain $F(s)$
1	$\frac{df(t)}{dt}$	$sF(s) - f(0)$
2	$\frac{d^n f(t)}{d t^n}$	$s^n F(s) - s^{n-1} f(0) - \ldots\ldots - f^{n-1}(0)$
3	$\int f(t)\, dt$	$\frac{1}{s} F(s)$
4	Unit impulse $\delta(t)$	1
5	Unit step 1	$\frac{1}{s}$
6	Unit ramp t	$\frac{1}{s^2}$
7	e^{-at}	$\frac{1}{s+a}$
8	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
9	$t - \frac{1}{a}(1 - e^{-at})$	$\frac{a}{s^2(s+a)}$
10	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
11	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
12	$\frac{1}{(\omega^2 - p^2)}\left[\sin(pt) - \frac{p}{\omega}\sin(\omega t)\right]$	$\frac{p}{(s^2 + p^2)(s^2 + \omega^2)}$
13	$\frac{1}{(\omega^2 - p^2)}[\cos(pt) - \cos(\omega t)]$	$\frac{s}{(s^2 + p^2)(s^2 + \omega^2)}$
14	$\frac{\omega}{\sqrt{1-\gamma^2}} e^{-\gamma\omega t} \sin(\omega t\sqrt{1-\gamma^2})$	$\frac{\omega^2}{s^2 + 2\gamma\omega s + \omega^2}$
15	$1 - \frac{e^{-\gamma\omega t}}{\sqrt{1-\gamma^2}} \sin(\omega t\sqrt{1-\gamma^2} + \phi)$	$\frac{\omega^2}{s(s^2 + 2\gamma\omega s + \omega^2)}$
16	$t - \frac{2\gamma}{\omega} - \frac{e^{-\gamma\omega t}}{\omega\sqrt{1-\gamma^2}} \sin(\omega t\sqrt{1-\gamma^2} + 2\phi)$	$\frac{\omega^2}{s^2(s^2 + 2\gamma\omega s + \omega^2)}$

where $\cos\phi = \gamma$ and $\gamma < 1$